SMTS-2
Theory of Structures

B.C. PUNMIA
ASHOK JAIN
ARUN JAIN

S.I. UNITS
Theory of Structures
STRENGTH OF MATERIALS
AND
THEORY OF STRUCTURES
(VOLUME II)

By

Dr. B.C. PUNMIA
B.E. (Hons.), M.E. (Hons.), Ph.D.
FORMERLY
PROFESSOR AND HEAD,
DEPARTMENT OF CIVIL ENGINEERING, &
DEAN, FACULTY OF ENGINEERING
J.N.V. UNIVERSITY,
JODHPUR

* ASHOK KUMAR JAIN
CONSULTING ENGINEER
ARIHANT CONSULTANTS, BOMBAY

* ARUN KUMAR JAIN
ASSISTANT PROFESSOR OF CIVIL ENGINEERING
M.B.M. ENGINEERING COLLEGE,
JODHPUR

IN SI UNITS

NINTH EDITION
(Thoroughly Revised and Enlarged)

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Preface to the Third Edition

In this volume, the author has attempted to present more advanced topics in strength of materials and mechanics of structures in a rigorous and coherent manner. Both volumes I and II cover the entire course for degree, diploma and AMIE examinations in this subject. Volume II has been divided into four parts. Part I contains four chapters on moving loads on statically determinate beams and frames, as well as on statically indeterminate beams. Part II deals with statically indeterminate structures and contains ten chapters, including a chapter on deflection of perfect frames. Part III contains five chapters on advanced topics on strength of materials. These chapters are mostly useful for mechanical engineering students. Part IV has three chapters on miscellaneous topics, including one on elementary theory of elasticity. The contents of the book are so designed that the book is equally useful to civil as well as mechanical and electrical engineering students.

The book is written entirely in metric units. Each topic introduced is thoroughly described, the theory is rigorously developed, and a large number of numerical examples are included to illustrate its application. General statements of important principles and methods are almost invariably given by practical illustrations. A large number of problems are available at the end of each chapter, to enable the student to test his reading at different stages of his studies.

The author is highly thankful to Prof. S.C. Goyal and Shri O.P. Kalani for their permission to reproduce some chapters and examples from the book 'Strength of Materials and Theory of Structures Vol. II' written by the author in their collaboration. Thanks are also due to the senate of the London University and to the Secretary of the Institution of Structural Engineers, London, for their kind permission to reproduce their examination questions. The London University is in no way responsible for the accuracy of the answers. In preparing this text, the author has freely consulted many excellent books on the subject and the help is gratefully acknowledged.

Every effort was made to eliminate errors in the book, but should the reader discover some, the author would appreciate
having these brought to his attention. Suggestions from the readers for improvement in the book will be most gratefully acknowledged.

Jodhpur,
January 17, 1971

B.C. PUNMIA

Preface to the Fourth Edition

In the fourth edition, the subject matter of the book has been updated. An appendix, containing questions from the AMIE section B examinations in "Theory of Structures", has been added. The author is thankful to Shri J.N. Shrivastava for his valuable suggestions for improvements in the book.

Jodhpur,
January 1, 1974

B.C. PUNMIA

Preface to the Sixth Edition

In the Sixth Edition, the subject matter has been revised and updated. A large number of Examples in SI units have been added at the end of the book.

Jodhpur,
15th June, 1982

B.C. PUNMIA

Preface to the Seventh Edition

In the Seventh Edition, the entire book has been rewritten using SI units. The old diagrams have been replaced by new ones.

Jodhpur
Deepawali
18.11.85

B.C. PUNMIA

Preface to the Eighth Edition

In the Eighth Edition, the subject matter has been revised and updated. A new chapter on Building Frames has been added at the end of the book.

Jodhpur
1st Sept., 1988

B.C. PUNMIA

Preface to the Ninth Edition

In the Ninth Edition, the book has been completely rewritten. The book has been divided into four sections. Section 1, containing five chapters, is on 'Moving Loads'. Section 2 on 'Statically Indeterminate Structures', contain eleven chapters. Section three is devoted to 'Advanced Topics in Strength of Materials' and has six chapters. Lastly, section 4 has six chapters on 'Miscellaneous Topics'. Thus in the Ninth Edition of the book, which has 28 chapters, five new chapters have been added.

In each chapter, the subject matter has been rearranged and new articles have been added. Many new advanced problems have been added which will be useful for competitive examinations.

Jodhpur
Mahaveer Jayanee
15-4-92

B.C. PUNMIA
ASHOK KUMAR JAIN
ARUN KUMAR JAIN
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MOVING LOADS

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3. INFLUENCE LINES FOR GIRDERS WITH FLOOR BEAMS
4. INFLUENCE LINES FOR FRAMES
5. THE MULLER-BRESLAU PRINCIPLE
11. INTRODUCTION

In the case of static or fixed load positions, the B.M. and S.F. diagrams can be plotted for a girder, by the simple principles of statics. In the case of rolling loads, however, the B.M. and S.F. at a section of the girder change as the loads move from one position to the other. The problem is, therefore, two-fold:

(i) to determine the load positions for maximum bending moment or shear force for a given section of a girder and to compute its value, and (ii) to determine the load positions so as to cause absolute maximum bending moment or shear force anywhere on the girder.

For every cross-section of girder, the maximum B.M. and S.F. can be worked out by placing the loads in appropriate positions. When these are plotted for all the sections of the girder, we get the maximum B.M. and maximum S.F. diagrams. The ordinate of a maximum B.M. or S.F. diagram at a section gives the maximum B.M. (or S.F.) at that section, due to a given train of loads.

We shall consider the following cases of loadings:
1. Single concentrated load.
2. Uniformly distributed load longer than the span of the girder.
3. Uniformly distributed load shorter than the span of the girder.
4. Two loads with a specified distance between them.
5. Multiple concentrated loads (train of wheel loads).

Sign Conventions

The following sign conventions will be followed for B.M. and S.F. at a given section (Fig. 1'1).
(1) A shear force having an upward direction to the right hand side of a section or downwards to the left of the section will be taken positive. Similarly, a negative S.F. will be one that has a downward direction to the right of the section or upward direction to the left of the section [Fig. 1-1(a, b)].

\[ + \text{VE S.F.} - \text{VE S.F.} - \text{VE B.M.} + \text{VE B.M.} \]

(a) (b) (c) (d)

Fig. 1-1.

(2) A B.M. causing concavity upwards will be taken as negative and will be called sagging B.M. Similarly a B.M. causing convexity upwards will be taken as positive, and will be called hogging bending moment [Fig. 1-1(c, d)].

1.2. SINGLE CONCENTRATED LOAD

Let us now consider a single concentrated load \( W \) travelling or rolling along a simply supported beam or girder \( AB \), of span \( L \), from left to right.

(a) MAXIMUM SHEAR FORCE DIAGRAMS

Consider a point \( C \), distant \( x \) from the left support \( A \). Let the distance of load \( W \) be \( y \) from \( A \). For any load position, the reaction

\[ R_b = \frac{Wy}{L} \quad \text{and} \quad R_d = \frac{W(L-y)}{L} \]

(ai) Load in \( AC \) \((y<x)\)

Let the load \( W \) be in \( AC \), such that \( y \) is lesser than \( x \).

Then \( F_x \) (at \( C \)) = \( +R_b = +\frac{Wy}{L} \) \( \ldots (1) \)

Thus, the shear force \( F_x \) increases as \( y \) increases till \( y=x \), in which case,

\[ F_{\text{max}} = +\frac{Wx}{L} \] \( \ldots (1-1) \)

This happens when the load is on the section \( (C) \) itself, thus, the value of positive shear force at the section.

MAX S.F.D

MAX B.M.D

Fig. 1-2.

For different values of \( x \) (i.e., for different position of the section \( C \)), the maximum positive shear force given by equation 1-1 will vary linearly with \( x \).

Thus, at \( x=0 \), \( F_{\text{max.}} \cdot (+)=0 \)

at \( x=L \), \( F_{\text{max.}} \cdot (+)=+\frac{WL}{L}=+W=F_{\text{max. max.}} \)

The absolute maximum positive S.F. therefore occurs at the right hand support, its value being equal to \( +W \).

The maximum +ve shear force diagram is represented by \( \text{abb} \) of Fig. 1-2(b).

(aii) Load in \( CB \) \((y>x)\)

We have seen that when the wheel load \( W \) reaches the section \( C \), maximum +ve S.F. of value \( +\frac{Wx}{L} \) occurs at the section. When the load moves further (i.e., when \( y \) becomes greater than \( x \)), we have

\[ F_x \) (at \( C \)) = -R_d = -\frac{W(L-y)}{L} \] \( \ldots (2) \)

Thus, the shear force changes sign immediately when the load crosses the section. The maximum negative S.F. occurs evidently when \( y=x \) (i.e., when \( y \) is least and the load is in \( CB \)).
Thus, \( F_{\text{max}} = -\frac{W(L-x)}{L} \)  

(1.2)

For different values of \( x \) (i.e., for different positions of the section \( C \)), the maximum negative shear force, given by equation 1.2 will vary linearly with \( x \).

Thus, \( \frac{dF_{\text{max}}}{dx} = 0 \) if \( x = 0 \).

\( \therefore \frac{dF_{\text{max}}}{dx} = 0 \) 

The maximum negative S.F. therefore occurs at the left hand support, its value being \(-W\).

The maximum negative S.F.D. is represented by \( a_{\text{b}} \) of Fig. 1.2(b).

(b) MAXIMUM BENDING MOMENT DIAGRAM

Let us now draw the maximum bending moment diagram for the beam \( AB \). It must be noted that a simply supported beam, under downward loads, bends causing concavity to the upper side. Hence the bending moment is always negative for all sections of the beam. Therefore, the maximum bending moment diagram will also be negative.

(bi) Load in \( AC \) (\( y < x \))

When the load is in \( AC \),

\[
M_x = -R_b (L-x) = -\frac{W}{L} (L-x)
\]

This increases as \( y \) increases. When the load reaches the section \( C \), \( y = x \), and the section has the maximum bending moment.

\[
M_{\text{max}} = -\frac{W}{L} (L-x)
\]

(1.3)

(bii) Load in \( CB \) (\( y > x \))

When the load \( W \) crosses the section \( C \),

\[
M_x = -R_{A, x} = -\frac{W(L-y)}{L} x
\]

This increases as \( y \) decreases. When the load is on the section \( C \), \( y = x \), and the section has the maximum bending moment,

\[
M_{\text{max}} = -\frac{W(L-x)x}{L}
\]

which is the same as equation 1.3.

Thus, the maximum bending moment at a section occurs when the load is on the section itself.

**ROLLING LOADS**

For different values of \( x \) (i.e., for different positions of section \( C \)) the maximum bending moment given by equation 1.3 will vary parabolically with \( x \). Fig. 1.2(c) shows the maximum bending moment diagram. For absolute maximum bending moment

\[
\frac{dM_{\text{max}}}{dx} = 0
\]

\[
\therefore -\frac{W}{L} (L-2x) = 0
\]

or

\[
x = \frac{L}{2}
\]

Thus, the absolute maximum bending moment occurs at the centre of the span, and its value is given by

\[
M_{\text{max}, \text{abs}} = -\frac{W}{L} \left( L - \frac{L}{2} \right) \frac{L}{2} = \frac{WL}{4}
\]

**1.3. UNIFORMLY DISTRIBUTED LOAD LONGER THAN THE SPAN OF THE GIRDER**

Let us now study the case of the uniformly distributed load \( w \) per unit length, longer than the span, and moving from left to right.

(a) MAXIMUM S.F. DIAGRAM

Let us consider a section \( C \) distant \( x \) from left support \( A \), as shown in Fig. 1.3(a). Let the head of the load be distant \( y \) from \( A \).

**Reaction**

\[
R_b = \frac{wy^2}{2L}
\]

When the load is in \( AC \) (i.e., \( y < x \)),

\[
F_x = +R_b = +\frac{wy^2}{2L}
\]

(1)

This evidently increases as \( y \) increases, until the head of the load reaches the section \( C \) (i.e., when \( y = x \)).

\[
F_{\text{max}, y} = +\frac{w(y^2)}{2L}
\]

(1.4)

When the load still moves further, it can be proved that the value of \( F_x \) given by equation 1.4 decreases. To prove this statement, let the head of the load move by a distance \( \delta x \) from \( C \) towards \( B \), and let \( \delta R_b \) be the corresponding increase in the reaction at \( B \).

Then

\[
R_b + \delta R_b = \frac{w(x+\delta x)^2}{2L}
\]

and

\[
F_x = (+R_b + \delta R_b) - w \cdot \delta x
\]

\[
= +\frac{w(x+\delta x)^2}{2L} - w \cdot \delta x
\]

\[
\therefore \frac{dF_x}{dx} = 0
\]

**The maximum bending moment**
Since \( \frac{4}{2} \rightarrow \frac{1}{2} \), less than 1, the expression inside the bracket is negative. Hence \( F_x \) given by (2) is less than \( F_x \) given by equation 1.4. Thus, the maximum positive shear at a section occurs when the head of the load reaches the section, (i.e. when the left portion AC is loaded and the right portion CB is empty).

The maximum positive S.F. diagram can be plotted by giving different values of \( x \) in equation 1.4.

Thus, at \( x=0 \), \( F_{max} = 0 \)

\[
\text{at } x=L, F_{max} = \frac{wL}{2} \]

The absolute maximum positive S.F. occurs at the right hand support. The maximum +ve S.F.D. is shown by \( ab \) in Fig. 1.3(b).

The S.F. at section C will continue to decrease as the load advances further. When the load covers the entire span,

\[
F_x = -R_d + wL = -\frac{wL}{2} + wx
\]  

This is negative for the section C to be in the left half of the portion.

### Rolling Loads

Let the load still move on so that the portion \( CB \) is fully loaded and portion \( AC \) is partially loaded, and we have

\[
F_x = +R_d - w(L-x)
\]  

In the above expression, the quantity \( w(L-x) \) remains constant as the load still moves further, while \( R_d \) diminishes. Thus, with the onward movement of the load, the negative value of \( F_x \) increases.

When the tail of the load reaches the section C, we have

\[
F_x = -R_d = -\frac{w(L-x)^2}{2L}
\]

This is the maximum value of negative shear force at C. As the load moves further, \( R_d \) decreases, and hence \( F_x \) decreases.

Thus,

\[
F_{max} = -\frac{w(L-x)^2}{2L}
\]

The maximum negative shear force thus occurs when \( AC \) is empty and \( CB \) is fully loaded. To plot the maximum negative S.F. diagram vary \( x \) from 0 to \( L \).

At \( x=0 \), \( F_{max} = -\frac{wL}{2} = -\frac{wL}{2} = F_{max \text{ max}} \text{ (ve)} \)

At \( x=L \), \( F_{max} = 0 \)

The maximum -ve S.F.D. is shown by \( ab \) in Fig. 1.3 (b).

(b) **MAXIMUM B.M. DIAGRAM**

Let the head of the load be in \( AC \), such that \( y < x \).

\[
R_b = \frac{wL}{2L}
\]

\[
M_x = -R_d(L-x) = -\frac{wL}{2L} (L-x)
\]

The value of \( M_x \) goes on increasing as \( y \) increases till the head of the load reaches the section C, and

\[
M_x = -\frac{wx}{2L} (L-x)
\]

Let the load now advance further by a small amount \( \Delta x \), and let \( \Delta B \) be the corresponding increase in the reaction at \( B \), such that

\[
R + \Delta R_b = \frac{w}{2L} (x+\Delta x)^2
\]

Hence

\[
M_x = -(R_b + \Delta R_b)(L-x) + wL \cdot \frac{\Delta x}{2}
\]

\[
= -\frac{w}{2L} (x+\Delta x)^2 (L-x) + \frac{wL}{2}(x+\Delta x)^2
\]
This is evidently more than that given by (2). Hence the B.M. at the section C continues to increase as the load moves further, till it occupies the whole span. In that case,

\[ M_x = -\frac{wx(L-x)}{2} \]  \hspace{1cm} (4)

As the load still moves further, so that portion AC is partially loaded, and portion CB is fully loaded, we have

\[ M_x = -R_B(L-x) + \frac{n(L-x)^2}{2} \]  \hspace{1cm} (5)

In the above expression, the quantity \( \frac{n(L-x)^2}{2} \) is constant, while \( R_B \) diminishes as the load moves further. Hence \( M_x \) decreases till the tail of the load reaches the section C.

When the load is in the portion CB only, AC is empty.

\[ M_x = -R_A \cdot x \]  \hspace{1cm} (6)

Since \( R_A \) decreases as the load moves further, \( M_x \) also decreases. It can thus be concluded that the values of \( M_x \) given by (5) and (6) are less than that given by (4). Thus, the maximum B.M. at the section occurs when the whole span is loaded, and its value is given by

\[ M_{max} = \frac{wx(L-x)}{2} \]  \hspace{1cm} (15)

The maximum bending moment diagram is evidently a parabola as shown in Fig 1.3 (c). The absolute maximum bending moment evidently occurs at the centre of the span \((x=L/2)\).

\[ M_{max} = \frac{w}{2} \left( L - \frac{L}{2} \right)^2 = \frac{wL^2}{8} \]  \hspace{1cm} (1.5)

1.4. UNIFORMLY DISTRIBUTED LOAD SHORTER THAN THE SPAN OF THE GIRDER

Let the uniformly distributed load \( w/\text{unit length} \) extend over a length \( a \) such that \( a < L \).

(a) MAXIMUM S.F. DIAGRAMS

(ai) Maximum Positive S.F.

(1) Let the position of the section C be such that \( x < a \).

When the head of load reaches the section C, the portion AC is fully loaded (since \( a > x \)).

\[ F_{max} = +R_B = +\frac{wx^2}{2L} \]  \hspace{1cm} (1.6a)

and

\[ \text{at } x=a, \ F_{max} = +\frac{wa^2}{2L} \]  \hspace{1cm} (2)

(ii) Let the position of the section C be such that \( x > a \).

When the load is in AC, the portion AC is partially loaded, and \( F_x = +R_A \), which goes on increasing as the head of the load approaches C. When the head of the load reaches C, we have

\[ F_{max} = +R_B = +\frac{wa}{L} \left( x - \frac{a}{2} \right) \]  \hspace{1cm} (1.6b)

This is a straight line relation.

At \( x = \frac{a}{2} \), \( F_{max} = 0 \).

At \( x = a \), \( F_{max} = +\frac{wa}{L} \left( a - \frac{a}{2} \right) = +\frac{wa^2}{2L} \).

At \( x = L \), \( F_{max} = +\frac{wa}{L} \left( L - \frac{a}{2} \right) \).

The maximum +ve S.F.D. thus consists of a parabola up to a distance of \( a \) from \( A \), and then straight line up to \( B \).

(aii) Maximum Negative S.F.

As the load moves further, the S.F. decreases. For a particular load position, it becomes zero, and then changes sign and becomes negative. As the load still moves further, the negative S.F. at C increases. For maximum negative shear force at C, the span AC
should be empty and the reaction at A should be a maximum. In other words, the tail of the load should be at C, and the load should extend from C towards B.

When the tail of the load is at C,

\[ F_{\text{max}} = -R_d = -\frac{w}{L} \left( L - \frac{a}{2} \right) \quad (1.7) \]

This is the equation of a straight line, and is valid for all values of \( x \) between 0 to \((L-a)\).

Thus, at \( x=0 \), \( F_{\text{max}} = -\frac{wa}{L} \left( L - \frac{a}{2} \right) \)

at \( x=(L-a) \), \( F_{\text{max}} = -\frac{w}{L} \left( L - \frac{a}{2} \right) - \frac{w(L-a)}{2L} \).

When the position of the section C is such that \( x>L-a \) [i.e. when \( x \) varies from \((L-a)\) to \(L\)]

\[ F_{\text{max}} = -R_d = -\frac{w(x)}{2} (L-x) \quad (1.8) \]

Thus, \( F_{\text{max}} \) is independent of \( a \), and varies parabolically.

At \( x=L-a \), \( F_{\text{max}} = -\frac{w}{2L} (L-x+\frac{a}{2})^2 = -\frac{w(L-a)}{2L} \) as before

At \( x=L \), \( F_{\text{max}} = -\frac{w}{2L} (L-x)^2 = 0 \).

Thus, the maximum negative shear force diagram is a straight line from \( x=0 \) to \( x=L-a \), and a parabola between \( x=L-a \) to \( x=L \).

The absolute maximum positive S.F. occurs at support B, when the head of the load is at B, and the absolute maximum negative S.F. is at A when the tail of the load is at A.

The maximum positive and negative S.F. diagrams have been shown in Fig. 1'4 (b).

(b) MAXIMUM B.M. DIAGRAM

Let the length of the U.D.L. be \( a \). When the load is in the portion \( AC \), the B.M. at the section \( C \) is given by

\[ M_x = -R_s(L-x) \]

This goes on increasing as the head of the load approaches the section C. When the head of the load crosses the section C, the B.M. still goes on increasing, till it attains maximum value at a specific load position. On further movement, the B.M. at the section C decreases.

ROLLING LOADS

For the maximum B.M. at the section C the load is to be so arranged that its C.G. is at a distance \( y \) from A, as shown in Fig. 1'5(a).

In this load position,

\[ R_s = \frac{wa}{L} \quad \text{MAX. B.M.D.} \]

Distance \( CB_1 = \left( y - x + \frac{a}{2} \right) \)

\[ M_x = R_s(L-x) + \frac{w(CB_1)^2}{2} = -\frac{wa}{L} (L-x) + \frac{w}{2} \left( y - x + \frac{a}{2} \right)^2 \quad (2) \]

For \( M_x \) to be maximum, differentiate it with respect to \( y \) and equate to zero. Thus, we have

\[ \frac{dM_x}{dy} = 0 = -\frac{wa}{L} (L-x) + \frac{w}{2} \left( y - x + \frac{a}{2} \right) \]

or \[ \frac{a}{L} (L-x) = \left( y - x + \frac{a}{2} \right) \quad (1.9) \]

In the above equation, \( a = AB \); \( L = AB \); \( L-x = CB \) and

\[ y - x + \frac{a}{2} = CB_1 \]

Hence equation 1.9 can be expressed geometrically as

\[ \frac{A_1B_1}{AB} \cdot CB = CB_1 \]

or \[ \frac{CB}{CB_1} = \frac{AB}{A_1B_1} = \frac{AB-CB}{A_1B_1-CB_1} = \frac{AC}{A_1C} \]
or
\[
\frac{AC}{CB_1} = \frac{AC}{CB} \quad (1'10)
\]

Thus, for maximum bending moment at a section, the load position is such that the section divides the load in the same ratio as it divides the span. This relation will be found to hold good generally, both for the point loads as well as the uniformly distributed loads.

Equation 1'9 is directly useful for the location of the U.D.L. for the maximum B.M.

For the maximum B.M. at C, we get, from equation 1'9,
\[
y = \frac{a}{L} (L - x) + x - \frac{a}{2} = \frac{a}{2} + x - \frac{ax}{L}
\]
and
\[
y - x + \frac{a}{2} = \frac{a}{L} (L - x)
\]

Substituting the values in (2), we get
\[
M_{\text{max}} = -\frac{wa}{L} (L - x) \left( \frac{a}{2} + x - \frac{ax}{L} \right) + \frac{w}{2} \left( \frac{a}{L} (L - x) \right)
\]
\[
= -\frac{wa}{L} (L - x) \left( 1 - \frac{a}{2L} \right) \quad (1'11)
\]

The maximum B.M. diagram can now be plotted by giving different values to x in equation 1'11. Absolute maximum B.M. occurs evidently at the centre, when \(x = L/2\).

Thus, from equation 1'11,
\[
M_{\text{max, max}} = -\frac{wa}{L} \cdot \frac{L}{2} \left( L - \frac{L}{2} \right) \left( 1 - \frac{a}{2L} \right) = -\frac{wa}{4} \left( L - \frac{a}{2} \right)
\]

The above value can also be obtained by considering Fig. 1'5, and applying the deduction of equation 1'10 independently.

Thus, for maximum bending moment at the centre of the span,
\[
\frac{A_1 C}{CB_1} = \frac{AC}{CB}
\]
where
\[
AC = CB = \frac{L}{2}
\]
\[
\therefore \quad \frac{A_1 C}{CB_1} = \frac{L/2}{L/2} = 1
\]
or
\[
\frac{A_1 C}{CB_1} = a/2
\]
In this position. \(R_a = R_b = \frac{wa}{2}\)

and
\[
M_{\text{max, max}} (\text{at centre}) = M_{\text{max, max}} = -\frac{wa}{2} \cdot \frac{L}{2} + \frac{w}{2} \left( \frac{a}{2} \right)^2
\]
\[
= -\frac{wa}{4} \left( L - \frac{a}{2} \right)
\]

which is same as before.

1'5. TWO POINT LOADS WITH A FIXED DISTANCE BETWEEN THEM

Let us now consider two point loads \(W_1\) and \(W_2\) at a fixed distance \(d\) apart, moving from left to right with \(W_1\) leading. Let the leading load \(W_1\) be smaller than \(W_2\).

(a) MAXIMUM POSITIVE SHEAR FORCE

For positive shear force at the section \(C\), we have to consider the three load positions:

(1) Both loads to the left of the section \(C\).
(2) Load \(W_1\) to the right of \(C\) and \(W_2\) to the left of it.
(3) Both loads to the right of \(C\): For this load position, there will be no positive shear (since \(F_x = -R_a\)), and hence we will consider only the first two load positions.

(1) Both Loads to the Left of \(C\)

For this load position,
\[
F_x = +R_a \quad (I)
\]

This increases as the leading load reaches near the section \(C\), and is maximum when \(W_1\) is just to the left of \(C\).

(1) When \(x < d\), only \(W_1\) will be on the girder and \(W_2\) will be off the span, with \(W_1\) at \(C\). Hence,
\[
F_{\text{max, max}} = +R_a = +\frac{W_1 x}{L} \quad (I) (1'12)
\]
(2) When \(x > d\), both \(W_1\) and \(W_2\) will be on the girder with \(W_1\) at \(C\). Hence
\[
F_{\text{max, max}} = +R_a = +\frac{W_1 x + W_2 (x - d)}{L} \quad (II) (1'13)
\]

(1) \(W_1\) to the Right of \(C\) and \(W_2\) to the Left of \(C\)

For this load position,
\[
F_x = +R_a - W_1 \quad (2)
\]

Since \(R_a\) increases as \(W_1\) and \(W_2\) reach near \(B\), the maximum S.F. occurs when the load is just to the left of \(C\).
When \((L-x)>d\), both \(W_1\) and \(W_2\) will be on the beam with \(W_2\) at \(C\). Hence
\[
F_{\text{max}} = +R_b - W_1 = \frac{W_2 x + W_1 (x+d)}{L} - W_1 \quad (\text{III}(1.14))
\]

When \((L-x)<d\), load \(W_1\) will be off the girder while \(W_2\) is at \(C\). Hence
\[
F_{\text{max}} = +R_b = +\frac{W_2 x}{L} \quad (\text{IV}(1.15))
\]

Thus, we have four equations for \(F_x\) (equations, I, II, III and IV) and one or the other of these equations will give maximum positive shear force depending upon the relative magnitudes of \(x\) and \(d\).

To find which of these four equations will give \(F_{\text{max}}\), we shall divide the beam in three zones:

(i) **Zone (1):** \(x=0\) to \(x=d\)
(ii) **Zone (2):** \(x=d\) to \(x=(L-d)\)
(iii) **Zone (3):** \(x=(L-d)\) to \(x=L\)

The first zone under consideration is from \(x=0\) to \(x=d\), and for this, both equations I as well as III will be applicable. For equation (I), \(W_1\) is at \(C\) while \(W_2\) is off the girder. For equation III, \(W_2\) is at \(C\) and \(W_1\) is to the right of it. Out of the two, equation I will give the larger value if
\[
\frac{W_1 x}{L} > \frac{W_2 x + W_1 (x+d)}{L} - W_1
\]
or
\[
x < \frac{W_1 (L-d)}{W_2} \quad (1.16)
\]

Thus, when \(x < \frac{W_1 (L-d)}{W_2}\), equation I will give greater \(F_{\text{max}}\).

Beyond this value (i.e., \(x = \frac{W_1 (L-d)}{W_2}\) to \(x=d\)) equation III will give greater \(F_{\text{max}}\).

(ii) **Zone (2):** \(x=d\) to \(x=L-d\)

The second zone under consideration is from \(x=d\) to \(x=L-d\), and for this both equations II as well as III will be applicable. For Eq. (II), \(W_1\) is at \(C\) and \(W_2\) is to the left of it. For Eq. (III), \(W_2\) is at \(C\) and \(W_1\) is to the right of it. Out of the two, equation (II) will give larger value if
\[
\frac{W_1 x + W_2 (x-d)}{L} > \frac{W_2 x + W_1 (x-d)}{L} - W_1
\]
or
\[
(W_1 + W_2) d < W_1 L \quad (1.17)
\]
or
\[
d < \frac{W_1 L}{W_1 + W_2}
\]
Thus when the value of $d$ is less than $\frac{W_1 L}{W_1 + W_2}$, max. +ve S.F. will occur when the leading load reaches the section. This is the standard case for which max. S.F.D. has been drawn in Fig. 1.6(b).

Thus, when $d < \frac{W_1 L}{W_1 + W_2}$, equation II will give $F_{max}$.

When $d > \frac{W_1 L}{W_1 + W_2}$, equation III will give $F_{max}$ and maximum ±ve S.F. will occur when the rear wheel load reaches the section.

Thus from Eq. I, when $x=0$, $F_{max} = 0$

when $x=d$, $F_{max} = \frac{W_1 d}{L}$

From Eq. II when $x=d$, $F_{max} = \frac{W_1 d}{L}$

when $x=L$, $F_{max} = W_1 + \frac{W_2(L-d)}{L}$

(b) MAXIMUM NEGATIVE SHEAR FORCE

In this case also, we will consider the three load positions for maximum negative shear force at the section (C):

1. Both loads to the right of C.
2. $W_2$, the right of C and $W_1$ to the left of C.
3. Both loads to the left of C: For this position, there will be no negative S.F. (Since $F_x = -Ra$) and hence we will consider only the first two load positions.

(i) Both loads to the right of C

For this load position, $F_x = -Ra$

This increases when $Ra$ increases. Hence the maximum value occurs when $W_2$ is just to the right of C.

1. When $(L-x) > d$, both $W_2$ and $W_1$ will be on the girder, with $W_1$ at C. Hence

\[ F_{max} = -Ra = -\frac{W_2(L-x) + W_1(L-x-d)}{L} \]

(2) When $(L-x) < d$, only $W_1$ will be at C and $W_1$ will be off the girder. Hence

\[ F_{max} = -Ra = -\frac{W_1(L-x)}{L} \]

ROLLING LOADS

(bii) $W_1$ to the right of C and $W_2$ to the left

In this case, maximum negative S.F. occurs when $W_1$ is just to the right of C.

3. When $x < d$, the load $W_2$ will be off the girder with load $W_1$ just to the right of C. Hence

\[ F_{max} = -Ra = -\frac{W_2(L-x)}{L} \]

(7)(1.20)

4. When $x > d$, both $W_1$ and $W_2$ will be on the girder, with $W_1$ just to the right of C. Hence

\[ F_{max} = -Ra = -\frac{W_2(L-x) + W_1(L-x+d)}{L} + W_2 \]

(8)(1.21)

Equations V, VI, VII and VIII are valid for appropriate range of $x$.

For the case when $W_1 < W_2$, equation V will give the maximum negative shear force, and is valid for all values of $x$ between 0 to $(L-d)$. Beyond this, $W_1$ is off the girder, and equation VI will be valid.

Thus, at $x=0$, $F_{max} = -\left\{ W_2 + W_1 \frac{(L-d)}{L} \right\}$

at $x=(L-d)$, $F_{max} = -\frac{W_2 d}{L}$

The complete maximum S.F.D. has been shown in Fig. 1.6(b). Eqs. VII and VIII will give maximum value only when $W_2 < W_1$, i.e., when the loads move in the reverse order.

Summary

1. The maximum positive S.F. occurs only when both the loads are to the left of section with $W_1$ just approaching it. This is valid if $d > \frac{W_1 L}{W_1 + W_2}$. The maximum +ve S.F.D. is governed by Eqs. I and II.

2. If $d > \frac{W_1 L}{W_1 + W_2}$, maximum positive S.F. occurs when $W_1$ is just to the left of C, and $W_2$ is to the right of C. The maximum +ve S.F.D. is governed by Eqs. I and III.

3. The maximum negative S.F. occurs only when both the loads are to the right of the section. The maximum negative S.F.D. is governed by Eqs. V and VI.
(4) If \( W_1 \) is greater than \( W \) (i.e., when the loads travel in reverse order), Eqs. VII and VIII are valid for maximum negative S.F.D.

See examples 1-2 and 1-3 for complete illustration.

(c) **Maximum Bending Moment Diagram**

When the two loads \( W_1 \) and \( W_2 \) are to the left of section \( C \),

\[
M_x = -R_a(L-x)
\]

This goes on increasing, as \( R_a \) increases, till \( W_1 \) reaches the section \( C \). Let \( ^1M_x \) be the bending moment at \( C \) when \( W_1 \) is on the section, and \( W_2 \) is to the left of it.

Then,

\[
^1M_x = -R_a(L-x)
\]

When both the loads are to the right of section \( C \),

\[
M_x = -R_a x
\]

This is evidently maximum when \( W_2 \) is at \( C \) and \( W_1 \) is ahead of it. Let \( ^2M_x \) be the bending moment at \( C \) when \( W_2 \) is on it and \( W_1 \) is to the right of it.

Then,

\[
^2M_x = -R_a x
\]

As the loads still move further, \( R_a \) decreases, and hence \( ^2M_x \) decreases.

As a third possibility of getting maximum bending moment at \( C \), let \( W_2 \) be to the right of \( C \), and \( W_1 \) to the left of it at a distance \( y \) from \( C \). Let \( ^3M_x \) be the bending moment at \( C \) for this loading. Then,

\[
^3M_x = -R_a(L-x) + W_1(d-y)
\]

The above equation may be rewritten in terms of \( ^1M_x \) and \( ^2M_x \) as under:

\[
^3M_x = ^1M_x - \frac{d-y}{L} \{ W_1 x - W_2(L-x) \}
\]

and

\[
^3M_x = ^2M_x + \frac{y}{L} \{ W_1 x - W_2(L-x) \}
\]

If \( W_1 x > W_2(L-x) \), \( ^1M_x > ^3M_x > ^2M_x \)

If \( W_1 x < W_2(L-x) \), \( ^3M_x > ^2M_x > ^1M_x \).

**Rolling Loads**

It is clear, therefore, that in either case, \( ^2M_x \) will not be maximum.

Hence maximum B.M. at the section is either \( ^1M_x \) or \( ^3M_x \), whichever is larger. Fig. 16(c) shows both the parabolas giving \( ^3M_x \) and \( ^2M_x \) governed by equations I and II respectively.

Now, \( ^3M_x \) is greater than \( ^2M_x \) if

\[
\frac{W_1 x + W_2(x-d)(L-x)}{L} > \frac{W_1(L-x-d) + W_2(L-x)x}{L}
\]

i.e.,

\[
x > \frac{W_1 d}{W_1 + W_2}
\]

For

\[
x < \frac{W_1 d}{W_1 + W_2}, \quad ^2M_x \text{ is maximum}
\]

For

\[
x > \frac{W_1 d}{W_1 + W_2}, \quad ^3M_x \text{ is a maximum.}
\]

Now, \( ^3M_x \) is zero at \( x = \frac{W_1 d}{W_1 + W_2} \), and at \( x = L \).

\( ^2M_x \) is zero at \( x = 0 \), and at \( (L-x) = \frac{W_1 d}{W_1 + W_2} \).

Both the parabolas cross each other at \( F' \), where \( ^1M_x = ^3M_x \).

To find the position of this section, put \( ^3M_x = ^2M_x \).

Then

\[
\frac{W_1 x + W_2(x-d)(L-x)}{L} = \frac{W_1(L-x-d) + W_2(L-x)x}{L}
\]

From which \( x \) (or \( AF \)) = \( \frac{W_2 L}{W_1 + W_2} \).

Thus, \( F \) divides \( AB \) in the ratio of \( W_1 : W_2 \). For all sections from \( A \) to \( F \), maximum B.M. is given by \( ^3M_x \), and for all sections from \( F \) to \( B \), the maximum B.M. is given by \( ^1M_x \).

The maximum value of \( ^3M_x \) in equation II will occur at

\[
x = \frac{AE}{2} = \frac{1}{2} \left( L - \frac{W_1 d}{W_1 + W_2} \right)
\]

When \( W_2 > W_1 \) then absolutely maximum B.M. anywhere in the girder occurs in the \( ^3M_x \) range at \( x = \frac{1}{2} AE \).

In case \( (L-x) < d \), \( W_2 \) is off the girder when \( W_2 \) is on the section. Let \( ^4M_x \) be the bending moment at \( C \) when \( W_2 \) is on it and \( W_1 \) is off, the girder.

Then

\[
^4M_x = -R_a(L-x) = -\frac{W_2 x}{L} (L-x) \quad \text{(III)} \quad \ldots (1.25)
\]
Similarly, when \( x < d \), \( W_t \) is off the girder when \( W_s \) is on the section. Let \( M^1_x \) be the bending moment at \( C \) when \( M_s \) is on it and \( W_t \) is off the girder.

Then \( M^1_x = -R_s(L-x) = -\frac{W_t x}{L} (L-x) \) \((1'26)\)

For some sections over a portion of the girder, \( M^2_x \) may sometimes be greater than \( M^1_x \). Fig. 1'6(c) and (d) show the B.M.D. for two such possibilities. See also examples 1'3 and 1'4 for such possibilities.

**Example 1'1.** A uniformly distributed load of 1 kN per metre run, 6 m long crosses a girder of 16 m span. Construct the maximum S.F. and B.M. diagram and calculate the values at sections at 3 m, 5 m and 8 m from the left hand support.

**Solution.**

(a) Maximum positive S.F. diagram.

The maximum positive and negative S.F. diagrams are plotted exactly in the same manner as explained in § 1'4.

For \( x \) up to 6 m (=a)

\[
F_{max} = +\frac{wx^2}{2L} = 1 \times \frac{w^2}{2 \times 16} \frac{x^2}{32} \text{ kN} \quad (1'6(a))
\]

(b) Maximum negative S.F. diagram

(1) For \( x \) between 0 to \( (L-a) = 16 - 6 = 10 \) m;

\[
F_{max} = -\frac{wx^2}{2L} = w \left( 16 - x - \frac{a}{2} \right) \frac{x^2}{16} \quad (1'7)
\]

At \( x=0 \), \( F_s = \frac{3}{8} (13 - 0) = \frac{39}{8} \text{ kN} \)

At \( x=3 \), \( F_s = \frac{3}{8} (13 - 3) = \frac{15}{4} \text{ kN} \)

At \( x=5 \), \( F_s = \frac{3}{8} (13 - 5) = -3 \text{ kN} \)

At \( x=8 \), \( F_s = \frac{3}{8} (13 - 8) = \frac{15}{8} \text{ kN} \)

At \( x=10 \), \( F_s = -\frac{3}{8} (13 - 10) = -\frac{9}{8} \text{ kN} \)

(5) For \( x \) between 10 m to 16 m:

\[
F_{max} = -\frac{wx^2}{2L} (L-x)^2 \quad (1'8)
\]

At \( x=10 \), \( F_s = -\frac{1}{32} (16-10)^2 = -\frac{36}{32} = -\frac{9}{8} \text{ kN} \) as before

At \( x=16 \), \( F_s = -\frac{1}{32} (16-16)^2 = 0. \)
(c) Maximum bending moment
For getting maximum bending moment at a section, the load should be so arranged that the section divides it in the same ratio as it divides the span.

Thus, from equation 1.10,

\[ \frac{A_1C}{CB_1} = \frac{AC}{CB} \]

Hence \( AC = x \); \( CB = L - x \).

\[ \therefore \frac{A_1C}{CB_1} = \frac{x}{L - x} \]

or \[ \frac{A_1C + CB_1}{CB_1} = \frac{A_1B_1}{CB_1} = \frac{x + L - x}{L - x} = \frac{L}{L - x} \]

or \[ CB_1 = \frac{L}{L - x} \times A_1B_1 = \frac{a(L - x)}{L} \]

(1) For \( x = 3 \) m

\[ CB_1 = \frac{a}{L} (L - x) = \frac{6}{16} (16 - 3) = \frac{39}{8} \text{ m} \]

\[ A_1C = 6 - \frac{39}{8} = \frac{9}{8} \text{ m} \]

\[ BB_1 = 16 - 3 - \frac{39}{8} = \frac{65}{8} \text{ m} \]

\[ R_A = (6 \times 1) \left( \frac{65}{8} + 3 \right) \frac{1}{16} = \frac{267}{64} \text{ kN} \]

\[ M_b = -\frac{267}{64} \times 3 + \frac{1}{2} \times \left( \frac{9}{8} \right)^2 = -11\frac{9}{9} \text{ kN-m} \]

Alternatively, from equation 1.11, we have

\[ M_{m_\text{ax}} = -\frac{wa}{L} (L - x) \left( 1 - \frac{a}{2L} \right) \]

\[ M_b = -\frac{1 \times 6 \times 3}{16} (16 - 3) \left( 1 - \frac{6}{2 \times 16} \right) \]

\[ = -11\frac{9}{9} \text{ kN-m} \]

which is the same as before.

(2) For \( x = 5 \) and \( x = 8 \)

For these sections also, equation 1.10 can be used to first locate the position of the U.D.L. and the maximum B.M. can then be calculated, or else equation 1.11 can be used to compute maximum B.M. directly. Thus, from equation 1.11,

\[ M_b = -\frac{1 \times 6 \times 5}{16} (16 - 5) \left( 1 - \frac{6}{2 \times 16} \right) \]

\[ = -16\frac{7}{9} \text{ kN-m} \]

ROLLING LOADS

and \[ M_b = -\frac{1 \times 6 \times 8}{16} (16 - 8) \left( 1 - \frac{6}{2 \times 16} \right) = -19\frac{5}{9} \text{ kN-m} \]

The results can now be summarised as below:

<table>
<thead>
<tr>
<th>Section</th>
<th>Max. +ve S.F.</th>
<th>Max. -ve S.F.</th>
<th>Max. B.M.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 m</td>
<td>\frac{9}{32} \text{ kN}</td>
<td>\frac{-15}{4} \text{ kN}</td>
<td>\frac{-119}{9} \text{ kN-m}</td>
</tr>
<tr>
<td>5 m</td>
<td>\frac{25}{32} \text{ kN}</td>
<td>\frac{-3}{4} \text{ kN}</td>
<td>\frac{-167}{7} \text{ kN-m}</td>
</tr>
<tr>
<td>8 m</td>
<td>\frac{15}{8} \text{ kN}</td>
<td>\frac{-15}{8} \text{ kN}</td>
<td>\frac{-195}{5} \text{ kN-m}</td>
</tr>
</tbody>
</table>

Example 1.2. Two point loads of 4 kN and 6 kN spaced 6 m apart cross a girder of 16 m span, the 4 kN load leading from left to right. Construct the maximum S.F. and B.M. diagrams, stating the absolute maximum values.

Solution

\[ W_1 = 4 \text{ kN}; W_2 = 6 \text{ kN}; d = 6 \text{ m}; L = 16 \text{ m} \]

\[ \frac{W_1L}{W_1 + W_2} = \frac{4 \times 16}{4 + 6} = 3.25 \]

\[ d = 6 < 3.25 \]

Thus the data is for the standard case, and the S.F.D. will be as that shown in Fig. 1.6(a).

(a) Maximum +ve S.F. diagram

Since \( d < \frac{W_1L}{W_1 + W_2} \), the maximum +ve S.F. at any section will occur under the leading load of 4 kN.

For \( x \) up to 6 m (=d), \( F_{m_{\text{ax}}} = + \frac{W_1x}{L} = + \frac{4x}{16} = + \frac{x}{4} \text{ kN} \)

\[ F_b = + \frac{6}{4} = +1.5 \text{ kN} \]

For \( x \) more than 6 m (=d), \( F_{m_{\text{ax}}} \) will be greater than \( F_{m_{\text{ax}}} \) and is given by equation 1.13, i.e.

\[ 1F_{m_{\text{ax}}} = + R_b = + \frac{W_1x + W_2(x - d)}{L} = + \frac{(W_1 + W_2)x - W_2d}{L} \]

\[ = \frac{(4 + 6)x - 6 \times 6}{16} = \frac{5x - 18}{8} \]

At \( x = L - 16 \), \( F_{m_{\text{ax}}} = + \frac{(5 \times 16) - 18}{8} = + \frac{31}{4} \text{ kN} \)
(b) Maximum $-\ve$ S.F. Diagram

For all sections between $x=0$ and $x=L-d=16-6=10$ m, the maximum negative S.F. will be when the load $W_1(=6 \text{ kN})$ is at the section, with $W_2(=4 \text{ kN})$ ahead of it. The variation is given by equation 1'18, i.e.,

$$F_{max} = -R_d = -\frac{W_1(L-x)+W_2(L-x-d)}{L}$$

...(1'18)

At $x=0$, $F_4=F_{max}=\frac{-6(L-0)+4(L-0-6)}{16} = -8.5 \text{ kN}$

At $x=10$, $F_{10} = -\frac{6(L-10)}{16} = -\frac{9}{4} \text{ kN}$

For all sections between $x=10$ to $x=16$, $W_1$ will be off the girder when $W_2$ is on it. Hence, max. $-\ve$ S.F. is given by equation 1'19, i.e.,

$$F_{max} = -R_d = -\frac{W_2(L-x)}{L}$$

...(1'19)

At $x=10$, $F_{10} = \frac{-6(L-10)}{16} = -\frac{9}{4}$, as before.

At $x=16$, $F_{20}=0$.

The max. $+\ve$ and $-\ve$ S.F.D. have been shown in Fig. 1'8(b).

(c) Maximum bending moment diagram

As discussed earlier, the maximum B.M. at a section may occur under any one of the following conditions:

(i) Max. B.M. under $W_1$ and $W_2$ behind it ($1M_x$).

(ii) Max. B.M. under $W_2$ with $W_1$ ahead of it ($2M_x$).

(iii) Max. B.M. under $W_2$ with $W_1$ off the girder ($02M_x$).

We shall investigate all the three possibilities.

(a) Max. B.M. under $W_1$ ($1M_x$):

From equation 1'22, we have

$$1M_x = \frac{W_1 x + W_2 (x-d)}{L} (L-x)$$

...(1'22)

This is zero at $x=3.6$ m and $x=16$ m.

Thus, $DB=16-3.6=12.4$ m.

$1M_{max}$ will occur at $x=3.6 + \frac{12.4}{2} = 9.8$ m, its value being.

$$1M_{max} = (36-9.8) \left(1-\frac{x}{16}\right) = -24.25 \text{ kN-m}$$

(b) Max. B.M. under $W_2$ ($2M_x$):

From equation 1'23, we have

$$2M_x = \frac{x}{L} \left\{W_2 (L-x-d) + W_1 (L-x)\right\}$$

$$= \frac{x}{16} \left\{4(L-x-6) + 6(L-x)\right\}$$

$$= \frac{x}{16} \left[136-10x\right]$$

This is zero at $x=0$, and $x=13.6$ m

Thus $AE=13.6$ m

$2M_{max}$ occurs at $x=\frac{AE}{2} = 6.8$ m, its value being

$$2M_{max} = \frac{6.8}{16} \left[136-68\right] = 28.8 \text{ kN-m}$$

Since $\frac{W_2 L}{W_1 + W_2} = \frac{6 \times 16}{6+4} = 9.6 \text{ m}$, for $x<9.6$, $2M_x$ is greater than $1M_x$, and for $x>9.6$ m, $1M_x$ is greater than $2M_x$. 

**Fig. 1'8**
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Mx and \( A_m \) are equal at \( x = 9.6 \) m (This is a check).

(c) Max. B.M. under \( W_2 \) when \( W_1 \) is off the girder \((A_{m}A_m)\)

From equation 1-25,

\[ A_{m}A_m = -\frac{W_2 x}{L} (L-x) = -\frac{6x}{16} (16-x) \]

and this is valid from \( x=L-d=16-6=10 \) m, to \( x=L=16 \) m. In this range \( A_{m}A_m \) is greater than \( 1_{m}M_x \) if

\[ -\frac{6x}{16} (16-x) > \left( 36-10x \right) \left( \frac{16-x}{16} \right) \]

or if

\[ 6x > 10x-36 \]

or

\[ 36 > 4x \]

or

\[ x < 9 \text{ m.} \]

But since the equation of \( A_{m}A_m \) is valid for \( x \) greater than 10 m, the above condition cannot be fulfilled, and hence \( A_{m}A_m \) is less than \( 1_{m}M_x \) between \( x=10 \) to \( x=16 \) m. \( A_{m}A_m \) has its maximum value at \( x = \frac{L}{2} = 8 \) m, its value being equal to

\[ A_{m}A_m = -\frac{6 \times 8}{16} (16-8) = -24 \text{ kN-m} \]

The maximum B.M.D. has been drawn in Fig. 18(c).

Example 1'3. Solve example 1'2 if \( W_1 = 4 \) kN; \( W_2 = 6 \) kN; \( d = 6 \) m and the span \( L = 12 \) m.

Solution.

For the present case

\[ W_1 \]

\[ W_1 + W_2 \]

\[ \frac{W_1}{L} = \frac{4 \times 12}{4+6} = 4.8 \text{ m.} \]

\[ d = 6 > \frac{W_1 L}{W_1 + W_2} \]

Thus the case is not the standard one, and the S.F.D. will not be similar to that of Fig. 16(b).

(a) Maximum +ve S.F. Diagram

(i) For \( x = 0 \) to \( x = d = 6 \) m, when only \( W_1 \) is on the span, and \( W_2 \) is off the span the maximum S.F., from equation 1-12, is given by

\[ F_{m} = + R_B = + \frac{W_2 x}{L} = + \frac{4x}{12} = + \frac{x}{3} \text{ kN} \]

(ii) For \( x = 6 \) to \( x = d = 6 \) m, when \( W_2 \) is just to the left of the section and \( W_1 \) ahead of it.

\[ F_{m} = + R_B = + \frac{W_2 x}{L} + \frac{W_2 (x+d)}{L} - W_1 = \frac{6x}{12} + \frac{4(x+6)}{12} - 4 = \left( \frac{5}{6} x - 2 \right) \]

Thus, for \( x = 0 \) to \( x = 6 \), \( F_{m} \) is given by both equation (1) and (2). \( F_{m} \) will be greater than \( F_{m} \) only if

\[ \left( \frac{5}{6} x - 2 \right) > \frac{x}{3} \]

or if \( 5x - 12 > 2x \)

or if \( 3x > 12 \)

or if \( x > 4 \)

Thus, \( F_{m} \) is equal to \( 4 \) kN, \( F_{m} \) (given by equation 1) gives the maximum S.F., while for \( x = 4 \) to \( x = 6 \), \( F_{m} \) gives the maximum values.

Thus, at \( x = 4 \), \( F_4 = + \frac{3}{2} \) kN

at \( x = 6 \), \( F_6 = + 3 \) kN

(iii) For \( x = d = 6 \) m (or \( L - d = 6 \) m) to \( x = L = 12 \); both \( W_1 \) and \( W_2 \) on the span:

The maximum S.F. is given by equation 1'13

\[ F_{m} = + R_B = + \frac{W_2 x + W_2 (x-d)}{L} \]

\[ = \frac{4x + 6(x-6)}{12} = \frac{5}{6} (x-3) \text{ kN} \]

(iv) For \( x = d = 6 \) m to \( x = L = 12 \), with \( W_2 \) on the section and \( W_1 \) off the girder:

The maximum S.F. is given by equation 1'5.

\[ F_{m} = + R_B = + \frac{W_2 x}{L} = + \frac{6x}{12} = + \frac{x}{2} \text{ kN} \]

Thus, for \( x = 6 \) to \( x = 12 \), the maximum S.F. is given by equation (3) and (4). Evidently \( F_{m} \) will be greater than \( F_{m} \) if

\[ \frac{x}{2} > \frac{5}{6} (x-3) \text{ kN} \]

or

\[ x < 9 \]

Thus, from \( x = 6 \) to \( x = 9 \), maximum S.F. will be governed by \( F_{m} \) (equation 4), while from \( x = 9 \) to \( x = 12 \), maximum S.F. will be governed by \( F_{m} \).
MAXIMUM BENDING MOMENT DIAGRAMS

The maximum bending moment may occur under any one of the following three conditions:

(i) Maximum bending moment under \( W_1 \) with \( W_1 \) ahead of it (\( M_2 \)).

(ii) Maximum bending moment under \( W_1 \) and \( W_2 \) behind it (\( M_3 \)).

(iii) Maximum bending moment under \( W_2 \) with \( W_1 \) off the span, \( M_4 \).

Let us investigate all the three possibilities:

(i) Maximum bending moment under \( W_1 \) with \( W_1 \) ahead of it (\( M_2 \)).

From equation 1.23,

\[ M_2 = -R_2 x = -\frac{x}{L} \left( W_1 (L-x-d) + W_2 (L-x) \right) \]

\[ = -\frac{x}{12} \left( 4 (12-x-6) + 6 (12-x) - 6 \right) \]

\[ = -\frac{x}{12} \left( 8 - \frac{5}{6} x \right) \]

This is zero at \( x = 3.6 \) m, and at \( x = 3\frac{3}{2} = 4.8 \) m.

\( M_2 \) will have its max. value at \( x = \frac{9.6}{2} = 4.8 \) m

\( M_2 = -4.8 \left( 3 - \frac{5}{6} \times 4.8 \right) = -19.2 \) kN-m.

(ii) Max. bending moment under \( W_1 \) with \( W_2 \) behind it (\( M_3 \)).

From equation 1.22, we have

\[ M_3 = -\frac{W_1 x + W_2 (x-d) (L-x)}{L} \]

\[ = -\frac{4x + 6(x-6)}{12} (12-x) = -\left( \frac{5}{6} x - 3 \right) (12-x) \]

This is zero at \( x = 3.6 \) m, and at \( x = 12 \) m.

\( M_3 \) is maximum at \( x = 3.6 + \frac{12-3.6}{2} = 7.8 \) m.

\[ M_3 = -\left( \frac{5}{6} \times 7.8 - 3 \right) (12-7.8) = -14.7 \) kN-m.

Now \( M_3 \) will be greater than \( M_2 \), for

\[ \frac{W_1 L}{W_1 + W_2} > 6 \times 12 \]

\[ \frac{W_1}{W_2 + W_2} > \frac{4 + 6}{4 + 6} > 7.2 \) m.

Thus for \( x = 0 \) to \( x = 7.2 \), max. B.M. will be governed by \( M_3 \).

---

**Diagram**

The diagram shows the bending moment distribution along the girder, with labels for different sections and maxima. The formulae and calculations for maximum bending moments are shown alongside. The diagram helps visualize the distribution of forces and bending moments along the span of the girder.
At \( x=7.2 \),
\[ ^1M_{x=7.2} = -\left(5 \times 7.2 - 3\right)(12 - 7.2) = 14.4 \text{ kN-m} \]
\[ ^2M_{x=7.2} = -7.2 \left(8 - \frac{5}{6} \times 7.2\right) = -14.4 \text{ kN-m} \]

(check)

(iii) Max. bending moment under \( W_1 \) with \( W_2 \) off the girder

From equation 1*25,
\[ ^3M_x = -6x(L-x) = -\frac{6x}{12} (12-x) = \frac{x}{6}(6-0.5x) \]

To get the section where \(^3M_x\) is equal to \(^4M_x\), we have
\[ x \left( \frac{5}{6} - 3 \right)(12-x) = x(6-0.5x) \]

or
\[ x = 6 \text{ m} \]

The common value of B.M. is given by
\[ ^2M_e = ^4M_e = -6(6-0.5\times6) = -18 \text{ kN-m} \]

To get the section where \(^4M_x\) and \(^5M_x\) are equal, we have
\[ \left( \frac{5}{6} x - 3 \right)(12-x) = x(6-0.5x) \]

which gives \( x = 9 \text{ m} \).

The common value of the max. B.M. is given by
\[ ^6M_e = -9(6-0.5\times9) = -13.5 \text{ kN-m} \]

The maximum value of \(^6M_e\) evidently occurs at \( x = \frac{L}{2} = 6 \text{ m} \)

its value being equal to \(^6M_e\) \( = -6(6-0.5\times6) = -18 \text{ kN-m} \)

Hence, to summarise:

(i) For \( x = 0 \), \( x = 6 \), max. B.M. is governed by \(^3M_x\).
(ii) For \( x = 6 \) to \( x = 9 \), max. B.M. is governed by \(^4M_x\).
(iii) For \( x = 9 \) to \( x = 12 \), max. B.M. is governed by \(^5M_x\).

The complete B.M. diagram is shown in Fig. 1*9(c). The absolute max. B.M. will be under \( W_2 \) at \( x = 4.8 \text{ m} \), its value being equal to 19.2 kN-m.

**Example 1*4.** Two point loads \( W_1 \) and \( W_2 \) \( (W_2 > W_1) \) spaced at a distance \( d \) travel from left to right across a simply supported girder, with \( W_1 \) leading. Prove that the limiting span below which the greatest bending moment anywhere in the girder will occur when the load \( W_2 \) has gone off the girder, is equal to \( \left(1 \pm \sqrt{\frac{W_1}{W_1+W_2}}\right)d \).

Hence, draw the max. B.M. diagram if \( W_1 = 4 \text{ kN} \); \( W_2 = 6 \text{ kN} \); \( d = 6 \text{ m} \) and the span \( L = 10 \text{ m} \).
Thus, \[ L_{\text{lim}} = \left[ \frac{1 \pm \sqrt{6}}{4 + 6} \right] 6 \]
\[ = (1 \pm 0.775) 6 \approx 1.35 \text{ m or } 10.65 \text{ m} \]
\[ = 10.65 \text{ (Taking the greater limiting value)} \]

Since our span is even lesser than the greater permissible value, \( 2M_x \) will be greater than \( 8M_{\text{max}} \).

Now \[ 0^1M_x = \frac{Wx}{L} (L - x) = - \frac{6x}{10} (10 - x) \]
This is maximum at \( x = \frac{L}{2} = 5 \text{ m} \)
\[ M_{\text{max}} = 0^1M_x = - \frac{6 \times 5}{10} \times 5 = -15 \text{ kN.m.} \]

To plot the max. B.M. diagram, let us again investigate all the three possibilities:

(a) Max. B.M. under \( W_2 \) with \( W_1 \) ahead of it (\( 0^1M_x \))
From Eq. 1'23,
\[ 0^1M_x = \frac{-1}{L} \left[ W_1(L - x - d) + W_2(L - x) \right] \]
\[ = \frac{-1}{10} \left[ 4(10 - x - 6) + 6(10 - x) \right] \]
\[ = -x(7.6 - x) \] (i)
This is zero at \( x = 0 \) and \( x = 7.6 \) m
(b) Max B.M. under \( W_1 \) with \( W_2 \) behind it (\( 1^1M_x \))
From Eq. 1'23,
\[ 1^1M_x = \frac{-1}{L} W_2 x + W_2(x - d) (L - x) \]
\[ = \frac{6 \times 10}{10} = 6 \text{ kN.m.} \]
\[ = 0^1M_x = 6 \text{ kN.m.} \]
This gives \( x = 9 \text{ m}. \)
Thus \[ 1^1M_x = 0^1M_x = -0.6 \times 9(10 - 9) = -5.4 \text{ kN.m.} \]
Hence, we get
(i) For \( x = 0 \) to \( x = 4 \) m, Max. B.M. is governed by \( 0^1M_x \)
(ii) For \( x = 4 \) m to \( x = 9 \) m, Max. B.M. is governed by \( 1^1M_x \).
(iii) For \( x = 9 \) m to \( x = 10 \) m, Max. B.M. is governed by \( 1^1M_x \).
The max. B.M.D. is shown in Fig. 1'10.

Example 1'5. Plot the maximum bending moment diagram for a simply-supported girder with the following data:
\[ W_1 = 3 \text{ kN (leading)} \]
\[ W_2 = 6 \text{ kN} \]
\[ d = 6 \text{ m} \]
\[ L = 10 \text{ m.} \]

Prove that maximum B.M. occurs under \( W_2 \) when \( W_1 \) is off the span.

Solution
\[ L_{\text{lim}} = \left[ \frac{1 \pm \sqrt{\frac{W_2}{W_1 + W_2}}} \right] d \]
\[ = \left[ \frac{1 \pm \sqrt{\frac{6}{3 + 6}}} \right] 6 = 10.89 \text{ m.} \]

Since \( L = 10 \text{ m.} \), maximum B.M. will occur under \( W_2 \) when \( W_1 \) is off the girder, i.e., \( 0^2M_{\text{max}} \) will be greater than \( 4^1M_{\text{max}} \).

To plot the maximum B.M. diagram, we will investigate all the possibilities.
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(a) Maximum B.M. under \( W_1 \) with \( W_1 \) ahead of it (\( \overline{M}_x \))

From equation 1.23, we have

\[
\overline{M}_x = -\frac{x}{L} \left\{ W_1 (L - x) + W_4 (L - x) \right\}
\]

(1.23)

\[
\overline{M}_x = -\frac{x}{L} \left\{ 3(10 - x) + 6(10 - x) \right\}
\]

(1)

It will be zero at \( x = 0 \) and \( x = 8 \) m.

(b) Maximum B.M. under \( W_1 \) with \( W_3 \) behind it (\( \overline{M}_x \))

From equation 1.22,

\[
\overline{M}_x = \frac{-W_1 x + W_4 (x - d)}{L} (L - x)
\]

(1.22)

\[
\overline{M}_x = \frac{-3x + 6(x - 6)}{10} (10 - x)
\]

(2)

It will be zero at \( x = 4 \) m and \( x = 10 \) m.

(c) Maximum B.M. under \( W_1 \) with \( W_1 \) off the girder (\( \overline{M}_x \))

From equation 1.25,

\[
\overline{M}_x = \frac{-W_3 x}{L} (L - x)
\]

(1.25)

\[
\overline{M}_x = \frac{-6x}{10} (10 - x) = -0.6x(10 - x)
\]

(3)

To find the section where \( \overline{M}_x \) and \( \overline{M}_x \) are equal, we have

\[
x(7.2 - 0.9x) = 0.6x(10 - x)
\]

which gives \( x = 4 \) m.

\[
\overline{M}_x = \overline{M}_x = -0.6 \times 4(10 - 4) = -14.4 \text{ kN-m.}
\]

For the rest of the span, \( \overline{M}_x \) will be greater than \( \overline{M}_x \) only if

\[
0.6x(10 - x) > (0.9x - 3.6) (10 - x)
\]

or

\[
0.6x > 0.9x - 3.6
\]

or

\[
x < 12 \text{ m.}
\]

However, since maximum value of \( x = L = 10 \) m, \( \overline{M}_x \) will always be greater than \( \overline{M}_x \).

The absolute maximum B.M. any where in the girder will evidently be governed by \( \overline{M}_x \). It will occur at \( x = \frac{L}{2} = 5 \) m and its value is

\[
M_{\max} = \overline{M}_x = -0.6 \times 5 \times 5 = -15 \text{ kN-m.}
\]

T6. SEVERAL POINT LOADS: MAXIMUM B.M.

Let us now take the case of a train of wheel loads \( W_1, W_2, \ldots, W_n \) crossing a simply supported girder. For getting the position and amount of maximum bending moment, we shall discuss the following two propositions.

PROPOSITION 1

When a series of wheel-loads cross a girder, simply supported at the ends, the maximum bending moment under any given wheel load occurs when the centre of the span is midway between the C.G. of the load system and the wheel load under consideration.

Thus, in Fig. 1.12, let us find the maximum bending moment under the wheel load \( W_k \), of the train of wheel loads \( W_1, W_2, \ldots, W_n \). Let \( W_k \) be the resultant of all loads to the left of \( W_k \) and \( W_R \) be the resultant of all loads to the right of \( W_k \) and inclusive of \( W_k \). Let \( W \) be the resultant of the load system, situated at \( a \) from \( W_L \), \( b \) from \( W_R \) and \( c \) from \( W_k \). For given load system \( a, b \) and \( c \) are constants.
To get the maximum B.M. under \( W_a \), let the load \( W_a \) be placed at a distance \( z \) from the centre \( C \) of the span. It is required to find the value of the variable \( z \).

Reaction \( R_d = \frac{W}{L} \left[ \frac{L}{2} + (c-z) \right] \)

B.M. under \( W_a \) is \( M = -R_d \left( \frac{L}{2} + z \right) + W_L(a+c) \)

\[ M = -\frac{W}{L} \left[ \frac{L}{2} + (c-z) \right] \left( \frac{L}{2} + z \right) + W_L(a+c) \]

\[ = -\frac{W}{L} \left( \frac{L^2}{4} + cz - z^2 + \frac{cL}{2} \right) + W_L(a+c) \]

For maximum \( M \),

\[ \frac{dM}{dz} = -\frac{W}{L} (c-2z) = 0 \]

or

\[ z = \frac{c}{2} \] ...(1.29)

Hence the centre of the span is midway between \( W \) and \( W_a \). This proves the proposition.

The above proposition can be used to find the maximum B.M. under desired wheel load. However, to get absolute maximum B.M. any where on the girder, several trials are to be made. Any one load must first be chosen and arranged according to the condition of equation 1.29 derived above, and the maximum B.M. is calculated. Another wheel load can then be chosen and the procedure repeated to get another value of maximum B.M. Two or three such trials may sometimes be needed, and the absolute maximum B.M. will be the greatest of these. However, to reduce the number of trials to a minimum, the following points must always be kept in mind:

1. The maximum B.M. always occurs under a wheel load, and not any where between two wheel loads.
2. Absolute maximum B.M. always occurs at a section near the centre of the span. (It never occurs at the centre unless the C.G. of the resultant load coincides with the centre line of some heavy wheel load).
3. The wheel load should be so selected that the centre of the span is midway between the C.G. of the load system and wheel load under consideration.
4. The absolute maximum B.M. generally occurs under the heavier wheel load—specially that which is very near to the C.G. of the load system.
and CB lighter, or vice versa. Hence the maximum B.M. at C will occur when \( \left( \frac{W_L}{x} - \frac{W_R}{L-x} \right) \) changes sign. The value \( \left( \frac{W_L}{x} - \frac{W_R}{L-x} \right) \) can change sign only when a load crosses C from left to right, thus increasing \( W_R \) and decreasing \( W_L \). Hence to get the value of maximum B.M. at a section, one of the wheel loads should be placed at the section, so that if that load is considered as a part of \( W_L \), the expression \( \left( \frac{W_L}{x} - \frac{W_R}{L-x} \right) \) is positive, but if considered as part of \( W_R \), the expression \( \left( \frac{W_L}{x} - \frac{W_R}{L-x} \right) \) becomes negative. If on rolling the loads from left to right, \( \left( \frac{W_L}{x} - \frac{W_R}{L-x} \right) \) does not change the sign from +ve to -ve, but instead, increases or remains positive, the loads should be rolled to the right so that next load comes over to the section. With this new load at the section, \( \left( \frac{W_L}{x} - \frac{W_R}{L-x} \right) \) should again be investigated for the two positions, as described above, till it changes sign. In the passage of a series of wheel loads, two or more positions of the load system may occur satisfying the above condition of change of sign from +ve to -ve. In such a case, the value of \( M_x \) at the section for each of these load positions must be calculated, and the greatest of these taken as the maximum B.M. at the section.

It must always be remembered that maximum B.M. at any section occurs when the wheel load is over it.

### 1.7. SEVERAL POINT LOADS: MAX. S.F. AT A SECTION

Let us now investigate the load position for getting maximum S.F. at a section due to several point loads \( W_1, W_2, ..., W_n \). The process of locating the load position for maxima is that of trial and error. However, the max. S.F. at the section occurs when one of the loads is on the section.

![Diagram](image)

**Fig. 1.14**

To get the max. +ve S.F. at C let the load \( W_1 \) be at the section C, and let another load \( W_2 \) be at d behind it. If the loads are rolled to the right by a distance \( d \), so that \( W_2 \) comes at C, the S.F. at the section C will be changed. This change (\( \delta B \)) consists of two components:

1. Increase \( \delta R_B \) (gradually as the loads roll)
   \[ \delta R_B = \frac{W_d}{L}, \text{ where } W=\text{resultant of all loads on the span}. \]

2. Sudden decrease or drop equal to \( W_1 \).
   \[
   \delta F = \delta R_B - W_1 = \frac{W_d}{L} - W_1 \quad \ldots (1.31)
   \]

If this change is positive, rolling will increase +ve S.F. In such a case, the rolling must be continued till equation 1.31 becomes negative.

The above discussion is true only if no load either enters or leaves the span when the system is rolled by the specified distance \( d \).

To discuss the most general case, let the load \( Q \) enter the span a distance \( a \), and load \( P \) move beyond \( B \) a distance \( b \), due to rolling. If \( W \) is the resultant load before the advance, we have

\[
\delta R_B = \frac{W_d}{L} + \frac{Qa}{L} - P \left( 1 + \frac{b}{L} \right)
\]

Hence
\[
\delta F = \delta R_B - W_1 = \frac{W_d}{L} + \frac{Qa}{L} - P \left( 1 + \frac{b}{L} \right) - W_1
\]

Since \( \frac{b}{L} \) and \( \frac{a}{L} \) are usually small compared with unity, the last two terms of the above expression may be neglected for approximation. Hence, we get

\[
\delta F = \frac{W_d}{L} - (W_1 + P) \quad \ldots (1.33)
\]

If this is +ve, rolling will increase the S.F. From the above, it is evident that the load entering the span does not change the S.F. appreciably while the load leaving the span does.

If all the loads are equal and equally spaced \( \left( \frac{W_d}{L} - W_1 \right) \) will always be negative and, hence, maximum S.F. at the section will occur when the first load reaches the section.

The absolute maximum +ve S.F. evidently occurs at the right support, for which the criterion of equation 1.33 must be tried.
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Equation 1.31 (or 1.33) can also be used for getting maximum positive S.F. at the section. However, in this case, the advance (or rolling) must be to the left till $8F$ is increased (or becomes positive). The first chosen wheel load $W_1$ is considered just to the right of the section before such movement. If the whole system is now moved to the left by distance, say $c$, negative S.F. will increase only if 
\[
\left(\frac{W_c}{L} - W_1\right)
\]
is positive. If it becomes negative, the load position before such movement gives maximum negative S.F. If it becomes positive, movement must be permitted till the expression becomes negative.

Example 1.6. The system of concentrated loads shown in Fig. 1.15(a) rolls from left to right across a beam simply supported over a span of 40 m, the 4 kN load leading. For a section 15 m from the left hand support, determine:
(a) The maximum bending moment.
(b) The maximum shearing force.

**Solution.**

![Diagram](image)

**Rolleing Loads**

(a) *Maximum B.M.*
By inspection, it is clear that the maximum B.M. at C will occur when the central 10 kN load is over the section, so that when the loads are rolled across the section, the condition loading in $AC-BC$ will alter from heavier-lighter to lighter-heavier. The loads are arranged as shown in Fig. 1.15 (b).

Give small movement to the left,
\[
\frac{W_L}{x} - \frac{W_R}{L-x} = \frac{6+6+10}{15} - \frac{10+4}{15} = 0.47 - 0.56 = -0.09.
\]

Since $\frac{W_L}{x} - \frac{W_R}{L-x}$ changes sign, the bending moment will decrease if the central 10 kN load is displaced from C. Hence the arrangement of the load shown in Fig. 1.15 (b) gives the maximum B.M.

Now $R_B = \frac{1}{40} \left[ (9.5 \times 6) + (12 \times 6) + (10 \times 15) + (10 \times 18) + (4 \times 20.5) \right]$

$= 13.525 \text{ kN}$

$M_c = - (13.525 \times 25) \times (4 \times 5) + (10 \times 3) = -286 \text{ kN-m}$

(b) *Maximum S.F.*

For maximum positive shear force, let us try with the first 4 kN load at the section C, with the load arranged as shown in Fig. 1.15 (c). Since the next load (i.e. 10 kN load) is at a distance $d = 2.5 \text{ m}$, let us roll the loads to the right by 2.5 m.

Here $W = \text{total load} = (6+6+10+10+4) = 36 \text{ kN}$

$W_1 = 4 \text{ kN}$

$d = 2.5$

$\therefore \quad 8F = \frac{Wd}{L} - W_1 = \frac{36 \times 2.5}{40} - 4 = 2.25 - 4 = -1.75$

Since it is negative, the S.F. decreases. Hence the maximum positive S.F. occurs when the 4 kN load is just to the left of the section C, as shown in Fig. 1.15 (c).

$\therefore \quad R_B = \frac{1}{40} \left[ (6 \times 4) + (6 \times 6.5) + (10 \times 9.5) + (10 \times 12.5) + (4 \times 15) \right]$

$= 8.575 \text{ kN}$

$\therefore \quad F_c = R_B = + 8.575 \text{ kN}$
Similarly, for negative S.F. at C, let us try with the last 6 kN load at C, as shown in Fig. 1·15(d). Since the next 6 kN load is at a distance \( d=2·5 \) m, let us roll the loads to the left by 2·5 m.

Here
\[
W=36 \text{ kN} \\
W_1=6 \text{ kN} ; \quad d=2·5 \text{ m} \\
8F=\frac{Wd}{L} - W_1 = \frac{36 \times 2·5}{40} - 6 = 22·5 - 6 = -3·75
\]

This shows that the negative S.F. will be decreased. Hence the maximum negative S.F. occurs when the 6 kN load is just to the right of section C, as shown in Fig. 1·15 (d).

\[
R_A = \frac{1}{40} \left[ (4 \times 14) + (10 \times 16·5) + (10 \times 19·5) + (6 \times 22·5) + (6 \times 25) \right] \\
= 17·525 \text{ kN} \\
F = -R_A = -17·525 \text{ kN}
\]

**Example 1·7. The following system of the wheel loads crosses a span of 25 m.**

**Wheel load**

\[
\begin{array}{cccccc}
16 & 16 & 20 & 20 & 20 \\
\end{array}
\]

**Distance between centre**

\[
\begin{array}{cccc}
3 & 3 & 4 & 4 \\
\end{array}
\]

Find the maximum value of bending moment and shearing force in the span.

**Solution**

\[\text{Fig. 1·16.}\]

(a) **Maximum B.M.**

Let us number the loads as 1, 2, 3, etc.

Then \[ W=20+20+20+16+16 = 92 \text{ kN}\]

Taking moment of all loads about load no. 5, we get

\[
92x = (16 \times 3) + (20 \times 6) + (20 \times 10) + (20 \times 14) \\
x = 7·04 \text{ m}
\]

(b) **Maximum S.F.**

\[
\text{Maximum S.F. values is either } R_A \text{ or } R_B. \text{ As the C.G. of the load can approach nearer to } B \text{ than to } A, R_B > R_A \text{ for limiting load position.}
\]

Keep the first load (i.e., 20 kN) just to the left of B. Since next load is at 4 m distance, give a movement of 4 m

Thus,

\[
\begin{align*}
W_1 &= 20 \text{ kN} \\
d &= 4 \text{ m} \\
P &= \text{load leaving the span} = 20 \text{ kN}
\end{align*}
\]

From equation 1·33,

\[
8F = \frac{Wd}{L} - (W_1 + P) = \frac{92 \times 4}{15} - (20 + 20) \\
= 24·5 - 40 = -15·5
\]

The negative sign shows that the shear force will decrease if the loads are moved. Hence the arrangement of the loads for maximum S.F. will be as shown in Fig. 1·16 (b).

Considering the first 20 kN load just to the left of B, we have

\[
R_B = \frac{1}{25} \left[ (16 \times 11) + (16 \times 14) + (20 \times 17) + (20 \times 21) + (20 \times 25) \right] \\
= 66·4 \text{ kN}
\]

\[
F_{\text{max}} = +R_B = +66·4 \text{ kN}
\]
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Example 18: A girder, simply supported over a span 20 m, is traversed by a moving load as shown in Fig. 1.17. Determine the maximum B.M. at 8 m from the left hand support.

Solution

Let us try with the second 3 kN load at the section C, as shown.

Giving slight motion to the left:

\[ \frac{W_L}{x} = \frac{3 + 3}{8} = 0.75 \]

And there must be a difference of 8 m.

Since \( \frac{W_L}{x} - \frac{W_R}{L-x} \) changes sign from +ve to -ve, the maximum will occur when the loads are arranged as shown.

\[ R_A = \frac{1}{20} \left( 8 \times 1 \times 4 + 3 \times 10 + 3 \times 12 + 3 \times 14 + 3 \times 16 \right) \]

\[ = 9.4 \text{ kN} \]

\[ M_{\text{max}} = -(9.4 \times 8) + (3 \times 4) + (3 \times 2) = -57.2 \text{ kN-m} \]

1.8. EQUIVALENT UNIFORMLY DISTRIBUTED LOAD

A given system of loading crossing a girder can always be replaced by uniformly distributed load, longer than the span, such that bending moment or S.F., due to this equivalent static load, every where is at least equal to that caused by the actual system of moving loads. Such a static load is known as equivalent uniformly distributed load (E.U.D.L.). The E.U.D.L. will be different for B.M. and S.F. The bending moment diagram for E.U.D.L. will be a parabola symmetrical about the base and must completely envelope the maximum bending moment diagram for the moving loads.

Let us now find the E.U.D.L. for the following cases, for B.M. purposes:

(a) Single point load.

(b) U.D.L. shorter than the span.

(c) Two point loads \( W_1 \) and \( W_2 \) at distance \( d \) apart.

(a) E.U.D.L. for Single Point Load:

The maximum B.M. at the section C, distant \( x \) from left support to single point load is given by equation 1.3,

\[ M_{\text{max}} = -\frac{W_x}{L} (L-x) \]  

(1)

If \( w' \) is E.U.D.L. over the whole span, B.M. at the section C is given by

\[ M = \frac{w'L}{2} x + \frac{w'}{2} (L-x) = \frac{w'L}{2} (L-x) \]  

(2)

Equating (1) and (2), we get

\[ w' = \frac{2W}{L} \]  

(1.34)

The same result could be obtained by equating the bending moment at the centre, i.e.

\[ \frac{w'L^3}{8} = \frac{wL}{4} \]

or

\[ w' = \frac{2W}{L} \]

which is the same as above.

(b) E.U.D.L. for U.D.L. shorter than the span:

The max. B.M. at the centre of the span, due to U.D.L. shorter than the span, is given by

\[ M_{\text{max}} = -\frac{w}{4} \left( L - \frac{a}{2} \right) \]

where \( a \) is the length of the U.D.L.

The B.M. at the centre of span, due to E.U.D.L. \( w' \) is

\[ M = -\frac{w'L^2}{8} \]

Equating the two, we get

\[ \frac{w'L^2}{8} = \frac{w}{4} \left( L - \frac{a}{2} \right) \]

or

\[ \frac{w'}{L^2} = \frac{2w}{L} \left( L - \frac{a}{2} \right) \]  

(1.35)

(c) E.U.D.L. for the point loads \( W_1 \) and \( W_2 \) at a distance \( d \) apart:

The E.U.D.L. for this must be such that the B.M.D. due to this completely envelopes \( M_x \), \( 2M_x \), and \( 3M_x \) diagrams. This can be
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there if the tangent to the curve of B.M. due to E.U.D.L. at the
support is equal to the greater of the tangents to \( M_x \) and \( 1M_x \) (or
diagrams at their corresponding ends).

Thus, in example 1-2, the equation of \( M_x \) is given by

\[
4M_x = y = \frac{x}{16} (136 - 10x)
\]

\[
\frac{dy}{dx} (at \ x = 0) = \frac{1}{16} (136) = 8.5
\]

The equation of \( M_x = (10x - 36) (1 - \frac{x}{16}) = (12.25x - 0.625x^2 - 36)
\]

\[
\frac{dy}{dx} (at \ x = 16) = 12.25 - 1.25 \times 16 = -7.75
\]

The minus sign simply shows that the inclination of the tangent is
in anticlockwise direction.

\[
\therefore \ \text{Greater} \ \frac{dy}{dx} \ \text{due to the actual loading} = 8.5
\]

The equation of \( M_x \) is given by

\[
M_x = \frac{w}{2} (L - x) L = \frac{w}{2} (16 - x)
\]

\[
\frac{dy}{dx} (at \ x = 0) = 8w'
\]

Equating this to the greater of (1) and (2), we get

\[
8w' = 8.5
\]

\[
w' = 8.5 \times \frac{8}{8} = 1.06 \text{ kN/m}
\]

This will give max. B.M. = \( \frac{w'L^2}{8} = \frac{8.5 \times 16 \times 16}{8} = 34 \text{ kN-m.} \)

The actual absolute Max. B.M. = 28.8 kN-m, as found in
example 1-2.

Similarly, the E.U.D.L. on the considerations of max. shear

1.9 COMBINED DEAD AND MOVING LOAD S.F. DIAGRAMS : FOCAL LENGTH

Let a girder \( AB \), simply supported over a span \( L \), carry a
uniformly distributed dead load \( w/\text{unit length} \). Also due to certain
system of moving loads, let \( w' \) be the E.U.D.L., based on shear
considerations.

ROLLING LOADS

Fig. 1.18(a) shows the S.F.D. due to dead load.
Fig. 1.18(b) shows the S.F.D. due to E.U.D.L. At any distance
\( x \), S.F. due to E.U.D.L. is given by

\[
F(x) = \frac{w' x^2}{2L} \quad (d)
\]

and

\[
F(x) = \frac{w' (L - x)^2}{2L} \quad (c)
\]

Fig. 1.18(c) shows the combined S.F.D., obtained after super-
imposing the two diagrams.

Thus, by combining -ve S.F. of (a) with +ve S.F. of (b), we get
final shear=ordinate \( C_2C_3 \). Similarly by combining -ve S.F. of
(a) with -ve S.F. of (b), we get final shear=ordinate \( C_1C_2 \). Hence
in the combined diagram, the final shear at any point is given by
vertical intercepts between dead load S.F. and the curves of
E.U.D.L.

From Fig. 1.18(c), we make the following observations :
At point \( C \), S.F. = \( C_1C_3 \) and \( C_1C_3 \) (both negative)
At point \( P \), S.F. = \( P_1P_3 ( = 0) \) and \( P_2P_4 \) (negative)
At point \( O \), S.F. = \( O_1Q_2 \) (positive) and \( O_1Q_2 = 0 \)
At point, D S.F. = $D_1 D_4$ and $D_4 D_3$ (both positive).

From the above, we make the following conclusions:

(a) For all sections to the left of $P$, the final S.F. is always negative.

(b) For all sections to the right of $Q$, the final S.F. is always positive.

(c) For all sections between $P$ and $Q$, the final S.F. is both positive and negative. That is, the S.F. changes sign as the load moves over the portion $PQ$ only. Such a portion of the girder, over which the final S.F. changes sign, is called the focal length. If such a girder is of lattice type, counter bracing is needed for this portion. In Fig. 118(c), thus, $PQ$ is the focal length of the girder.

Example 1'9. Calculate the focal length of a girder of 16 m span carrying a dead load of 3 kN/m and E.U.D.L. of 6 kN/m for shear.

**Solution:** (Fig. 1'18)

Let $F_d$ = S.F. due to dead load, at any section.

Then, $F_d = \frac{-wL}{2} + wx = -\frac{3x}{2} + 3x = -24 + 3x$  

$F_i(+ve) = +\frac{w^2 x^2}{2L} + \frac{6x}{2} + 3x = \frac{31x}{16}$  

At the point $P$ (Fig. 1'18(c)), $F_d + F_i (+ve) = 0$  

$\therefore \quad -24 + 3x + \frac{31x}{16} = 0$

which gives $x = AP = 5'85$ m

By symmetry $BQ = AP = 5'85$ m

$\therefore$ Focal length $PQ = AB - 2AP = 16 - 2 \times 5'85 = 4'3$ m.

Example 1'10. Calculate the focal length of a girder of 16 m span, carrying a dead load of 3 kN/m and a uniform live load of 2 kN/m, 4 m long, travelling from left to right.

**Solution.** (Fig. 1'18)

$F_d = \frac{-wL}{2} + wx = -\frac{3x}{2} + 3x = -24 + 3x$  

For $x > 5$ m, $F_i(+ve) = +\frac{w}{L} \left( x - \frac{a}{2} \right)$ ...(see equation 1'6)

$= +\frac{2 \times 4}{16} \left( x - \frac{4}{2} \right) = 0'5(x - 2)$

At the point $P$ (Fig. 1'18(c)), $F_d + F_i (+ve) = 0$

$\therefore \quad -24 + 3x + 0'5(x - 2) = 0$

which gives $x = AP = 7'14$ m

By symmetry, $QR = AP = 7'14$ m

$\therefore$ Focal length $PQ = 16 - 2 \times 7'14 = 1'72$ m

Example 1'11. Calculate the focal length of the girder of example 1'2 if it also carries a dead load of intensity 3 kN/m over the whole span.

**Solution**

For the given girder : $L = 16$ m ; $W_i = 4$ kN  

$W_a = 6$ kN ; $d = 6$ m

For any section distant $x$ from $A$,

$F_d = -\frac{wL}{2} + wx = -\frac{3x}{2} + 3x = 24 + 3x \quad (1)$

(a) For $x > 6$, max. $+ve$ S.F. due to live load is given by

$F_i(+ve) = F_{X} = \frac{W_i x + W_a x - d}{L} = \frac{+(W_i + W_a) x - W_i d}{L}$

$= \frac{(4 + 6) x - 6 x}{16} = \frac{5 x - 18}{8} = \frac{5 x - 22.5}{8}$

At the point $P$ (Fig. 1'18(c)), we have

$F_d + F_i (+ve) = 0$

or

$-24 + 3x + \frac{5}{8} x - 2.25 = 0$

which gives $x = AP = 7'24$ m.

(b) Again, for $x > 8 < 10$, we have

$F_i(-ve) = F_{X} = \frac{W_a (L - x) - W_i (L - x - d)}{L}$

$= \frac{(6(16 - x) + 4(16 - x - 6))}{16}$

$= -8'5 + \frac{5}{8} x \quad (3)$

For the point $O$, we have

$F_d + F_i (-ve) = 0$

$\therefore \quad -24 + 3x - 8'5 + \frac{5}{8} x = 0$

which gives $x = AQ = 8'96$ m.

Hence focal length $AQ - AP = 8'96 - 7'24 = 1'72$ m
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PROBLEMS

1. A single rolling load of 10 kN rolls along a girder of 20 m span. Draw the diagrams of maximum B.M. and maximum S.F. positive and negative. What will be the absolute maximum (+) S.F. and B.M.? (L.U.)

2. A uniform load of 1 kN/m, 4 m long crosses a girder of 16 m span. Construct the maximum S.F. and B.M. diagrams and calculate values at section 6 m and 8 m from left hand support.

3. Two concentrated rolling loads of 12 and 6 kN, placed 4.5 m apart, travel along a simply-supported girder of 16 m span. Sketch the graphs of maximum shearing force and maximum bending moment and indicate the position and magnitudes of the greater value.

4. A simply-supported girder has a span of 40 m. A moving load consisting of a uniformly distributed load of 1 kN/m over a length of 8 m preceded by a concentrated load of 6 kN moving at a fixed distance of 2 m in front of the distributed load, crosses the beam.

Find (a) the point of the beam at which the greatest bending moment occurs, (b) the position of the load where it occurs, (c) the value of the greatest B.M.

5. A simply-supported beam is traversed by a train of wheel loads of irregular spacing and unequal weights. State and prove a rule giving the train position for the bending moment under a particular load to have its maximum value, and (b) a rule giving the train position for the bending moment at a given point on the beam to its maximum value.

6. A freely supported gantry girder of effective span L carries a travelling crane with two wheel loads, each w at spacing a, this spacing being less than \( \frac{L}{2} \). Find, from first principles, the maximum bending moment induced by the loads.

If the spacing a is increased, find the maximum value of a (in terms of L) for which the maximum bending moment will occur at the centre of the span with only one wheel on the girder. (U.L.)

7. A system of moving loads cross a girder of 36 m span which is simply supported at its ends. The loads and their distances are as follows:

<table>
<thead>
<tr>
<th>Wheel loads (kN)</th>
<th>2</th>
<th>6</th>
<th>6</th>
<th>5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance between centres</td>
<td>1-5</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

Determine
(a) The maximum bending moment at the quarter span.
(b) The maximum bending moment in the girder.

For each case, make a sketch of the girder showing clearly the section where the bending moment occurs and the corresponding position of the loads.

8. The following system of concentrated loads roll from left to right on a span of 15 m, 4 kN load leading:

<table>
<thead>
<tr>
<th>Load</th>
<th>2</th>
<th>6</th>
<th>6</th>
<th>5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance</td>
<td>1-5</td>
<td>1-5</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

For a section 4 m from the left hand support, determine (a) the maximum bending moment, (b) maximum S.F. (L.U.)

ROLLING LOADS

9. The following system of wheel loads crosses a plate girder of 30 m span:

<table>
<thead>
<tr>
<th>Wheel load</th>
<th>8</th>
<th>18</th>
<th>18</th>
<th>15 kN</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance between centres</td>
<td>4-5</td>
<td>3-5</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Determine the maximum value of the shearing force which may be produced at the middle point of the span. Also, find the equivalent uniformly distributed load which could produce the same maximum bending moment at midspan.

10. A simply supported beam of span L is crossed by a uniformly distributed load of length m and of total weight W. If L is greater than m, obtain from first principles an expression for the maximum bending moment at any point at distance a from one support. Hence show that a single point load of \( W \left( 1 - \frac{a}{L} \right) \), travelling across the span will give the same maximum moment everywhere along the beam as the above uniformly distributed load.

11. The following arrangement of axle load is carried by a single bridge girder across a clear span of 30 m.

<table>
<thead>
<tr>
<th>Axle loads</th>
<th>5</th>
<th>5</th>
<th>10</th>
<th>10</th>
<th>10 kN</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spacing</td>
<td>2-5</td>
<td>2-5</td>
<td>2-5</td>
<td>2-5 m</td>
<td></td>
</tr>
</tbody>
</table>

Determine the maximum bending moment and maximum S.F. at section distant 10 m from left hand abutment. The 5 kN load leads, and the system may pass over the bridge from either side.

12. A beam, simply supported over a span L is traversed by a uniformly distributed load of intensity \( \frac{w}{2} \) and length \( \frac{L}{5} \). If the beam also carries a dead load, uniformly distributed over the span, of intensity \( \frac{w}{2} \) indicate on the diagram the length of the beam for which there is reversal of shear force.

Answers

1. S.F.: \( \pm 10 \) kN; B.M.: \( -50 \) kN.m.
2. 13-12 kN-m; 14-10 kN-m.
3. \( P_{max} = (+a) = 14-62 \) kN at right hand support.
4. \( P_{max} = (-a) = 16-31 \) kN at left support.
5. \( M_{max} = 59-1 \) kN at 7.25 m from left support.
6. (a) 20-26 m from left support.
7. (b) Tail of load at distance of 13-17 m from support.
8. (c) \( M_{max} = 118-5 \) kN-m.
9. \( M_{max} = \frac{W}{8L} (2L - a)^4; a = 0-5862 \).
10. \( M_{max} \) at left quarter span = 373-8 kN-m.
11. \( M_{max} \) at right quarter span = 396-8 kN-m.
12. \( M_{max} \) at 51-9 kN-m with central wheel load placed 0-48 m off the centre.
13. \( M_{max} \) at 12-1 kN when 4th load is on the section.
14. (a) 19-4 kN; (b) \( w = 3-11 \) kN/m.
15. 216-7 kN-m; \( -21-25 \) kN.
16. 0-2286 L.
2

Influence Lines

2.1. DEFINITION

An influence line for any given point or section of a structure is a curve whose ordinates represent to scale the variation of a function, such as shear force, bending moment, deflection, etc., at the point or section as unit load moves across the structure. In other words, an influence line for any given point C on a structure is such a curve that its ordinate at any point D gives the bending moment, shear force or similar quantity at C when a unit load is placed at D. For statically determine structures, the influence lines for B.M., shear, or stress are composed of straight lines, while they are curvilinear for statically indeterminate structures. The influence lines are very useful in the speedy determination of the value of a function at the given section under any complex system of loading. These also help to determine, in an easy manner, the disposition of the load system so as to cause the maximum value of the function at the section.

The difference between a curve of B.M. or S.F. (as discussed in the previous chapter) and an influence line of B.M. or S.F. must be clearly understood at this stage. The ordinate of a curve of B.M. or S.F. gives the value of the B.M. or S.F. at the section where the ordinate has been drawn, while in the case of an influence line, the ordinate at any point gives the value of the B.M. or S.F. only at the given section (for which the influence line has been drawn) and not at the point at which the ordinate has been drawn. Also, there is one single B.M. or S.F. curve of the whole beam under the action of a given set or train of loads, while there are infinite number of influence lines, one for each section of the beam, drawn for a unit rolling load.

2.2. INFLUENCE LINE FOR SHEAR FORCE

Let us consider a simply supported beam AB of span L, and construct the influence line for S.F. at a section C distant x from the left support. The position of the section is fixed, while the unit load moves from left to right. The problem is to plot the variation of S.F. at the given section C, as the unit load moves along the beam.
At any instant, let the unit load be at a distance $aL$ from the support $A$. Then, $R_b=+a$, and $R_a=(1-a)$.

1. Shear force at $C=F_c=+R_b=+a$ (1)
The variation is linear, and is valid for all positions of load between 0 to $x$ from $A$.

When the load is at $A$, $aL=0$, \[ F_c=0 \]

When the load is at $C$, $aL=x$
\[ F_c=+a=+\frac{x}{L} \]

When the unit load crosses the section $C$, $aL>x$, and hence
\[ F_c=-R_a=-(1-a) \] (2)

Thus, the S.F. changes sign as the unit load crosses the section. The variation is linear, and is valid for all load positions between $x$ to $L$ from $A$.

When the unit load is slightly to the right of $C$, $aL=x$
\[ F_c=-(1-a)=-(1-\frac{x}{L})=-\frac{L-x}{L} \]

When the unit load is at support $B$, $aL=L$ or $x=1$
\[ F_c=-(1-a)=-(1-1)=0 \]

The complete influence line diagram for the S.F. at $C$ is given in Fig. 21(b).

As per definition, the ordinate $+y_1$ at a point gives the S.F. at $C$, due to unit load at the point where the ordinate $y_1$ is measured. Hence if a load $W_1$ is acting at that point, and $y_1$ is the ordinate of I.L. under it, the S.F. at $C=W_1y_1$. Similarly, if a load $W_2$ is acting at a certain point, and $-y_2$ is the ordinate of the influence line under the point of application of the load, the S.F. at $C$ will be $-W_2y_2$. If $W_1$ and $W_2$ are acting simultaneously, the S.F. at $C=W_1y_1-W_2y_2$. Hence if the beam is being acted upon by loads $W_1, W_2, W_3, \ldots, W_n$, and $y_1, y_2, \ldots, y_n$ are the corresponding influence ordinates under them, the S.F. at $C$ is
\[ F_c=W_1y_1+W_2y_2+W_3y_3+\ldots+W_ny_n=\sum W y \] (2.1)

In the above equation, the numerical value of the ordinate $y$ is to be substituted with its proper algebraic sign, i.e. $+ve$ if it is of positive diagram, and $-ve$ if of negative portion of the influence line diagram.

Let us now take the case of U.D.L. (w) of a length $a$, placed in the position shown in Fig. 21(c). Let us consider a length $8x$ of the load, and the corresponding elementary load $8W=w.8x$. Hence S.F. at $C$, due to the elementary load $8W$ is
\[ \delta F_c=8W.y \]

(where $y$ is the influence line ordinate under $8W$)

**Influence Lines**

\[ \delta F_c=w.8x.y \] (3)

\[ =w.\text{area of the elementary strip of the I.L. diagram [shown in Fig. 21(d)]} \]

Therefore, the shear force at $C$, due to total U.D.L. of length $a$ given by
\[ F_c=\sum (\delta x.y)=w.\sum \delta x.y \] (2.2)

\[ =w.\text{area of I.L. diagram under the U.D.L. [shown shaded in Fig. 21(d)]} \]

Hence the S.F. at $C$, due to U.D.L. of length $a$ is equal to the area of the I.L. diagram under the U.D.L. multiplied by the intensity of the load.

Fig. 21(e) shows the U.D.L. extending to both the sides of the section $C$. In this case, the S.F. at $C$ is obtained by multiplying the net area by the intensity of the load.

Thus,
\[ F_c=w(a_1-a_2) \]

where
- $a_1$ = area of the positive S.F. diagram under the U.D.L.
- $a_2$ = area of the negative S.F. diagram under the load.

If
\[ a_1=a_2, \quad F_c=0. \]

**Influence Line for the Reactions**

If the section $C$ is located at the support $B$, the value of $x=L$, and hence the ordinate of $+ve$ I.L. diagram under $B=\frac{x}{L}=\frac{L}{L}=1$.

Thus, the I.L. for reaction at $B$ is the I.L. for shear at $C$ when $x=L$, and is a triangle having a maximum ordinate of unity under $B$. However, the I.L. for reactions at $A$ and $B$ can be plotted independently as under:

When the load is at a distance $aL$ from $A$.

\begin{align*}
R_b &= +a \quad \text{and} \quad R_a = +(1-a) \\
\text{When the load is at } A, & \quad a=0 \\
& \quad R_b = 0 \quad : \quad R_a = +1 \\
\text{When the load is at } B, & \quad aL=L \ ; \ or \ a=1 \\
& \quad R_a = +a = +1 \quad : \quad R_a = +(1-a) = 0 \\
\end{align*}

Hence the I.L. for $R_b$ consists of a triangle having zero ordinate at $A$ and unit ordinate at $B$. Similarly, the I.L. for $R_a$ consists of a triangle having unit ordinate at $A$ and zero ordinate at $B$ as shown in Fig. 21(g) and (h) respectively.
2.3. INFLUENCE LINE FOR BENDING MOMENT

Let us now construct the I.L. for B.M. at C.

When the unit load is at a distance \( aL \) from \( A \), such that \( aL < x \), we have \( R_a = a \) and \( R_a = (1-a) \).

\[ M_C = -R_a(L-x) = -a(L-x) \]

The variation is linear, and is valid for load position distant 0 to \( x \) from \( A \).

When the unit load is at \( A \), \( aL = 0 \).

\[ M_C = 0 \]

When the unit load is at \( C \), \( aL = x \).

\[ M_C = -\frac{x}{L} (L-x) \quad (2.3) \]

When the unit load is at \( C \), \( aL > x \).

\[ M_C = -R_a \cdot x = -(1-a)x \] \( (1) \)

The variation is linear, and is valid for load position distant \( x \) to \( L \) from \( A \).

When the load is at \( C \), \( aL = x \).

\[ M_C = -\left( 1 - \frac{x}{L} \right) x = -\frac{L-x}{L} x \]

which is the same as equation 2.3.

Thus, the I.L. diagram for \( M_C \) is a triangle having a maximum ordinate of \( \frac{x}{L} (L-x) \) under the section as shown in Fig. 2.2(b).

Influence Lines

If there are two loads \( W_1 \) and \( W_2 \) acting, and if \( y_1 \) and \( y_2 \) are the influence line ordinates under these loads, we have by definition

\[ M_C = -(W_1 y_1 + W_2 y_2) \]

Hence, if there are number of point loads \( W_1, W_2, \ldots, W_n \) and the corresponding I.L. ordinates under them are \( y_1, y_2, \ldots, y_n \) we have

\[ M_C = (W_1 y_1 + W_2 y_2 + \cdots + W_n y_n) = -\sum W y \] \( (2.4) \)

Let there be an U.D.L. of intensity \( w \), and length \( \alpha \), as shown in Fig. 2.2(c). Consider an elementary length \( \delta x \) of the load, such that the elementary load \( \delta W = w \delta x \). Let \( y \) be the average ordinate under the elementary load. Then the B.M. at \( C \) due to this elementary load is given by

\[ \delta M_C = \delta W \cdot y = -w \delta x \cdot y \]

\[ = -w x \cdot \text{area of the elementary strip of the I.L. diagram [shown thick in Fig. 2.2(d)].} \]

Hence the B.M. at \( C \), due to the total U.D.L. of length \( \alpha \) is

\[ M_C = -\int w (\delta x \cdot y) = -w x \cdot \text{area of I.L. diagram under U.D.L. [shown shaded in Fig. 2.2 (d)]} \] \( (2.5) \)

Thus, the B.M. at \( C \), due to U.D.L. of length \( \alpha \) is equal to the intensity of load multiplied by the area of I.L. diagram under the uniformly distributed load.

2.4. LOAD POSITION FOR MAXIMUM S.F. AT A SECTION

In chapter 1 on rolling loads, we have derived the load positions for maximum S.F. at a given section. We will now use the influence line for determination of the position of loads for maximum S.F. at the section \( C \). We shall take different loading conditions.

1. Single point load

Let a single point load of magnitude \( W \) roll from left to right. Referring to I.L. of S.F. at the section \( C \) distant \( x \) from \( A \) [Fig. 2.1 (b)], maximum positive S.F. will occur when the load is just to the left of \( C \), and maximum -ve S.F. will occur when the load is just to the right of \( C \).

Thus, \( Fc(\max. \text{ +ve}) = \frac{W x}{L} \)

and \( Fc(\max. \text{ -ve}) = -\frac{W (L-x)}{L} \).
2. U.D.L. longer than the span

From the I.L. for S.F. at C, Fig. 2-1(b), it is clear that the max. +ve S.F. will occur when the span AC is loaded and CB is empty and max. −ve S.F. will occur when the span CB is loaded and AC is empty.

Thus, \( F_c(\text{max.}\,+\text{ve}) \equiv w \cdot \frac{1}{2} \cdot \frac{x}{L} = \frac{wx^2}{2L} \)

\( F_c(\text{max.}\,-\text{ve}) \equiv w \cdot \frac{1}{2} \left( L-x \right) \left( \frac{L-x}{L} \right) = \frac{w(L-x)^2}{2L} \)

3. U.D.L. shorter than the span

Let the U.D.L. of length \( a \) travel from left to right. From Fig. 2-1(b), maximum +ve S.F. at C will occur when the head of the load reaches C, while maximum −ve S.F. will occur when the tail of the load is at C.

4. Several Point Loads

For several point loads, we may use the same criterion, as discussed in the previous chapter. Thus, if \( a \) load \( W_1 \) is at the section \( C \), with other loads in appropriate position, and the loads are moved by a distance \( d \) such that next load comes over \( C \), the change \( \delta F_c \) is given by

\[ \delta F_c = \frac{wd}{L} - W_1. \]

If the above expression is positive, it indicates an increase in S.F. and the loads must be permitted to roll to get greater S.F. The procedure must be repeated until the above expression changes sign, which indicates that greatest peak has been passed.

2.5. LOAD POSITION FOR MAXIMUM B.M. AT A SECTION

Here also, we shall consider all the loading conditions:

1. Single Point Load

Let a single point load \( W \) roll from left to right. Since the I.L. diagram for B.M. at \( C \) has the maximum ordinate under \( C \) itself (see Fig. 2-2(b)), maximum B.M. will occur when the load is at \( C \) itself.

Thus \( M_c(\text{max.}) = W \cdot \frac{x}{L} \cdot (L-x) \)

2. U.D.L. greater than the span

Refer to Fig. 2-2(b) maximum B.M. will occur when the U.D.L. occupies the whole span.

Thus \( M_c(\text{max.}) = w \times \text{area of I.L. diagram} \)

\[ = w \times \frac{1}{2} \times L \times \frac{x}{L} \left( L-x \right) = \frac{wx(L-x)}{2} \]

3. U.D.L. Shorter than the span

Let the uniformly distributed load be of length \( a \). The load has to be arranged, with respect to section \( C \), in such a way that the area of the I.L. diagram under the load is maximum.

Influence Lines

Thus \( M_c(\text{max.}) = w \times \text{area of I.L. diagram} \)

\[ = \frac{wx(L-x)}{2} \]
Hence the maximum bending moment at a section occurs when the section divides the U.D.L. in the same ratio as it divides the span.

\[ M_{c}(\text{max.}) = w \times \text{area of I.L. diagram under the load} \]

\[ = w \left[ \frac{(a_1+cc_1)}{2} \right] \]

\[ = \frac{wa}{2} \left\{ \frac{x(L-x)}{L} - \frac{a}{L} + \frac{x(L-x)}{L} \right\} \]

\[ = \frac{wa}{2} \left( \frac{x(L-x)}{L} \right) \left( \frac{L-a}{L} + \frac{x(L-x)}{L} \right) \]

\[ = \frac{wax(L-x)}{2L^2} \left( 2L + a \right) \]

\[ (2.7) \]

4. Several Point Loads
Let the loads be so arranged that \( W_L \) is the resultant of the loads to the left of the section \( C \), and \( W_R \) is the resultant of the loads to the right of \( C \). Let \( y_1 \) and \( y_2 \) be the ordinates under \( W_L \) and \( W_R \) respectively.

The B.M. at \( C \), for this arrangement is given by

\[ M_c = W_1 \cdot y_1 + W_R \cdot y_2 \]

\[ \text{Fig. 2.4.} \]

This will be maximum only if a small movement \( \delta d \) of the loads either to the left or to the right, will decrease its value. Let the loads be given a movement \( \delta d \) to the right, and let the new ordinates under \( W_L \) and \( W_R \) be \( (y_1 + \delta y_1) \) and \( (y_2 - \delta y_2) \) respectively. The corresponding change \( \delta M_c \) is given by

\[ \delta M_c = [W_L(y_1 + \delta y_1) + W_R(y_2 - \delta y_2)] - [W_L y_1 + W_R y_2] \]

\[ = W_L \delta y_1 - W_R \delta y_2 \]

Example 2.1. Two wheel loads of 16 and 8 kN, at a fixed distance apart of 2 m, cross a beam of 10 m span. Draw the influence line for bending moment and shear force for a point 4 m from the left abutment, and find the maximum bending moment and shear force at that point.

Solution. (Fig. 2.5).

\[ \text{Fig. 2.5.} \]

(a) Max. B.M. at \( C \)
The I.L. for B.M. at \( C \) distant 4 m from \( A \) is shown in Fig. 2.5(b).

The maximum ordinate under \( C = \frac{x(L-x)}{L} = \frac{4 \times 6}{10} = 2.4 \).

The B.M. at \( C \) is maximum when \( \delta W y \) is maximum. By inspection, \( M_{\text{max}} \) occurs when the loads are as in the position shown.
Ordinate under 16 kN load = 2·4.
Ordinate under 8 kN load = \( \frac{2·4 \times 4}{6} = 1·6 \)
\[ M_c = (16 \times 2·4) + (8 \times 1·6) = 51·2 \text{ kN}\cdot\text{m} \]

(b) Max. S.F. at C
The I.L. for S.F. at a section C distant 4 m from A is shown in Fig. 2.5(c).
The ordinate under C are 
\[ \frac{x}{L} = +\frac{4}{10} = +0·4 \]
and 
\[ \frac{L-x}{x} = -\frac{6}{10} = -0·6 \]

By inspection of the I.L., max. S.F. occurs when the 16 kN load is just to the right of C, and the 8 kN load is ahead of it. Ordinate under 16 kN load = +0·6. Ordinate under 8 kN load = -0·4.
\[ F_c = -[16 \times 0·6 + 8 \times 0·4] = 12·8 \text{ kN} \]

It can be shown that the max. +ve S.F. at C will be lesser than 12·1 kN.

Example 2.2. Make neat diagram of the influence lines for shearing force and B.M. at a section 3 m from one end of a simply supported beam, 12 m long. Use the diagram to calculate the maximum shearing force and the maximum bending moment at this section due to a uniformly distributed rolling load, 5 m long and of 2 kN per metre intensity.

Solution. (Fig. 2.6)

\[ a_1 a_2 = \frac{2·25}{3} \times 1·75 = 1·3125 \]
\[ b_1 b_2 = \frac{2·25}{9} \times 5·25 = 1·3125 \]
\[ M_c = \frac{1·3125 + 2·25}{2} \times (1·25 + 3·75) \times 2 = 17·81 \text{ kN}\cdot\text{m} \]

(b) I.L. for S.F.

The I.L. for S.F. at C is shown in Fig. 2.6(c). The ordinates under C are: \( +\frac{3}{12} = +0·25 \), and \( -\frac{9}{12} = -0·75 \). By inspection, maximum S.F. at C will occur when the tail of the load is at C. The ordinate under the head of load \( \frac{-0·75 \times 4}{9} = \frac{1}{3} \).

Then \[ F_c = w \times (\text{Shaded area of I.L. under U.D.L.}) = 2 \times \frac{5}{2} \left(0·75 + \frac{1}{3}\right) = 5·42 \text{ kN.} \]

Example 2.3. A simply supported girder has a span of 25 m. Draw on squared paper the influence line for shearing force at a section 10 m from one end, and using the diagram determine the maximum shearing force due to the passage of a knife-edge load of 4 kN, followed immediately by a uniformly distributed load of 2·4 kN per metre extending over a length of 5 m. The loads may cross in either direction.
Solution. (Fig. 2.7)

The I.L. ordinate \( cc_1 = \frac{10}{25} = +\frac{2}{5} \)

\( cc_2 = -\frac{15}{25} = -\frac{3}{5} \)

For maximum +ve S.F., the 5 kN load will be just to the left of C, and the U.D.L. behind or to the left of it. In this position, the ordinate \( aa_1 \) under the tail of the U.D.L. is

\[ aa_1 = \frac{2}{5} \times \frac{5}{10} = \frac{1}{5} \]

\[ Fc(\text{+ve}) = \left( 5 \times \frac{2}{5} \right) + 2.4 \left( \frac{2}{5} + \frac{1}{5} \right) \frac{5}{2} = +5.6 \text{ kN} \]

For maximum +ve S.F. at C, the 5 kN load will be just to the right of C, with U.D.L. to the right of it. In this position, the ordinate \( bb_1 \) under the tail of the load is

\[ bb_1 = \frac{3}{5} \times \frac{10}{15} = \frac{2}{5} \]

\[ Fc(-\text{ve}) = \left( 5 \times \frac{3}{5} \right) + 2.4 \left( \frac{3}{5} + \frac{2}{5} \right) \frac{5}{2} = -9 \text{ kN} \]

Hence the maximum S.F. at the section is the greater of the two. Its value is, therefore, 9 kN.

Example 2.4. Four wheel loads of 6, 4, 8 and 5 kN cross a girder of 20 m span, from left to right followed by U.D.L. of 4 kN/m and 4 m long with the 6 kN load leading. The spacing between the loads in the same order are 3 m, 2 m and 2 m. The head of the U.D.L. is at 2 m from the last 5 kN load. Using influence lines, calculate the S.F. and B.M. at a section 8 m from the left support when the 4 kN load is at centre of the span.

\[ 4 \text{kN/m} \]
\[ 5 \text{kN} \]
\[ 8 \text{kN} \]
\[ 4 \text{kN} \]
\[ 6 \text{kN} \]

\[ 4 \text{m} \]
\[ 2 \]
\[ 2 \]
\[ 3 \]
\[ 7 \text{m} \]

\[ 8 \text{m} \]

\[ 20 \text{m} \]

(a) Bending Moment

The ordinate of I.L. for B.M. at \( \frac{8 \times 12}{20} = 4.8 \)

When the 4 kN load is at the centre of the beam, the arrangement of the other loads will be as shown in Fig. 2.8 (a).

\[ \therefore \text{Ordinate under 6 kN load} = \frac{4.8}{12} \times 7 = 2.8 \]

Ordinate under 4 kN load = \( \frac{4.8}{12} \times 10 = 4 \)

Ordinate under 5 kN load = \( \frac{4.8}{8} \times 6 = 3.6 \)

Ordinate under head of U.D.L. = \( \frac{4.8}{8} \times 4 = 2.4 \)

\[ \therefore Mc = -\Sigma My = -[(6 \times 2.8)+(4 \times 4)+(8 \times 4.8)+(5 \times 3.6)] 
\[ + (\frac{1}{2} \times 2.4 \times 4 \times 4) \]
\[ = -108.4 \text{ kN-m} \] (1)

(b) Shear Force

The ordinate of I.L. for S.F. at C are \( \frac{8}{20} = \frac{2}{5} = +0.4 \) and

\[ \frac{12}{20} = \frac{3}{5} = -0.6 \]

Hence ordinate under 6 kN load = \( \frac{0.6}{12} \times 7 = 0.35 \)

ordinate under 4 kN load = \( \frac{0.6}{12} \times 10 = 0.5 \)
ordinate under 5 kN load = \( \frac{0.4}{8} \times 6 = 0.3 \)

ordinate under head of U.D.L. = \( \frac{0.4}{8} \times 4 = 0.2 \)

(i) maximum -ve S.F.
For maximum -ve S.F., consider the 8 kN load just to the left of C.
Then \( F_C = + \left\{ (8 \times 0.4) + (5 \times 0.3) + \left( \frac{1}{2} \times 4 \times 0.2 \times 4 \right) \right\} \)
\[ = +2.2 \text{ kN} \]

(ii) Maximum -ve S.F.
For maximum -ve S.F., consider the 8 kN load just to the right of C.
Then \( F_C = -\left\{ (6 \times 0.35) + (4 \times 0.5) + (8 \times 0.6) \right\} + (5 \times 0.3)
\quad \left( + \frac{1}{2} \times 4 \times 0.2 \times 4 \right) \)
\[ = -5.8 \text{ kN} \]

Hence the maximum S.F. at C, under the given load positions, will be the greater of the two, i.e. 5.8 kN.

**Example 2.5.** A horizontal beam ABC is hinged at A and simply supported at B. The span is 15 m. The cantilevered portion BC is 6 m long. Draw the influence line for bending moment for the points D and E respectively 12 m from A and 4 m from C. Hence find the maximum ± bending moments at D and the maximum bending moment at E due to a load of 1 kN/m of a length 3 m. State the corresponding position of the load.

**Solution**

Since the beam is hinged at A, it is statically determinate.

(a) I.L. for B.M. at E
Let the unit load roll from left to right. When the unit load is between A to E, the B.M. at E is equal to zero (since there is no load to the right of E).

When the unit load is in EC, distant \( x \) from E,
\[ M_E = +1 \times x = +x. \]

The variation is linear,
when \( x = 0 \), \( M_E = 0 \).
when \( x = 4 \text{ m} \), \( M_E = +4 \).

Hence the ordinate of I.L. diagram is zero under E and +4 units under C. The I.L. for B.M. at E is shown in Fig. 2.9(b).

For maximum value of \( M_E \), the head of the load should be at C. In this position, ordinate under the tail of the U.D.L. = \( \frac{4}{4} \times 1 \)
\[ = 1. \]

Hence \( M_E = +1 \times \frac{3}{2} \times (1 + 4) = +7.5 \text{ kN-m} \)

(b) I.L. for B.M. at D
Let the unit load travel from left to right. When the load is at a distance \( x \) from A, such that \( x < 15 \).
\[ R_A = \frac{15 - x}{15} \uparrow \text{ and } R_B = \frac{x}{15} \uparrow \]

When \( x < 12 \text{ m} \), \( M_D = -R_B \times 3 = -\frac{x}{15} \times 3 = -\frac{x}{5} \)

when the load is at A, \( x = 0 \), \( \therefore M_D = 0 \)
when the load is at D, \( x = 12 \), \( \therefore M_D = -\frac{12}{5} = -2.4 \)

When \( 12 < x < 15 \text{ m} \), \( M_D = -R_A \times 12 = -\frac{15 - x}{15} \times 12 \)

-when the load is at B, \( x = 12 \), \( \therefore M_D = -2.4 \)
when the load is at A, \( x = 15 \), \( \therefore M_D = 0 \)
when the load is in BC, distance \( x \) from A, \( R_B = \frac{x}{15} \uparrow \)

\[ \therefore M_D = +1 \times (x - 12) - R_B \times 3 = (x - 12) - \frac{x}{15} \times 3 = (0.8x - 12) \]

-when the load is at E, \( x = 15 \), \( \therefore M_D = 0 \)
when the load is at C, \( x = 21 \), \( \therefore M_D = -4.8 \text{ KN-m} \).

The complete I.L. for B M., at D is shown in Fig. 2.9(c).
Now, for maximum positive B.M., the head of the load should be at C. In this position, the ordinate under the tail of the load

\[ = \frac{4.8}{6} \times 3 = 2.4 \]

Hence \( M_0 \) (positive max.) \( = -\frac{1}{2} \times 3 \times (2.4 + 4.8) = -10.8 \text{ kN-m} \).

For maximum negative B.M., the load should be partially to the left and partially to the right of D such that the ordinate \( a = b \).

Let the tail of the load be a distance \( x \) from D. Then, from equation 2.6

\[
\frac{AD}{DB} = \frac{A_D}{D_B}
\]

or

\[
\frac{12}{3} = \frac{x}{3-x}
\]

From which

\[
x = 2.4 \text{ m.}
\]

\[ \therefore \text{ Ordinate } a = b = \frac{2.4}{12} \times 9.6 = 1.92 \]

\[ \therefore \text{ Moment } M_0 \text{ (negative max.)} = -\frac{1}{2} \times (1.92 + 2.4) \times 3 = -6.48 \text{ kN-m.} \]

Example 2.6. A beam ABC is supported at A, B and C, and has a hinge at D distant 3 m from A. AB = 7 m and BC = 10 m. Draw the influence lines for:

(i) reactions at A, B and C.
(ii) S.F. at a point just to the right of B.
(iii) B.M. at a section 1 m to the right of B.

Hence, if a U.D.L. of intensity 2 kN/m, and length 3 m, travels from left to right, calculate above quantities from which I.L. are drawn.

Solution

The direction of any reaction will be +ve if it is acting in upward direction (↑).

(a) I.L. for reaction at \( R_a \)

Let the unit load roll from left to right. When the load is in \( AD \), distant \( x \) from A,

\[ R_a = \frac{3-x}{3} \uparrow \] (since \( M_D = 0 \))

When the load is at A, \( x = 0 \), \( \therefore R_a = 1 \)

When the load is at D, \( x = 3 \), \( \therefore R_a = 0 \).

As the load enters DB, the reaction \( R_a \) is always zero, since \( M_D \) has to be zero due to the hinge. Hence the I.L. for \( R_a \) consists of a triangle having a maximum ordinate of unity under A, and zero under D, as shown in Fig. 2.10(b).

Fig. 2.10.

For maximum \( R_a \), the U.D.L. of length 3 m occupy the whole portion \( AD \).

\[ R_a = \frac{1}{2} \times 3 \times 2 = 3 \text{ kN} \uparrow \]

(b) I.L. for reaction at \( B(R_b) \)

When the unit load is in \( AD \), distant \( x \) from A, 

\[ R_b = \left(1 - \frac{x}{3}\right) \downarrow \]

and hence pressure on \( DBC \) at \( D = \frac{x}{3} \downarrow \)

Taking moments about C,

\[ R_b \times 10 = \text{pressure at } AD \times 14 \]

\[ R_b = \frac{x}{3} \times 14 = \frac{14x}{3} \uparrow \] (2)
When the unit load is at \( A \), \( x=0 \), \( \therefore \ R_B=0 \)
When the unit load is at \( D \), \( x=3 \), \( \therefore \ R_B=+1'4 \).

Now, let the load be in \( DBC \), distant \( x \) from \( A \). \( R_A \) will be zero for this range of load position. Hence, taking moments about \( C \), we have

\[
1 \times (17-x) = R_B \times 10
\]

From which

\[
R_B = 1'7 - \frac{x}{10} \quad (3)
\]

When the load is at \( D \), \( x=3 \ m \), \( \therefore \ R_B=+1'4 \)
When the load is at \( B \), \( x=7 \ m \), \( \therefore \ R_B=+1 \)
When the load is at \( C \), \( x=17 \), \( \therefore \ R_B=0 \).

Thus the I.L. for \( R_B \) is triangle as shown in Fig. 2'10(c). For maximum \( R_B \), let the tail of the U.D.L. be at \( x \) from \( D \), so that shaded area is maximum. The criterion, given by Eq. 2'6 is

\[
\frac{AD}{DC} = \frac{A_1D}{DC_1}
\]

or

\[
\frac{3}{14} = \frac{x}{3-x}, \text{ from which } x = \frac{9}{17} \ m
\]

\( \therefore \) ordinate \( a_{a_1} = c_{c_1} = \frac{1'4}{3} \left( 3 - \frac{9}{17} \right) = 1'15 \)

\[
R_B = 2 \times \frac{1}{2} \times 1'15 \times 3 = 7'65 \ kN \uparrow
\]

(c) I.L. for Reaction at \( C (R_c) \)

When the unit load is \( AD \), distant \( x \) from \( A \), \( R_A = \left( 1 - \frac{x}{3} \right) \uparrow \)
and hence pressure on \( DBC \) at \( D = \frac{x}{3} \downarrow \)

Taking moment about \( B \),

\[
\Sigma M_B = 0 = \frac{x}{3} \times 4 + R_C \times 10
\]

\( \therefore \ R_C = - \frac{4x}{30} = - \frac{2x}{15} \quad (i.e. \ R_C = - \frac{2x}{15} \downarrow) \quad (4) \)

When the load is at \( A \), \( x=0 \), \( \therefore \ R_C=0 \)
When the load is at \( D \), \( x=3 \ m \), \( \therefore \ R_C = -0'4 \)

Now, let the load be in \( DBC \), distant \( x \) from \( A \). \( R_A \) will be zero for the range of load position. Hence taking moments about \( B \), we have

\[
\Sigma M_B = 0 = 1 \times (7-x) + R_C \times 10
\]

or

\[
R_C = - \left( 0'7 - \frac{x}{10} \right) \quad (5)
\]

When the load is at \( A \), \( x=0 \), \( \therefore \ R_C=0 \)
When the load is at \( D \), \( x=3 \), \( \therefore \ R_C=+1'4 \)
When the load is at \( C \), \( x=17 \), \( \therefore \ R_C=+1 \)

The I.L. for \( R_C \) is shown in Fig. 2'10(d).

By inspection, maximum \( R_C \) will occur when the head of the U.D.L. is at \( C \). In this position, ordinate, under the tail of the U.D.L. = \( + \frac{1}{10} \times 7 = +0'7 \)

\( \therefore \ R_C = \frac{2}{2} \times (1 + 0'7)3 = +5'1 \ kN \quad (i.e. 5'1 \ kN \uparrow) \)

(d) I.L. for S.F. at a section just to the right of \( B \)

When the load is between \( A \) to \( B \),

\[ F_B = + R_C, \quad \text{and hence the variation of } F_B \text{ will be the same as that of } R_C. \]

Hence I.L. for \( F_B \) will have zero ordinate under \( A \) and \( B \), and ordinate of \( -0'4 \) under \( D \).

When the load is in \( BC \), at distance \( x \) from \( A \), \( R_A=0 \) and

\[ R_B = 1'7 - \frac{x}{10} \quad \text{from Eq. 3 above).} \]

Hence \[ F_B = - R_B = - \left( 1'7 - \frac{x}{10} \right) \quad (6) \]

When the load is just to the right of \( B \), \( x=7 \ m \).

\[ F_B = - \left( 1'7 - \frac{7}{10} \right) = -1. \]

When the load is at \( C \), \( x=17 \ m \)

\[ F_B = - \left( 1'7 - \frac{17}{10} \right) = 0. \]

The complete I.L. diagram is shown in Fig. 2'10 (c). It must be noted that the S.F. is always negative at this section. By inspection, maximum S.F. will occur when the tail of the load is at \( B \). In this position, the ordinate under the head of the U.D.L. = \( \frac{1}{10} \times 7 = 0'7 \).

\[ F_B(\max.) = - \frac{2}{2} \times (1 + 0'7)3 = -5'1 \ kN. \]

(e) I.L. for B.M. at \( E \), 1 m to the right of \( B \)

When the unit load is in \( AD \), distant \( x \) from \( A \).

\[ R_C = - \frac{2x}{15} \quad \text{(from equation 4 above)} \]

\[ \therefore \ M_E = - R_C \times 9 = - \left( -\frac{2x}{15} \times 9 \right) = +1'2x \quad (7) \]
when the load is at \( A \), \( x=0 \) \( \because M_E=0 \)

when the load is at \( D \), \( x=3 \text{ m} \) \( \because M_E=+3'6 \text{ kN-m} \)

When the load is in \( DE \), distant \( x \) from \( A \),

\[
M_E=-R_c \times 9=+(0'7-\frac{x}{10})9=+6'3-0'9x \tag{8}
\]

when the load is at \( D \), \( x=3 \text{ m} \) \( \because M_B=+3'6 \text{ kN-m} \)

when the load is at \( B \), \( x=7 \text{ m} \) \( \because M_E=0 \)

when the load is at \( E \), \( x=8 \text{ m} \) \( \because M_E=-0'9 \text{ kN-m} \)

When the load is in \( EC \), distant \( x \) from \( A \), \( R_A=0 \) and

\[
M_E=-R_c \times 9=-(0'7-\frac{x}{10})9=-6'3+0'9x \tag{9}
\]

when the load is at \( E \), \( x=8 \text{ m} \) \( \because M_E=-0'9 \text{ kN-m} \)

when the load is at \( C \), \( x=17 \text{ m} \) \( \because M_E=0 \)

The complete I.L. for \( M_E \) is shown in Fig. 2'10 (f).

For maximum \( M_E \), let the tail of the U.D.L. be at \( x \) from \( D \).

The corresponding area of I.L. diagram is shown shaded. Using criterion of equation 2'6, we have

\[
\frac{AD}{DB} = \frac{A_D}{DB_1}
\]

or

\[
\frac{3}{4} = \frac{x}{3-x}, \text{ from which } x = \frac{9}{7} \text{ m}
\]

\[
\therefore \text{Ordinate } aA = bb_1 = \frac{3}{4} \left(3 - \frac{9}{7}\right) = 2'06
\]

\[
M_E = \frac{2}{2} (2'06 + 3'6)(3) = +16'98 \text{ kN-m}
\]

Example 2'7. Draw dimensioned influence lines for the reactions at \( A \) and \( C \) and for the bending moment at \( E \), the mid-point of the lower beam \( CF \) of the simply supported beam system shown in Fig. 2'11. By the use of these influence lines, calculate the greatest value of \( R_A, R_c \) and \( M_E \) due to the passage of two \( 10 \text{ kN} \) rolling loads, 2 m apart which travel across the upper beam \( AB \).

\[
R_A = \frac{6-x}{6} = 1 - \frac{x}{6} \tag{1}
\]

When the load is at \( A \), \( x=0 \) \( \therefore R_A=+1 \)

When the load is at \( D \), \( x=6 \text{ m} \) \( \therefore R_A=0 \).

\[
R_A = (10 \times 1) + \left(10 \times \frac{2}{3}\right) = +50 \frac{2}{3} = +16'67 \text{ kN}
\]

By inspection, maximum \( R_A \) will be obtained when one \( 10 \text{ kN} \) load is at \( A \) and other ahead of it at 2 m.

\[
\therefore \text{Ordinate under next } 10 \text{ load } = \frac{1}{6} \times 4 = \frac{2}{3}
\]

\[
R_A = 10 \times (1 + \frac{2}{3}) = +50 \frac{2}{3} = +16'67 \text{ kN}
\]

\[
R_A = 10 \times \left( 1 + \frac{2}{3} \right) = +50 \frac{2}{3} = +16'67 \text{ kN}
\]

(b) I.L. for reaction at \( C \)

When the load is at a distance \( x \) from \( A \)

\[
R_B = \frac{x}{6} \tag{3}
\]

Thus, for the lower beam \( CE \), the downward load at \( D = \frac{x}{6} \).
\[ R_c = \frac{6RD}{8} = \frac{6x}{8} = \frac{x}{8} \]  
(4)

When the load is at \( A, x = 0 \), \( R_c = 0 \)
When the load is at \( B, x = 8 \) m, \( R_c = +1 \).

The I.L. for \( R_c \) is shown in Fig. 2'11(c).

By inspection, maximum \( R_c \) will be obtained when one 10 kN load is at \( B \), and other behind it by 2 m (i.e. at \( D \)).

The ordinate under \( D = \frac{1}{8} \times 6 = + \frac{3}{4} \)

\[ R_c = (10 \times 1) + \left( \frac{10 \times 3}{4} \right) = +17.5 \text{ kN}. \]

(c) I.L. for B.M. at \( E \)
When the load is at a distance \( x \) from \( A \),

\[ R_D = \frac{x}{6} \]  
(3)

Thus, for the lower beam \( CF \), the downward load at \( D = \frac{x}{6} \).

\[ R_F = \frac{2RD}{8} = \frac{2}{8} \times \frac{x}{6} = + \frac{x}{24} \]

\[ M_E = R_F \times 4 = - \frac{4x}{24} = - \frac{x}{6} \]  
(4)

When the load is at \( A, x = 0 \), \( M_E = 0 \)
When the load is at \( B, x = 8 \) m, \( M_E = - \frac{8}{6} = - \frac{4}{3} \)

The I.L. for \( M_E \) is shown in Fig. 2'11(d).

By inspection, maximum B.M. at \( E \) will occur when one point load is at \( B \), and other 2 m behind it (i.e. at \( D \)). In this position, ordinate under \( D = \frac{4}{3} \times \frac{1}{8} \times 6 = 1 \)

\[ M_E = - \left\{ \left( \frac{10 \times 4}{3} \right) + (10 + 1) \right\} = -23.33 \text{ kN-m}. \]

PROBLEMS

1. Draw the influence lines for S.F. and B.M. at a section 5 m from one end of a simply supported beam, 25 m long. Hence calculate the maximum B.M. and S.F. at this section due to a uniformly rolling load of 8 m long and of intensity 1 kN/m.

2. A beam has a span of 20 m. Draw the I.L. for B.M. and S.F. for a section 8 m from the left hand support and determine the maximum B.M. and S.F. for this section due to two point loads of 8 and 4 kN at a fixed distance of 2 m apart rolling from left to right with either of the loads leading.

3. When the load is at \( A, x = 0 \), \( R_c = 0 \)
When the load is at \( B, x = 8 \) m, \( R_c = +1 \).

The I.L. for \( R_c \) is shown in Fig. 2'11(c).

By inspection, maximum \( R_c \) will be obtained when one 10 kN load is at \( B \), and other behind it by 2 m (i.e. at \( D \)).

The ordinate under \( D = \frac{1}{8} \times 6 = + \frac{3}{4} \)

\[ R_c = (10 \times 1) + \left( \frac{10 \times 3}{4} \right) = +17.5 \text{ kN}. \]

(c) I.L. for B.M. at \( E \)
When the load is at a distance \( x \) from \( A \),

\[ R_D = \frac{x}{6} \]  
(3)

Thus, for the lower beam \( CF \), the downward load at \( D = \frac{x}{6} \).

\[ R_F = \frac{2RD}{8} = \frac{2}{8} \times \frac{x}{6} = + \frac{x}{24} \]

\[ M_E = R_F \times 4 = - \frac{4x}{24} = - \frac{x}{6} \]  
(4)

When the load is at \( A, x = 0 \), \( M_E = 0 \)
When the load is at \( B, x = 8 \) m, \( M_E = - \frac{8}{6} = - \frac{4}{3} \)

The I.L. for \( M_E \) is shown in Fig. 2'11(d).

By inspection, maximum B.M. at \( E \) will occur when one point load is at \( B \), and other 2 m behind it (i.e. at \( D \)). In this position, ordinate under \( D = \frac{4}{3} \times \frac{1}{8} \times 6 = 1 \)

\[ M_E = - \left\{ \left( \frac{10 \times 4}{3} \right) + (10 + 1) \right\} = -23.33 \text{ kN-m}. \]

4. Draw the influence lines for S.F. and B.M. at a section 5 m from one end of a simply supported beam, 25 m long. Hence calculate the maximum B.M. and S.F. at this section due to a uniformly rolling load of 8 m long and of intensity 1 kN/m.

5. Draw the influence lines for S.F. and B.M. at a section 8 m from the left hand support and determine the maximum B.M. and S.F. for this section due to two point loads of 8 and 4 kN at a fixed distance of 2 m apart rolling from left to right with either of the loads leading.

6. A girder \( AB \) of length 30 m is simply-supported at \( C \) and \( D \) which are 5 and 20 m respectively from \( A \). Draw the influence lines for B.M. and S.F. for the mid point of the girder and obtain the maximum B.M. and S.F. at this point when the girder is crossed by a uniformly distributed load \( w \) kN/metre which can occupy the whole or any part or parts of the span.

7. Draw the influence lines for the reactions at \( A, B \) and \( C \), and for the bending moment at \( B \) in the structure shown in Fig. 2'12. Calculate the maximum value of each reaction and of the bending moment at \( B \) when a long uniformly distributed load of intensity \( w \) per/metre crosses from \( A \) to \( C \). The beam is hinged at mid-point of \( BC \).

8. One span of a road bridge \( ABCD \) consists of two cantilevers projecting from abutments \( A \) and \( D \), and carrying a suspended span \( BC \) between them. \( AB = CD = 3L \); \( BC = 4L \). Draw the influence lines for : 
(a) B.M. at \( A \) and at centre of \( BC \).
(b) S.F. at \( B \) and at \( D \).

9. Draw the general type of influence line for shear force at a point on a simply supported beam of span \( L \), and deduce therefrom, giving a figured sketch, the diagram of maximum shear force of both signs due to the passage
over the span of a uniformly distributed load of intensity \( w \) and length \( \frac{L}{5} \).

If the beam also carries a dead load, uniformly distributed over the span of intensity \( \frac{w}{2} \), indicate on the diagram the length of beam for which there is reversal of shearing force. (U.L.)

Answers

1. S.F. 5.12 kN; H.M. 26.9 kN.
2. (a) 6.8 kN
   (b) 54.4 kN m.
3. (a) 91.25 kN-m at 11.94 m from left.
   (b) 74.75 kN-m at 14.19 m from the left.
   (c) Both moments equal at 18.75 m from left; value = 61.5 kN-m.
4. 424 kN-m.
5. 216.7 kN-m; -21.25 kN.
6. \( F_{max} = + \frac{20}{3} w; - \frac{5w}{3} \) kN.

\[ M_{max} = 37.5 \text{ kN-m} \]

7. \( P_A = 46.1 \text{ kN} \) (tension).
8. \( R_A = 5w \); \( R_B = 15w \); \( R_C = 2.5w \); \( M_B = 25w \) (hoggling).
9. Focal length = 6.2286 L.

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### 3.1. INTRODUCTION

We have seen in the previous chapter that an influence line for any given point or section of a structure is a curve whose ordinates...
represent to scale the variation of a function, such as shear force, bending moment, deflection etc., at a point or section, as a unit load moves across the structure. In the case of a beam, the unit load actually crosses each and every section and hence the influence line ordinate changes from point to point, as the unit load moves. However, in nearly all framed structures and girders with floor beams, the loads (whether knife edge load or uniformly distributed load) are applied at the nodes or joints only, so that we have the case of the loads applied at definite points [Fig. 3'1(a), (b)]. Due to this, the S.F. and B.M. for a panel, between the nodal points, are constant. Hence influence lines for S.F. (or B.M.) are plotted for a panel and not for a particular section of a girder.

3'2. Influence Line of S.F. for Girder with floor beams

Fig. 3'2(a), (b) shows such a system in which the loads are transmitted to the girder at definite points. For a given position of unit load, the S.F. for the whole of the panel, situated between the nodal points, is constant. Hence I.L. is plotted for S.F. of a panel and not for S.F. at a particular section of the girder.

Let such a girder or frame consist of *n* panels, each of length *d* such that the total length *L* = *nd*.

Let us plot the I.L. for S.F. in a panel CD [Fig. 3'2(b)]. Let there be *m* panels to the left of CD, and (*n* - *m* - 1) panels to its right. Thus CD is (*m* + 1)th panel from the left support.

(a) Load in AC
When the load is at *A*, *R* = 0, \( \therefore F_{CD} = 0 \)

When the load is at *C*, \( R = \frac{m}{n} \)

\[ F_{CD} = +R = +\frac{m}{n} \]

(b) Load in DG
Now, let the load be in portion DG.

When the load is at *D*, \( R = \frac{(n-m-1)d}{nd} = \frac{n-m-1}{n} \)

\[ F_{CD} = -R = -\frac{n-m-1}{n} \]

(c) Load in CD
When the load is at a distance *x* from *C*, load transmitted at the panel point \( C = \frac{d-x}{d} \), and load transmitted at the panel point \( D = \frac{x}{d} \).

\[ F_{CD} = +R = -\frac{x}{d} = \frac{md+x}{d} - \frac{x}{d} \]

The variation is linear.

When the load is at *C*, *x* = 0, \( \therefore F_{CD} = +\frac{m}{n} \), as before

When the load is at *D*, *x* = *d*.

\[ F_{CD} = \frac{md+d}{nd} - \frac{d}{d} = -\frac{n-m-1}{n} \]

Thus the ordinate \( CC = +\frac{m}{n} \)

ordinate \( dd = -\frac{n-m-1}{n} \)

The ordinate is zero at some point *O* between *C* and *D*. The I.L. for *Fcd* is shown in Fig. 3'2(c).
LOAD POSITIONS FOR MAXIMUM S.F.

Let us now determine the load positions for maximum S.F. in panel CD.

1. **Single Point Load**

   Maximum +ve S.F. will occur when the point load is at C and maximum -ve shear will occur when the load is at D.

   \[ F_{CD} (+ve \max.) = \frac{Wm}{n} \quad [3'1(a)] \]
   \[ F_{CD} (-ve \max.) = \frac{W(n-m-1)}{n} \quad [3'1(b)] \]

2. **U.D.L. greater than the Span**

   Maximum +ve S.F. will occur when \( ao \) is fully loaded and \( og \) is empty, and maximum -ve S.F. will occur when \( og \) is fully loaded and \( ao \) is empty, \( o \) being the point of zero ordinate of the I.L. for \( F_{CD} \). The position of the point \( o \) can very easily be located by the consideration of the triangles \( cc_1o \) and \( dd_1o \). Thus,

   \[ \frac{cc_1}{dd_1} = \frac{cd}{do} = \frac{co}{cd-co} \]

   Since \( cc_1, dd_1, \) and \( cd \) are known, \( co \) can be calculated, and hence \( a \) can be located.

3. **U.D.L. shorter than the span**

   Let a U.D.L. of length \( a \) travel from left to right such that \( a < Ao \), and also \( a < Og \). For getting maximum positive shear, the shaded area of the +ve portion of the I.L. should be maximum. For this, the ordinate \( pp_1 \) should be equal to ordinate \( qq_1 \). Hence applying the criterion of equation 2'6,

   \[ AC = \frac{PC}{\alpha} = \frac{PC}{\alpha - \varphi} \]

Since \( AC, CO \) and \( a \) are known, \( PC \) can be computed, and then the shaded area can be known.

Similarly, for maximum negative shear, the ordinate \( rr_1 \) should be equal to \( ss_1 \). Hence, from equation of criterion 2'6,

\[ \frac{OD}{RD} = \frac{RD}{DG} = \frac{a}{a - RD} \]

Knowing \( OD, DG \) and \( a, RD \) can be computed, and the shaded area can be known.

3. **Irregular Load System**

   Let a train of wheel loads travel from left to right. Let the arrangement of the load [Fig. 3'2(d)] be such that \( W_1 \) is the resultant of the load to the right of \( CD, W_2 \) is the resultant of the load to the left of \( CD, \) and \( W_3 \) is the resultant of the load on the panel CD itself. This arrangement will give maximum \( F_{CD} \) only if a small movement of the load system decreases the S.F.

   Inclination \( \theta \) of \( eC_1 \) or \( gd_1 \) is given by

   \[ \tan \theta = \frac{m/n}{md} = \frac{1}{n} \]

   Inclination \( \phi \) of \( e_1d_1 \) is given by

   \[ \tan \phi = \frac{cc_1+dd_1}{cd} = \frac{(ac+dg)\tan \theta}{cd} = \frac{L-d}{d} \cdot \frac{1}{L} \]

   By giving a small movement \( 8x \) to the right, the ordinates \( y_1 \) and \( y_2 \) are increased and ordinate \( y_1 \) is decreased. However, the decrease of \( y_1 \) increases the +ve S.F. Hence, the change in S.F. is given by

   \[ \frac{\delta F_{CD}}{dx} = (W_3+W_1) \tan\theta \cdot W_2 \tan \phi \]

   \[ \therefore \frac{\delta F_{CD}}{dx} = \frac{W_2}{L} \cdot \frac{W_3(L-d)}{L} = \frac{W_2+W_3}{L} \cdot \frac{W_3}{L} \]

   (where \( W \) is the total load).

   Hence, the maximum, \( \frac{W_2}{L} - \frac{W_3}{n} \) (or \( \frac{W_2}{n} - \frac{W_3}{n} \)) should change sign. In the limiting case, when the loads are very near, \( W_2 = \frac{W}{n} \)

   Hence the maximum S.F. in a panel occurs when the load in that panel is equal to the load divided by the number of panels.
3'4. INFLUENCE LINE OF B.M. FOR GIRDER WITH FLOOR BEAMS

Let us now draw the I.L. for bending moment at a point P, distant \( x \) from \( A \), in the panel \( CD \).

When the unit load is in \( AC \),
\[
M_P = -R_0(L-x)
\]  
(1)

Again when the unit load is in \( DG \),
\[
M_P = -R_d \cdot x
\]  
(2)

Both these variations are same as for a girder without floor beam. It must be remembered that \( x \) is a fixed quantity in the above equation.

When the load is at \( C \),
\[
R_0 = \frac{md}{nd} = \frac{m}{n}
\]

\[
\therefore \quad M_P = \text{Ordinate } cc_1 = \frac{m}{n}(L-x)
\]  
(3)

When the load is at \( D \),
\[
R_d = \frac{(n-m-1)d}{nd} = \frac{n-m-1}{n}
\]

\[
\therefore \quad M_P = \text{Ordinate } dd_1 = \frac{n-m-1}{n} \cdot \frac{x}{m}
\]  
(4)

If the girder were without the floor beam, the ordinate \( pp_1 \) under the section would have been \( \frac{x}{L} (L-x) \), and the corresponding ordinates \( cc_1 \) and \( dd_1 \) would have been
\[
cc_1 = \frac{x}{L} (L-x) \times \frac{1}{x} \times md = \frac{m}{n} (L-x)
\]

which is the same as found above in Eq. 3

and
\[
dd_1 = \frac{x}{L} (L-x) \times \frac{1}{x} \times (n-m-1) \cdot \frac{d}{n} = \frac{n-m-1}{n} \cdot \frac{x}{m}
\]

which is the same as found above in Eq. 4.

Hence the portion \( cc_1 \) and \( dd_1 \) of I.L. for a girder with floor beams can be obtained by constructing the I.L. for the beam assuming it to be without floor beam, making the central ordinate \( pp_1 = \frac{x}{L} (L-x) \) joining \( p_1 \) to \( a \) and \( b \), as shown in Fig. 3'4 (b).

To plot the portion of I.L. diagram under the bay \( CD \), consider the unit load at a distance \( a \) from \( C \). The panel point load transferred to \( C \) and \( D \) will be \( \frac{d-a}{d} \) and \( \frac{a}{d} \) respectively.

\[
M_P = cc_1 \left( \frac{d-a}{d} \right) + dd_1 \left( \frac{a}{d} \right)
\]

This is a linear function of \( a \). When the load is at \( C \), \( a=0 \), and hence \( M_P = cc_1 \left( \frac{d}{d} \right) = cc_1 \). Similarly, when the load is at \( D \), \( a=d \), and \( M_P = dd_1 \left( \frac{d}{d} \right) = dd_1 \). Hence the I.L. portion under panel \( CD \) is obtained by joining \( c_1 \) and \( d_1 \) by a straight line. The figure \( ac_1d_1b \) is thus the complete I.L. diagram for B.M. at point \( P \) in the panel \( CD \).

The I.L. for the point \( P \) in any other panel can also be found in a similar manner. However, when the point \( P \) coincides with some panel or node point, such as \( C \), the I.L. diagram will be a triangle. Fig. 3'4 (d) shows the I.L. for B.M. at \( C \). The ordinate \( cc_1 \) under \( C \) [Fig. 3'4 (d)] = \( \frac{md(n-m)d}{nd} = \frac{m}{n} (n-m)d \). When \( n=6 \) and \( m=2 \),

\[
cc_1 = \frac{2}{6} (6-2) d = \frac{4}{3} d
\]

3'5. LOAD POSITIONS FOR MAXIMUM B.M.

1. Single Point Load

From the inspection of the I.L. for \( M_P \), it is clear that \( M_P_{\text{max}} \) will be obtained by putting the load either at \( C \) or at \( D \), depending upon whether ordinate \( cc_1 \) is bigger or \( dd_1 \) is bigger.

Thus
\[
M_{P_{\text{max}}} = \frac{Wm}{n} (L-x)
\]  
3'4 (a)
or

\[ \sigma = \frac{W(n-m-1)}{n}x. \]

2. U.D.L. longer than the Span

Maximum B.M. will evidently occur when the load occupies the whole span. In that case, \( M_p = W \times \text{shaded area of I.L. diagram.} \)

3. Irregular Load System

Let \( W_1 \) be the resultant of the loads to the right of \( D \), \( W_2 \) the resultant of the loads to the left of \( C \), and \( W_3 \) the resultant of the loads on the panel \( CD \). Let the section \( P \) be at a distance \( b \) from \( C \) (such that \( b = x-md \)). Then, it can be proved that the maximum B.M. at \( P \) occurs when the expression

\[ \frac{W(L-x)}{L} - \left\{ \frac{W_2 (d-b)}{d} + W_1 \right\} \text{ changes sign.} \]

As a rule, \( b = \frac{d}{2} \), so that the above criterion reduces to

\[ \frac{W(L-x)}{L} - \left( \frac{W_2}{2} + W_1 \right) \text{ changes sign} \quad \ldots (3'5) \]

Hence, to get the maximum value of \( M_p \), the procedure is as follows: Place the load on the span such that the span is fully covered (if the load system is long) and with one load at \( D \). First consider this load as part of \( W_3 \) and then as part of \( W_1 \). If this causes the expression of equation \( 3'5 \) to change sign, the position is the maximum required. If not, move the load on until another load comes at \( C \) or \( D \), and apply the above criterion again.

Example 3.1. A Pratt girder shown in Fig. 3'5 consists of eight panels each 3'5 m square, the loading being on the lower boom. Draw the influence line for the force in the member \( EC \) and determine the maximum tension and maximum compression in \( EC \) due to

(a) a concentrated rolling load of 20 kN,
(b) a uniform live load of 10 kN/m and 10 m long. Indicate clearly for each of the four required values the corresponding load positions.

Solution.

If we pass a section 1-1, it is clear that force in \( EC \) is equal to shear in \( BC \times \sec 45^\circ = F_{BC} \times \sqrt{2} \).

Hence \( P_{EC} = F_{BC} \sqrt{2} \).

Also, when the load is in \( ao \), S.F. in panel \( BC \) will be positive, and hence force in \( EC \) will be compressive. Similarly, when the load is in \( od \), S.F. in panel \( BC \) will be negative, and hence force in \( EC \) will be tensile.
STRENGTH OF MATERIALS AND MECHANICS OF STRUCTURES

Then \[ F_{bc} (\text{+ve max.}) = w \times \text{Area } ob \]
\[ = 10 \times \frac{1}{2} \times 8 \times \frac{1}{4} = 10 \text{ kN} \]
\[ \therefore \quad P_{EC} = F_{bc} \sqrt{2} = 10\sqrt{2} \text{ kN (compressive)} \]

For maximum negative shear, the load should be so arranged that the area of the I.L. diagram under it (shown dotted) is maximum. This will happen when ordinate \( pp = qq \) (i.e., point \( c \) divides the load \( pq \) in the same ratio as it divides the base \( \text{od of the triangle } oc, d \)). Hence applying the criterion of equation 2.6, we have

\[ \frac{oc}{cd} = \frac{pc}{eq} = \frac{pe}{10-pc} \]
or,
\[ \frac{2.5}{20-2.5} = \frac{pc}{10-pc} \]

From which
\[ pe = 1.25 \text{ m} \]
Also,
\[ op = oc - pe = 2.5 - 1.25 = 1.25 \text{ m} \]

Hence
\[ qq = pp = 5 \times \frac{1}{8} \times 2.5 \times 1.25 = \frac{5}{16} \]

Hence \( F_{bc} (\text{-ve max.}) = \frac{1}{2} \left( \frac{5}{8} + \frac{5}{16} \right) \times 10 \times 10 = 46.88 \text{ kN} \)
\[ \therefore \quad P_{EC} = F_{bc} \sqrt{2} = 46.88 \sqrt{2} = 66.2 \text{ kN (tensile)} \]

PROBLEMS

1. A N-girder bridge (Fig. 3.6) has cross-girders at the lower panel points. The diagonals are at 45°. A live load of 6 kN/m (per girder), longer than the span, cross the bridge. Find the maximum forces in the three members \( AB, AD \) and \( CD \).

![Fig. 3.6](image)

Answers

1. \( P_{AD} = 126 \text{ kN (com.)} \); \( P_{CD} = 96 \text{ kN (tensile)} \)
\( P_{AD} = 46.1 \text{ kN (tension)} \).

Influence Lines for Stresses in Frames

4.1. PRATT TRUSS WITH PARALLEL CHORDS

![Pratt truss with parallel chords](image)
Fig. 41 shows a Pratt truss with 6 panels, each of length 4 m and of height 5 m. Let us draw the influence lines for stresses in members of panels HI and II. The truss is statically determinate.

\[
\sin \theta = \frac{5}{\sqrt{4^2 + 5^2}} = \frac{5}{\sqrt{41}} = 0.78
\]
\[
\cos \theta = \frac{4}{\sqrt{41}} = 0.625; \csc \theta = 1.28
\]

(1) **Influence line for \( P_{BC} \)**

In order to find stress \( P_{BC} \) in member \( BC \), pass a section \( aa \) as shown. Evidently:

\[
P_{BC} = \frac{M_1}{5} \text{ (compression)}
\]

where \( M_1 \) = bending moment at joint \( I \).

The influence line for bending moment at chord point \( I \) will be a triangle having a maximum ordinate equal to \( \frac{8 \times 16}{24} = \frac{16}{3} \). Hence influence line for \( P_{BC} \) will also be a triangle having a maximum ordinate of \( \frac{1}{5} \times \frac{16}{3} = \frac{16}{15} = 1.067 \) under \( I \) as shown in Fig. 41 (b).

The minus sign indicates compression.

(2) **Influence line for \( P_{CI} \)**

\( P_{CI} \) = Shear in panel \( IJ \)

Thus the I.L. for \( P_{CI} \) is the same as the I.L. for shear in the panel \( IJ \). From § 3.2, the ordinates of the I.L. diagram are as under:

*Under point \( I \) ordinate \( = \frac{8}{24} = \frac{1}{3} \)
*Under point \( J \) ordinate \( = \frac{12}{24} = \frac{1}{2} \)

The I.L. diagram for \( P_{CI} \) is shown in Fig. 41 (c). When the load traverses the panel \( IJ \) there is reversal of shear in member \( CI \).

When the load is to the left of \( I \), the S.F. is positive and the force in \( CI \) is tensile, while when the load is to the right of \( J \), the S.F. is negative and the force in \( CI \) is compressive. Thus a positive S.F. gives tension while a negative S.F. gives compression.

(3) **Influence line for \( P_{HI} \)**

Pass a section \( bb \),

\[
P_{HI} = \frac{M_B}{5} \text{ (tension)}
\]

**Influence Lines for Stresses in Frames**

Hence the I.L. for \( P_{HI} \) will be a triangle, having an ordinate of \( \frac{4 \times 20}{24} \times \frac{1}{5} = \frac{2}{3} \) under \( H \), as shown in Fig. 41 (c).

(4) **Influence line for \( P_{BH} \)**

When the unit load is at \( A \), \( P_{BH} = 0 \).

When the unit load is at \( H \), \( P_{BH} = \frac{1}{3} \text{ (tension)} \).

When the load is at \( I \) or to the right of \( I \), \( P_{BH} = 0 \).

The influence line for \( P_{BH} \) will therefore be a triangle having a maximum ordinate of unity under \( H \), as shown in Fig. 41 (c).

(5) **Influence line for \( P_{AB} \)**

The load in \( AB \) can be found by resolution of the forces at \( A \) in the vertical direction.

When the unit load is at \( A \), \( R_A = 1 \), and hence \( P_{AB} = 0 \). When the unit load is at \( H \), \( R_A = 20 \times \frac{5}{24} = \frac{5}{6} \), and

\[
P_{AB} = R_A \cos \theta = \frac{5}{6} \times 1.28 = 1.07 \text{ (comp.)}
\]

When the load is at \( G \), \( R_A = 0 \). \( \therefore P_{AB} = 0 \)

The I.L. for \( P_{AB} \) is shown in Fig. 41 (f).

**4.2. Pratt Truss with Inclined Chords**

Fig. 42 (a) shows Pratt truss with inclined chords, consisting of 6 panels each of 4 m length.

(1) **Influence line for \( P_{HI} \)**

Pass a section \( aa \) cutting three members

\[
P_{HI} = \frac{M_B}{BH} = \frac{1}{3} \text{ (tension)}
\]

The influence line diagram will therefore be a triangle having a maximum ordinate \( = \frac{1}{3} \left( \frac{4 \times 20}{24} \right) = \frac{10}{9} = 1.111 \) under \( H \) as shown in Fig. 42 (b).

(2) **Influence line for \( P_{BC} \)**

\[
P_{BC} = \frac{M_I}{x} \text{ (compression)}
\]

where \( x \) is the perpendicular distance between point \( I \) and \( BC \). In order to find \( x \), prolong \( CB \) back to meet \( IA \) produced, in \( O \).

\[
\tan \alpha = \frac{5 - \frac{1}{2}}{\frac{3}{2}} = \frac{9}{2} \quad \therefore \alpha = 26^\circ 34' \]

\[
\sin \alpha = 0.447; \cos \alpha = 0.894
\]

\[
OH = \frac{3}{\tan \alpha} = \frac{3}{\frac{3}{5}} = 6 \text{ m}
\]
Now \( \theta = \sin \alpha = 10 \times 0.447 = 4.47 \) m

For Fig. 4.2

Pratt truss with inclined chords.

The influence line diagram will be a triangle having a maximum ordinate \( = \frac{1}{\frac{4.47}{24}} (\frac{8 \times 16}{24}) = 1.93 \), as shown in Fig. 4.2 (c).

(3) Influence line for \( P_{BI} \)

\[
P_{BI} = \frac{M_0}{y'}
\]

where \( y' \) = perpendicular distance of point \( O \) from \( BI \)

\[
= OI \sin \theta = 10 \times \frac{3}{4} = 6 \text{ m}
\]

\[
\therefore \quad P_{BI} = \frac{M_0}{6}
\]

where \( M_0 \) is the moment about \( O \), of the forces to the left of section \( aa \).

When the unit load is at \( A \), \( R_A = 1 \). Hence considering the forces to the left of the section \( aa \), \( M_0 = 0 \). Hence \( P_{BI} = 0 \). When the unit load is at \( H \), \( R_A = \frac{20}{24} = \frac{5}{6} \).

\[
\therefore \quad P_{BI} = \frac{M_0}{6} = \frac{1}{6} \left( (1 \times OI) - (R_A \times AO) \right) \text{ (comp.)}
\]

\[
= \frac{1}{6} \left( (1 \times 6) - \left[ \frac{5}{6} \times 2 \right] \right) = 0.722 \text{ (comp.)}
\]

When the unit load is at \( I \), \( R_A = \frac{16}{24} = \frac{2}{3} \)

\[
\therefore \quad P_{BI} = \frac{M_0}{6} = \frac{1}{6} \left( R_A \times OA \right) = (tension)
\]

\[
= \frac{1}{6} \left( \frac{2}{3} \times 2 \right) = \frac{2}{9} = 0.222 \text{ (tension)}
\]

Thus, there is a reversal of stress in \( BI \) as the load traverses the panel \( HI \).

When the load is \( G \), \( R_A = 0 \); Hence \( M_0 \) and \( P_{BI} \) are zero.

The complete I.L. diagram for \( P_{BI} \) is shown in Fig. 4.2 (d).

(4) Influence line for \( P_{CI} \)

Pass a section \( bb \) to cut the three members \( BC \), \( CI \) and \( JL \).

Since \( BC \) and \( JL \) meet at \( O \), when produced, we have

\[
P_{CI} = \frac{M_0}{OI} = \frac{M_0}{10}
\]

where \( M_0 \) is the moment, about \( O \), of all forces to the left of section \( bb \).

When the unit load is at \( A \), \( R_A = 1 \).

\[
\therefore \quad M_0 = 0 \text{ and hence } P_{CI} = 0
\]

When the unit load is between \( A \) and \( I \), \( R_A \) is less than unity, and hence the net moment \( M_0 \) is clockwise. Hence \( P_{CI} \) will give an anti-clockwise moment, giving tensile force in it.

When the load is at \( I \), \( R_A = \frac{16}{24} = \frac{2}{3} \)

\[
\therefore \quad P_{CI} = \frac{M_0}{10} = \frac{1}{10} \left( (1 \times OI) - (R_A \times OA) \right) \text{ (tension)}
\]

\[
= \frac{1}{10} \left( (1 \times 10) - \left( \frac{2}{3} \times 2 \right) \right) = 0.867 \text{ (tension)}
\]

When the unit load is at \( J \), \( R_A = \frac{4}{4} \).
\[ P_{CI} = \frac{M_a}{10} = \frac{1}{10} \left( R_d \times OA \right) \text{ (compression)} \]

\[ = \frac{1}{10} \times \frac{4}{2} \times 2 = 0.1 \text{ (compression)} \]

Thus, there is reversal of stress in CI as the unit load traverses the span IJ.

When the unit load is at \( G \), \( R_d = 0 \) and hence \( M_a \) and \( P_{CI} \) are zero. The I.L. for \( P_{CI} \) is shown in Fig. 4.2(e).

### 4.3. WARREN TRUSS WITH INCLINED CHORDS

![Warren Truss with Inclined Chords Diagram](image)

Fig. 4.3 shows a Warren girder with inclined chords. There are six panels each of 4 m span.

(1) **Influence line for \( P_{BC} \)**

Pass a section \( aa \) to cut members \( BC, CI \) and \( IJ \)

\[ P_{BC} = \frac{M_t}{x} \text{ (compression)} \]

\[ x = \text{perpendicular distance of } I \text{ from } BC \]

\[ = \text{of } \sin \alpha \]

But \( \tan \alpha = \frac{4-2}{2} = \frac{1}{2} \), \( \therefore \alpha = 26^\circ 34' \); \( \sin \alpha = 0.447 \)

\[ OJ = 2 + \frac{2}{4} = 2 \text{ m} \]

\[ x = OJ \sin \alpha = 6 \times 0.447 = 2.68 \text{ m} \]

\[ P_{BC} = \frac{M_1}{2.68} \text{ (compression)} \]

The influence line for \( P_{BC} \) will be a triangle having a maximum ordinate of \( \frac{4 \times 20}{24} \times \frac{1}{2.68} = 1^2.24 \) under I, as shown in Fig 4.3 (a).

(2) **Influence line for \( P_{IJ} \)**

\[ P_{IJ} = \frac{M_c}{4} \text{ (tension)} \]

When the load is at \( A \), \( R_d = 0 \); hence \( M_c \) and \( P_{IJ} \) are zero.

When the load is at \( I \), \( R_d = 4 \times \frac{1}{24} = \frac{1}{6} \)

\[ \therefore \ P_{IJ} = \frac{M_c}{4} = \frac{1}{4} \left( \frac{1}{6} \times \frac{18}{24} \right) = 0.75 \text{ (tension)} \]

When the load is at \( J \), \( R_d = 1 \times \frac{16}{24} = \frac{2}{3} \)

\[ \therefore \ P_{IJ} = \frac{M_c}{4} = \frac{1}{4} \left( \frac{2}{3} \times \frac{6}{24} \right) = 1 \text{ (tension)} \]

When the load is at \( H \), \( R_d = 1 \) and \( R_d = 0 \)

Hence \( M_c \) and \( P_{IJ} \) are zero.

The influence line diagram for \( P_{IJ} \) has zero ordinates under \( A \) and \( H \) and ordinates of 0.75 and 1.0 under \( I \) and \( J \), as shown in Fig. 4.3 (c).

(3) **Influence line for \( P_{IC} \)**

\[ P_{IC} = \frac{M_0}{r} \]

where \( r = \text{perpendicular distance of } O \text{ from } CI \)

\[ = \text{of } \sin \beta \]

But \( \text{of } 6 \text{ m}; \sin \beta = \frac{4}{\sqrt{4^2 + 4^2}} = 0.894 \)

\[ \therefore \ r = 6 \times 0.894 = 5.36 \text{ m} \]

\[ \therefore \ P_{IC} = \frac{M_0}{5.36} \]
When the unit load is at $A$, $R_A=1$; Hence $M_0$ and $P_{IC}$ are zero.

When the unit load is at $I$, $R_A=-\frac{1 \times 20}{24} = -0.833$

\[ P_{IC} = \frac{1}{5 \cdot 36} \left\{ (1 \times 6) - (0.833 \times 2) \right\} \]
\[ = 0.808 \text{ (tension)} \]

When the unit load is $J$, $R_A=\frac{1 \times 16}{24} = 0.667$

\[ P_{IC} = \frac{1}{5 \cdot 36} (0.667 \times 2) = 0.248 \text{ (compression)} \]

Thus, there is reversal of stress $IC$ when the unit load crosses the panel $IC$. The I.L. for $P_{IC}$ is shown in Fig. 4.3(d).

4.4. **K-TRUSS**

(1) **Influence line for $P_{CD}$**

Pass a section $ab$ as shown in Fig. 4.4(a). Considering the equilibrium of the portion to the left of section $aa$ and taking moments about $O$, we get

\[ P_{CD} = \frac{M_0}{6} \text{ (compression)} \]

(2) **Influence line for $P_{ON}$**

\[ P_{ON} = \frac{M_C}{6} \text{ (tension)} \]
The I.L. for \( P_{ON} \) will be a triangle having a maximum ordinate of 
\[
\frac{8 \times 24}{32} \times \frac{1}{6} = 1 \text{ under } O, \text{ as shown in Fig. 4'4(c).
}
(3) **Influence line for \( P_{OD} \) and \( P_{ON} \)

\( QD \) and \( QN \) have the same inclination with the vertical. Hence they will carry equal but opposite stresses. Thus, numerically,

\[
P_{OD} = P_{ON}
\]

Pass a section \( bb \) and consider the equilibrium of the left portion. Resolving vertically,

\[
P_{OD} \sin \theta + P_{QN} \sin \theta = 0 \text{ shear in panel } ON = F_{ON}
\]

When the unit load is at \( A, \) \( R_A = 1, \) and hence shear in panel \( ON \) is zero. Therefore, \( P_{OD} \) and \( P_{ON} \) are zero.

When the unit load is at \( O,
\[
R_A = \frac{1 \times 24}{32} = 0.75
\]

\[
\therefore 2P_{OD} \sin \theta = F_{ON} = 1 - 0.75 = 0.25 \text{ (tension)}
\]
or

\[
P_{OD} = \frac{0.25}{2 \sin \theta} \quad \text{[But } \sin \theta = \frac{3}{5} = 0.6]\n\]

and

\[
P_{ON} = 0.208 \text{ (compression)}
\]

When the unit load is at \( N,
\[
R_A = \frac{1 \times 20}{32} = \frac{5}{8} = 0.625 = F_{ON}
\]

\[
\therefore P_{OD} = \frac{F_{ON}}{2 \sin \theta} = \frac{0.625}{2 \times 0.6} = 0.52 \text{ (comp.)}
\]

and

\[
P_{ON} = 0.52 \text{ (tension)}
\]

When the unit load is at \( I, R_A = 0. \) Hence \( F_{ON} \) is zero. Therefore, \( P_{OD} \) and \( P_{ON} \) are zero.

The I.L. for \( P_{OD} \) and \( P_{ON} \) are shown in Fig. 4'4 (d) and (e) respectively.

(4) **Influence line for \( P_{RN} \)**

Pass a section \( cc \), cutting members \( CD, QD, RN \) and \( NM, \) and consider the equilibrium of the left portion.

(i) When the unit load is at \( O,
\[
R_A = \frac{1 \times 24}{32} = 0.75 \text{ and } P_{OD} = 0.208 \text{ (tension)}
\]

\[
\therefore P_{OD} \sin \theta + P_{RN} = 1 - R_A = 1 - 0.75 = 0.25
\]

\[
\therefore P_{RN} = 0.25 - P_{OD} \sin \theta = 0.25 - (0.208 \times 0.6)
\]

\[
= 0.125 \text{ (tension)}
\]

\[
(i) \text{ When the unit load is at } N,
\]

\[
R_A = \frac{1 \times 20}{32} = 0.625 \text{ and } P_{OD} = 0.52 \text{ (comp.)}
\]

\[
\therefore -P_{OD} \sin \theta + P_{RN} = 1 - R_A = 1 - 0.625 = 0.375
\]

or

\[
P_{RN} = 0.375 + (0.52 \times 0.6) = 0.687 \text{ (tension)}
\]

(ii) When the unit load is at \( F, \)

\[
R_A = \frac{1 \times 16}{32} = 0.5 \text{ and } P_{OD} = 0.416 \text{ (comp.)}
\]

\[
P_{RN} = R_A - P_{OD} \sin \theta = 0.5 - (0.416 \times 0.6)
\]

\[
= 0.25 \text{ (comp.)}
\]

The I.L. for \( P_{RN} \) is shown in Fig. 4'4 (f). (5) **Influence line for \( P_{DR} \)**

Pass a section \( dd, \) cutting members \( DE, DR, QN \) and \( ON. \) Out of these four, stresses in members \( QN \) and \( ON \) are known. Consider the equilibrium of the portion to the left of section \( dd. \)

(i) When the unit load is at \( O,
\[
R_A = 0.75 \text{ and } P_{GN} = 0.208 \text{ (comp.)}
\]

\[
P_{DR} = 1 - R_A - P_{GN} \sin \theta
\]

\[
= 1 - 0.75 - (0.208 \times 0.6)
\]

\[
= 0.125 \text{ (comp.)}
\]

(ii) When the unit load is at \( N
\[
R_A = 0.625 \text{ and } P_{GN} = 0.52 \text{ (tension)}
\]

\[
P_{GN} = R_A - P_{GN} \sin \theta = 0.625 - (0.52 \times 0.5)
\]

\[
= 0.313 \text{ (tension)}
\]

The I.L. for \( P_{GN} \) is shown in Fig. 4'4(g). The influence lines for stresses in other members can similarly be plotted.

4'5. **BALTIMORE TRUSSES WITH SUB-TIES : THROUGH TYPE**

Simple trusses become uneconomical when the span exceeds 80 to 100 m. Earlier, multiple web systems were used in long span bridges. However, they are expensive and highly indeterminate, and are no longer used. The modern trend is to use some form of sub-divided trusses or K-trusses. A sub-divided truss is obtained by placing in every panel of the truss some secondary members of diagonals. In contrast with primary members, which are stressed with all positions of the loads, secondary members are stressed only by the loads in certain limited positions.

Fig. 4'5 (a) shows a Baltimore truss (through type), with sub-ties. The load moves on the lower chords.

(i) **Influence line for \( P_{HT} \) and \( P_{TI} \)**

Pass a section \( aa, \) cutting members \( BC, BN \) and \( HT \)

\[
P_{HT} = P_{TI} = \frac{M_b}{15} \text{ (tension)}
\]
The influence line will be a triangle having a maximum ordinate of $\frac{12 \times 60}{72} \times \frac{1}{15} = \frac{2}{3}$ under $H$, as shown in Fig. 4-5 (b).

(2) **Influence line for $P_{BC}$**

$$P_{BC} = \frac{M_A}{15} \text{ (compression)}$$

When the unit load is at $H$,

$$R_A = \frac{1 \times 60}{72} = \frac{5}{6}$$

$$P_{BC} = \frac{1}{15} \left( \frac{5}{6} \times 24 - 1 \times 12 \right)$$

$$= \frac{1}{15} = 0.0533 \text{ (comp)}$$

When the unit load is at $T$,

$$R_A = \frac{1 \times 54}{72} = 0.75$$

$$P_{BC} = \frac{1}{15} \left( 0.75 \times 24 \right) = 1.2 \text{ (comp.)}$$

When the load is at $G$, $R_A = 0$

$$P_{BC} = 0$$

The influence line for $P_{BC}$ is shown in Fig. 4-5 (c).

(3) **Influence line for $P_{BH}$**

Pass a section $bb$. Consider equilibrium of the left portion

$$P_{BH} = \frac{M_A}{12} \text{ (tension)}$$

$M_A$ and hence $P_{BH}$ are zero when the unit load is at $A$. When the unit load is $H$,

$$P_{BH} = \frac{1}{12} \left( 1 \times 12 \right) = 1 \text{ (tension)}$$

When the unit load is at $T$ or beyond $T$ on right side there is no external force to the left of section $bb$ except $R_A$. Hence $M_A = 0$. Therefore, $P_{BH}$ is zero. The I.L. for $P_{BH}$ is, therefore, a triangle having zero ordinates under $A$ and $T$, and a maximum ordinate of unity under $H$, shown in Fig. 4-5 (d).

(4) **Influence line for $P_{BN}$**

Considering the equilibrium to the left of section $aa$,

$$P_{BN} \sin \theta = \text{shear in panel } HT = F_{HT}$$

$$\sin \theta = \frac{15}{\sqrt{15^2 + 12^2}} = 0.781; \cos \theta = \frac{12}{\sqrt{15^2 + 12^2}} = 0.625$$

When the unit load is at $H$, $R_A = \frac{1 \times 60}{72} = 0.833$

$$F_{HT} = 1 - R_A = 1 - 0.833 = 0.167$$

$$P_{BN} = \frac{F_{HT}}{\sin \theta} = \frac{0.167}{0.781} = 0.214 \text{ (compression)}$$
When the unit load is at $T$, $R_A = \frac{1}{72} \times \frac{54}{1} = 0.75 = F_{HT}$

\[ \therefore P_{BN} = \frac{F_{HT}}{\sin \theta} = \frac{0.75}{0.781} = 0.96 \text{ (tension)} \]

When the unit load is at $A$ or $G$, $F_{HT}$ and $P_{BN}$ are zero. The I.L. for $P_{BN}$ is shown in Fig. 4.5 (e).

(5) **Influence line for $P_{NT}$**

$NT$ is secondary member.

- When the unit load is at $H$ or to the left of $H$, $P_{NT} = 0$
- When the unit load is at $T$, $P_{NT} = 1$ (tension)
- When the unit load is at $I$ or to the right of $I$, $P_{NT} = 0$

The I.L. for $P_{NT}$ is shown in Fig. 4.5 (f).

(6) **Influence line for $P_{NC}$**

$NC$ is a sub-tie, and is thus a secondary member. Pass a horse shoe section $ee$ cutting five members. Out of these, four members pass through $I$ when produced. Hence take the moments about $I$. Consider the equilibrium of the portion enclosed by the horse shoe section $ee$.

- When the unit load is at $H$ or $I$, $M_i = 0$, and hence $P_{NC} = 0$.
- When the unit load is at $T$, we get, by taking moment about point $I$:
  \[ P_{NC} \times (15 \cos \theta) = 1 \times 6 \text{ (tension)} \]
  \[ \therefore P_{NC} = \frac{6}{15 \cos \theta} = \frac{6}{15 \times 0.625} = 0.64 \text{ (tension)} \]

Thus, I.L. for $P_{NC}$ is a triangle, as shown in Fig. 4.5 (g), and is similar to I.L. for $P_{NT}$. The vertical component of $P_{NC}$ is equal to $0.64 \sin \theta = 0.64 \times 0.781 = 0.5$. Hence it is very interesting to note that in general, for both parallel and non-parallel chord trusses, where the secondary has the same slope as the main diagonal, the vertical components of stress in the secondary diagonal will be equal to one half of the load applied at the joint.

(6) **Influence line for $P_{NI}$**

Pass a section $cc$ cutting four members. Consider the equilibrium of the portion to the left of section $cc$.

Resolving forces vertically:
\[ P_{NC} \sin \theta + P_{NI} \sin \theta = \text{S.F. in panel } TI = F_{TI} \]
\[ P_{NC} + P_{NI} = F_{TI} \cos \theta \]

The member $NC$ will have stress only when the load is in span $HI$.

- (i) When the unit load is at $P$, $P_{NC} = 0$ and $R_A = 0.833$. 

**Influence lines for stresses in frames**

\[ F_{TI} = 1 - 0.833 = 0.167; \cos \theta = \frac{F_{TI}}{0.781} \]
\[ P_{NI} = \frac{0.167}{0.781} = 0.214 \text{ (compression)} \]

(ii) When the unit load is at $T$, $P_{NC} = 0.64$ (tension) and $R_A = 0.75$

\[ P_{NI} = (0 - 0.74 - 0) \cos \theta = 0.25 x 0.64 = 0.16 \text{ (tension)} \]

(iii) When the unit load is at $I$, $P_{NC} = 0$ and $R_A = 0.667$

\[ P_{NI} = 0.667 \cos \theta = \frac{0.667}{0.781} = 0.854 \text{ (tension)} \]

The I.L. for $P_{NI}$ is shown in Fig. 4.5 (h).

(7) **Influence line for $P_{CI}$**

Pass a section $dd$ cutting four sections. Consider equilibrium of left portion. Resolving the forces vertically,

\[ P_{CI} + P_{NC} \sin \theta = \text{shear in panel } IU = F_{IU} \]

Member $NC$ has stress when the load is in panel $HI$ only.

- (i) When the unit load is at $H$, $R_A = 0.833$ and $P_{NC} = 0$
  \[ P_{CI} = F_{IU} = (1 - 0.833) = 0.167 \text{ (tension)} \]

- (ii) When the unit load is at $T$, $R_A = 0.75$
  \[ P_{NC} = 0.64 \text{ (tension)} \]

\[ P_{CI} = (0.75 - 0.1) + P_{NC} \sin \theta = -0.25 + (0.64 \times 0.781) \]
\[ = 0.25 \text{ (compression)} \]

- (iii) When the unit load is at $I$, $R_A = 0.667$ and $P_{NC} = 0$
  \[ P_{CI} = (1 - 0.667) = 0.333 \text{ (tension)} \]

- (iv) When the unit load is at $U$, $R_A = 0.583$ and $P_{NC} = 0$
  \[ P_{CI} = 0.583 \text{ (compression)} \]

The I.L. for $P_{CI}$ is shown in Fig. 4.5 (i).

4.6. **Baltimore Truss with Sub-ties: Deck Type**

Fig. 4.6 (a) show a deck type Baltimore truss with sub-ties. The load moves on the top chord members.

(1) **Influence line for $P_{NM}$**

Pass a section $aa$ to cut three members.

\[ P_{NM} = \frac{M_c}{15} \text{ (tension)} \]

The influence line will be triangle having a maximum ordinate of $\frac{1}{15} \left( \frac{12 \times 65}{72} \right) = 2 \frac{2}{3} = 0.667$ under C, as shown in Fig. 4.6 (b).
(2) Influence line for $P_{cd}$

\[ P_{cd} = \frac{M_m}{15} \text{ (compression)} \]

When the unit load is at $C$, $R_A = \frac{1 \times 60}{72} = \frac{5}{6} = 0.833$

\[ P_{cd} = \frac{1}{15} \left( \frac{5}{6} \times 24 - 1 \times 12 \right) = -\frac{8}{15} = 0.533 \text{ (compression)} \]

When the unit load is at $Q$, $R_A = \frac{1 \times 4}{72} = \frac{3}{4}$

\[ P_{cd} = \frac{1}{15} \left( \frac{3}{4} \times 24 \right) = \frac{6}{5} = 1.2 \text{ (compression)} \]

The I.L. for $P_{cd}$ is shown in Fig. 4-6(c).

(3) Influence line for $P_{cr}$
Consider equilibrium of the portion to the left of $aa$ and resolve the forces vertically. Then

\[ P_{cr} \sin \theta = \text{shear in panel } CQ = F_{cq} \]

\[ P_{cr} = F_{cq} \csc \theta \]

But $\sin \theta = \frac{15}{\sqrt{15^2 + 12^2}} = 0.781$; $\cos \theta = 0.625$

\[ \csc \theta = 1.28 \]

\[ P_{cr} = 1.28 F_{cq} \]

(i) When the unit load is at $C$, $R_A = 5/6 = 0.833$

\[ F_{cq} = 1 - 0.833 = 0.167 \]

\[ P_{cr} = 1.28 \times 0.167 = 0.214 \text{ (compression)} \]

(ii) When the unit load is at $Q$, $R_A = \frac{2}{7} = 0.75$ and $F_{cq} = 0.75$

\[ P_{cr} = 1.28 \times 0.75 = 0.96 \text{ (tension)} \]

Let I.L. for $P_{cr}$ is shown in Fig. 4-6(d).

(4) Influence line for $P_{qr}$

$QR$ is a secondary member and hence it will be stressed when the load is in panel $CD$. The influence line for $P_{qr}$ is shown in Fig. 4-6(e), having zero ordinate under $C$ and $D$, and an ordinate of unity under $Q$.

(5) Influence line for $P_{rd}$

$RD$ is a secondary member, and hence it will be stressed only when the load is in panel $CD$. Pass a horse shoe section $bb$ cutting four members. Out of these, the line of action of forces in three members pass through point $C$. Hence take the moment of forces about point $C$. Consider the equilibrium of the portion enclosed by the section $bb$.

When the unit load is at $C$ or $D$, $M_C$ is zero, and hence $P_{rd}$ is zero.

When the unit load is at $Q$, we get

\[ M_C = 1 \times 6 = P_{rd} \times (12 \sin \theta) \]
\[ P_{RD} = \frac{1}{12} \csc \theta = \frac{1}{4} \csc \theta = \frac{1}{4} \times 1.28 = 0.64 \text{ (tension)} \]

The I.L. for \( P_{RD} \) is shown in Fig. 4.6 (f).

(6) **Influence line for \( P_{RM} \)**

Pass a section \( ee \) cutting four members \( QD, RD, RM \) and \( NM \). The stress in \( RD \) is known. Consider the equilibrium of the left portion. Resolving the forces vertically, we get

\[ P_{RD} \sin \theta + P_{RM} \sin \theta = \text{shear in panel } QD = f_{QD} \]

In the above relation, member \( RD \) will have stress only when the load is in \( CD \).

(i) When the unit load is at \( C, P_{RD} = 0 \) and \( R_A = 5/6 \times 0.833 \)

\[ P_{RM} = \csc 6 = \frac{5}{6} \times 1.28 = 0.853 \text{ (tension)} \]

The I.L. for \( P_{RM} \) is shown in Fig. 4.6(g). It will be seen that the influence lines for \( P_{CR} \) and \( P_{RM} \) are exactly the same for load positions between \( B \) and \( C \) and between \( D \) and \( H \).

(7) **Influence line for \( P_{DM} \)**

Pass a section \( dd \), cutting four members. Considering the equilibrium of the left portion and resolving the forces vertically, we get, in general:

\[ P_{DM} + P_{RD} \sin \theta = \text{shear in panel } QD = F_{QD} \]

Member \( RD \) will have stress only when the load is in \( CD \).

(i) When the unit load is at \( C, R_A = 0.833 \) and \( P_{RD} = 0 \)

\[ P_{DM} = F_{QD} = 1 - 0.833 = 0.167 \text{ (tension)} \]

(ii) When the unit load is at \( Q, R_A = 0.75 \)

\[ P_{DM} = 0.64 \text{ (tension)} \]

(iii) When the unit load is at \( D, P_{RD} = 0 \) and \( R_A = \frac{5}{6} \)

\[ P_{DM} = 0.64 \times 0.64 = 0.409 \text{ (compression)} \]

The I.L. for \( P_{DM} \) is shown in Fig. 4.6(h).

4.7 **BALTIMORE TRUSS WITH SUB-STRUTS : THROUGH TYPE**

(1) **Influence line for \( P_{BC} \)**

Pass a section \( aa \) cutting three members. Considering equilibrium of the left portion,

\[ P_{BC} = \frac{M_K}{15} \text{ (compression)} \]

The influence line diagram for \( P_{BC} \) is thus a triangle having a maximum ordinate of \( \frac{24 \times 48}{72} \times \frac{1}{15} = 1.067 \text{ (comp.)} \) under \( K \) as in Fig. 4.7(b).

(2) **Influence line for \( P_{Lk} \)**

\[ P_{Lk} = \frac{M_B}{15} \text{ (tension)} \]

When the unit load is at \( L, R_A = 0.72 \times 0.833 \)

\[ P_{Lk} = \frac{1}{15} (0.833 \times 12) = 0.667 \text{ (tension)} \]

When the unit load is at \( T, R_A = 0.54 \times 0.72 \)

\[ P_{Lk} = \frac{1}{15} (0.75 \times 12 + 1 \times 6) = 1 \text{ (tension)} \]

When the unit load is at \( K, R_A = 0.48 \times 0.72 \)

\[ P_{Lk} = \frac{1}{15} (0.667 \times 12) = 0.533 \text{ (tension)} \]

The I.L. diagram for \( P_{Lk} \) is shown in Fig. 4.7(c).

(3) **Influence line for \( P_{NK} \)**

Resolving the forces vertically,

\[ P_{NK} \sin \theta = \text{shear in panel } TK = F_{TK} \]

or

\[ P_{NK} = F_{TK} \csc \theta \]

\[ \sin \theta = \frac{15}{\sqrt{15^2 + 12^2}} = 0.781 \text{ ; } \csc \theta = 1.28 \text{ ; } \cos \theta = 0.625 \]

When the unit load is at \( T, R_A = 0.75 \)

\[ P_{NK} = (1 - 0.75) \times 1.28 = 0.32 \text{ (comp.)} \]

When the unit load is at \( K, R_A = 0.667 \)

\[ P_{NK} = 0.667 \times 1.28 = 0.853 \text{ (tension)} \]

The I.L. diagram for \( P_{NK} \) is shown in Fig. 4.7(d).

(4) **Influence line for \( P_{NT} \)**

\( NT \) is a secondary member, and hence it will be stressed only when the load is in panel \( LK \). Evidently, I.L. for \( P_{NT} \) will be a triangle having zero ordinates under \( L \) and \( K \) and an ordinate of unity under \( T \), as shown in Fig. 4.7(e).
(5) Influence line for $PLN$
Pass a horse shoe section $bc$, cutting five members $BN, NK, LN, LT$ and $TK$. Out of these, line of action of forces in four members pass through point $K$. Hence take the moment of the forces, about $K$ and consider the equilibrium of the portion enclosed by the horse shoe section. Note that $LN$ is a secondary member, and hence it will be stressed only when the load is in panel $LK$.

*When the unit load is at $L$, $M_K=0$ and hence $PLN=0$*

*When the unit load is at $T$, $M_K=1 \times 6 = PLN \times (12 \sin \theta)$*

\[ PLN = \frac{6}{12 \sin \theta} = \frac{1}{2} \csc \theta = \frac{1}{2} \times 1.28 = 0.64 \text{ (comp.)} \]

*When the unit load is at $K$, $M_K=0$, and hence $PLN=0$*

The I.L. diagram for $PLN$ is shown in Fig. 3.7 (f).

(6) Influence line for $PBn$
Pass a section $cc$, cutting four members. Considering the equilibrium of the left portion, we get, in general,

\[ PBn \sin \theta = PLN \sin \theta = \text{shear in } LT = FLT \]

or

\[ PBn + PLN = FLT \csc \theta = 1.28 \text{ FLT} \]

The member $LN$ will be stressed only when the load is in $LK$.

*When the unit load is at $L$, $R_A=0.833$ and $PLN=0$*

\[ PBn = (1 - 0.833) \times 1.28 = 0.24 \text{ (comp.)} \]

*When the unit load is at $T$, $R_A=0.75$ and $PLN=0.64$ (comp.)*

\[ PBn = 0.64 \times 0.75 = 0.48 \text{ (tension)} \]

or

*When the unit load is at $K$, $R_A=0.867$ and $PLN=0$*

\[ PBn = 0.867 \times 1.28 = 1.12 \text{ (tension)} \]

The I.L. diagram for $PBn$ is shown in Fig. 4.7 (g).

(7) Influence line for $PBl$
Pass a section $dd$, cutting four members $BM, BL, LN$ and $LT$. Out of these, the lines of action of forces in two members pass through $A$. Hence take the moment of all the forces, about $A$ and consider the equilibrium of the left portion.

*When the unit load is at $A$, $PBl$ is evidently zero.*

*When the unit load is at $L$, $PLN=0$.*

\[ PBl \times 12 = 1 \times 12 \]

or

\[ PBl = 1 \text{ (tension).} \]

*When the unit load is at $T$, $PLN=0.64$ (comp.)*

\[ PBl \times 12 = (PLN \sin \theta) \times 12 \]

\[ PBl = 0.64 \times 0.781 = 0.5 \text{ (tension)} \]

*When the unit load is at $K$, $PLN=0$.*

\[ PBl \times 12 = \text{zero.} \text{ or } PBl \text{ is zero} \]

The I.L. for $PBl$ is shown in Fig. 4.7 (h).
4.8. PENNSYLVANIA OR PETTIT TRUSS WITH SUB-TIES

(1) Influence line for $P_{CD}$

Pass a section $aa$ cutting three members

$$P_{CD} = \frac{M_t}{r} \text{(compression)}$$

where $r$ = Perpendicular distance of $J$ from $CD$.

Prolong $DC$ backward to meet $GA$ produced in $O_1$. Let $\theta_i$ be

the inclination of $CD$, as shown in Fig. 3'8 (a).

$$\tan \theta_i = \frac{15-13}{12} = \frac{1}{6} = 0.1667; \theta_i = 9^\circ 28'$$

$$\sin \theta_i = 0.165; \cos \theta_i = 0.986$$

$$O_{1J} = \frac{DJ}{\tan \theta_i} = 15 \times 6 = 90 \text{ m}; O_{1A} = 90 - 36 = 54 \text{ m}$$

Now, $r = O_{1J} \sin \theta_i = 90 \times 0.165 = 14.85$ m

$$\therefore P_{CD} = \frac{M_t}{14.85} \text{(compression)}$$

When the unit load is at $K$, $R_a = \frac{48}{72} = 0.667$

$$\therefore P_{CD} = \frac{1}{14.85}[1.667 \times 36 - 1 \times 12] = 0.808 \text{(comp.)}$$

When the unit load is at $U$, $R_a = \frac{42}{72} = 0.583$

$$\therefore P_{CD} = \frac{1}{14.85}[0.583 \times 36] = 1.412 \text{(comp.)}$$

The I.L. diagram for $P_{CD}$ is shown in Fig. 4'8 (b).

(2) Influence line for $P_{KJ}$

$$P_{KJ} = \frac{M_c}{13} \text{(tension)}$$

The I.L. diagram for $P_{KJ}$ will, therefore, be a triangle having

a maximum ordinate of $\frac{24 \times 48}{72} \times \frac{1}{13} = 1.23$ under $K$ as shown in

Fig. 4'8 (c).

(3) Influence line for $P_{CO}$

Consider the equilibrium of the portion to the left of section

$aa$. Take moment about $O_1$ where two members $CD$ and $KJ$ meet

when produced.

$$P_{CO} = \frac{M_{OJ}}{X}$$

where $X$ = perpendicular distance of $CO$ from $O_1$

$= O_{1J} \sin \theta$

$\theta$ = inclination of $CJ$ with $O_{1J}$

Fig. 4'8.

Pettit truss with sub-ties
the equilibrium of the portion to the left of section cc and take moments about O₁.

(i) When the unit load is at K, \( R_A = 0.677 \) and \( P_{OD} = 0 \)

\[ (P_{OJ} \sin \theta_1) \times O_J = 1 \times O_A \times R_A = 0.667 \times O_A \]

\[ P_{OJ} = \frac{1}{0.734 \times 90} \times [(1 \times 78 - 0.667 \times 54)] = 0.636 \text{ (comp.)} \]

(ii) When the unit load is at \( U \), \( R_A = 0.583 \) and \( P_{OD} = 0.694 \) (tension).

Taking moments about \( O_1 \), we get

\[ (P_{OJ} \sin \theta_1) \times O_J = (1 \times O_A) - (R_A \times O_A) \]

\[ (P_{OJ} \sin \theta_1) \times O_J = 0.5 \times 54 \]

\[ P_{OJ} = 0.734 \times 90 = 0.408 \text{ (tension)} \]

The I.L. diagram for \( P_{OJ} \) is shown in Fig. 4.8 (g).

(7) **Influence line for \( P_{NC} \)**

The influence line for \( P_{NC} \) can be drawn exactly in the same manner as that for \( P_{OD} \). The I.L. diagram is a triangle with a central ordinate of 0.8 as shown in Fig. 4.8 (h). Reader is advised to compute this ordinate.

(8) **Influence line for \( P_{CK} \)**

Pass a section \( dd \) cutting four members.

If \( \alpha \) is the inclination of member \( NC \) with the vertical

\[ \sin \alpha = \frac{6}{\sqrt{8.5^2 + 6^2}} = \frac{6}{10.4} = 0.577 \text{; } \cos \alpha = \frac{8.5}{10.4} = 0.818 \]

Prolong \( CB \) backwards to meet \( GA \) produced in \( O_A \). Let \( \theta_2 \) be the inclination of \( CB \) with horizontal.

Then, \( \tan \theta_2 = \frac{13 - 9}{12} = \frac{4}{13} = \frac{1}{3} \); \( \theta_2 = 18^\circ 26' \)

\[ \sin \theta_2 = 0.316 \text{ and } \cos \theta_2 = 0.949 \]

\[ O_A K = \frac{CK}{\tan \theta_2} = \frac{13 \times 3}{3} = 39 \text{ m} \]

\[ O_A A = 39 - 24 = 15 \text{ m} \]

Take the moments about \( O_A \), of all unbalanced forces to the left of the section \( dd \).
When the unit load is at \( L, R_A = \frac{60}{72} = 0.833 \) and \( P_{NC} = 0 \)

\[
P_{CK} = \frac{M_{O_2}}{O_2 K} = \frac{M_{O_2}}{39} = \frac{1}{39} \left\{ (1 \times 27) - (0.833 \times 15) \right\}
= 0.372 \text{ (tension)}.
\]

When the unit load is at \( T, R_A = 0.75 \) and \( P_{NC} = 0 \)

\[
P_{CK} = \frac{P_{NC} \cos \alpha \times O_2 K - (P_{NC} \sin \alpha \times 13) + (R_A \times O_2 A)}{- (1 \times O_2 T)}
\]

or
\[
P_{CK} = \frac{1}{39} \left\{ (0.8 \times 0.818 \times 39) - (0.8 \times 0.577 \times 13) + (0.75 \times 15) \right\}
= 0.058 \text{ (tension)}.
\]

When the unit load is at \( K, R_A = 0.667 \) and \( P_{NC} = 0 \)

\[
P_{CK} = \frac{1}{39} \left\{ (1 \times 39) - (0.667 \times 15) \right\} = 0.744 \text{ (tension)}.
\]

When the unit load is at \( U, R_A = 0.583 \) and \( P_{NC} = 0 \)

\[
P_{CK} = \frac{1}{39} (0.583 \times 15) = 0.224 \text{ (comp.)}
\]

The influence line diagram for \( P_{CK} \) is shown in Fig. 4.8 (i).

The Müller-Breslau Principle

5.1. Introduction

The Müller-Breslau principle or Müller-Breslau influence theorem is the most important tool in obtaining influence lines for statically determinate as well as statically indeterminate structures. The method is based on the concept of the influence line as a deflection curve. While developing the method about twenty years after the influence line was first introduced by Winkler (1867), Müller-Breslau became aware of the great values of Maxwell's theorem of reciprocal displacement. In fact, Müller-Breslau principle is the straight application of Maxwell's reciprocal theorem. For a detailed study of the reciprocal theorem, the reader is advised to read articles 7.3 and 7.4 of chapter 7.

5.2. The Müller-Breslau Principle

The Müller-Breslau principle may be stated as follows:

"If an internal stress component, or a reaction component is considered to act through some small distance and thereby to deflect or displace a structure, the curve of the deflected or displaced structure will be, to some scale, the influence line for the stress or reaction component."

To prove the validity of the above statement, let us consider a two span continuous beam ABC, freely supported at A and C, and continuous over support B and plot the influence line for reaction \( R_B \) at \( B \).

Let a unit load act at a point \( X \) distant \( x \) from end \( A \). If the support at \( B \) is removed, the beam will deflect as shown in Fig. 5.1 (a). Remove the unit load as well as the redundant reaction \( R_B \), and place a downward unit load at \( B \). The beam will deflect under the unit load, as shown in Fig. 5.1 (c).
Let $y_{BB} =$ deflection at $B$ due to unit load at $B$
$y_{XB} =$ deflection at $X$ due to unit load at $B$
$y_{BX} =$ deflection at $B$ due to unit load at $X$.

(The first suffix to $y$ denotes the point where the deflection is reckoned and the second suffix denotes the position of the unit point load).

Thus, when the unit load is acting at $X$, the deflection of point $B$, in the absence of $R_B$, will be equal to $y_{BX}$. However, as the support at $B$ is at the same level as $A$ and $C$, the upward deflection at $B$ due to $R_B$ is to neutralise this downward deflection $y_{BX}$. Hence we get, from consistent deformation (chapter 7):

$R_B y_{BB} = y_{BX}$

By Maxwell's reciprocal theorem (7·3)

$y_{BX} = y_{XB}$

Hence

$R_B = \frac{y_{XB}}{y_{BB}}$  \hspace{1cm} (5·1)

Thus, the reaction at $B$, due to unit load at any point $X$ is proportional to the deflection at the point $X$ due to the unit load acting at $B$. In other words, the deflection curve shown in Fig. 5·1 (c) represents, to some scale, the influence line for $R_B$.

If the deflection $y_{BB}$ in the direction of unit load at $B$, is selected as unity, the deflection curve will directly give influence line for $R_B$.

5·3 INFLUENCE LINES FOR STATICALLY DETERMINATE BEAMS

The Müller-Breslau Principle is applicable both for statically determinate beams as well as for statically indeterminate beams. Let us first take statically determinate beam.

The Müller-Breslau influence theorem for statically determinate beams may be stated as follows:

The influence line for an assigned function of a statically determinate beam may be obtained by removing the restraint offered by that function and introducing a directly related generalised unit displacement at the location and in the direction of the function.
1. I.L. For reaction $R_a$ and $R_b$

The I.L. for reaction ($R_a$) at $A$ can be found by lifting the beam off the support $A$ by a unit distance, as shown in Fig 5.2 (b). The deflected shape gives the I.L. for $R_a$. This can be easily proved by applying the principle of virtual work to the rigid body motion of the beam shown in Fig. 5.2 (b). The total virtual work ($\delta W$) must be equal to zero since the resultant of the force system is zero. Thus, if the ordinate under the unit load is $y$, we have

$$\delta W = R_a(1 \cdot 0) - 1 \cdot 0(y) = 0$$

which gives

$$y = R_a$$

which proves the proposition.

When the unit load is at a distance $xL$ from $A$, the magnitude of $y$ is given by the relation

$$\frac{y}{L} = \frac{x}{L-x} \Rightarrow \frac{y}{L} = 1 - \frac{x}{L}$$

Similarly, the I.L. for reaction $R_b$ can be found, as shown in Fig. 5.2 (c).

2. I.L. for S.F. at $C$

Let us find the I.L. for S.F. ($F_c$) at $C$. We know that S.F. ($F_c$) acts to both sides of the section and is represented by $(\pm \pm \pm)$. Hence cut the beam at $C$ into two parts $AC$ and $CB$. The free body diagram of the two parts is shown in Fig. 5.3 (b). Let the beam go through rigid body motions of parts $AC$ and $CB$, as shown in Fig. 5.2 (d), so that the total movement $C_1C_2 =$ unity. The deflected shape will then give the influence line for $F_c$. This can be very easily proved by applying the principle of virtual work. Thus, it is the ordinate of the I.L. under the unit load, we have

$$\delta W = R_a(0) - 1 \cdot 0(y) - Mc(0) + Fc(CC_1) + Fc(CC_2) + Mc(0) + R_d(0) = 0$$

or

$$y = Fc(CC_1 + CC_2) = Fc(C_1C_2), \text{ where } C_1C_2 = 1$$

$$y = Fc$$

which proves the proposition.

Now, if the section $C$ is at a distance $x$ from $A$, $CC_1$ will be equal to $x/L$, and $CC_2$ will be equal to $(L-x)/L$. Similarly, the ordinate $y$ under the unit load is given by

$$y = \left(\frac{x}{L}\right) \times \frac{x}{L} = x$$

3. I.L. for B.M. ($M_c$) at $G$

For obtaining I.L. for $M_c$, introduce a hinge at $C$, and let the system go through rigid body motions of $AC$ and $CB$ as shown in Fig. 5.2 (e). Then

$$\delta W = R_a(0) - 1 \cdot 0(y) - Fc(CC_1) + Mc(0) + Fc(CC_1) + Mc(0_1) + R_d(0) = 0$$

or

$$y = Mc(0_1 + 0_2), \text{ where } 0_1 + 0_2 = 1$$

$$\therefore y = Mc$$

Now

$$CC_1 = x0_1 = (L-x)0_1$$

Thus,

$$\theta_2 = \frac{x}{L-x}0_1$$

But

$$0_1 + 0_2 = 1$$

$$\therefore 0_1 + \frac{x}{L-x}0_1 = 1$$

$$\therefore 0_1 \left(\frac{L}{L-x}\right) = 1, \therefore \theta_1 = L-x$$

Hence

$$CC_1 = x0_1 = \frac{x}{L}(L-x)$$

Also, ordinate $y$ is given by

$$\frac{y}{xL} = \frac{CC_1}{x} \Rightarrow y = aL \times \frac{1}{x} \times \frac{x}{L} \times (L-x) = a(L-x)$$

When the unit load is at $C$, $aL=x$, or $a=x/L$

$$y = CC_1 = \frac{x}{L}(L-x)$$

**Example 5.1** A two span beam $ABC$ has internal hinges at $D$ and $E$. Using Muller-Breslau influence theorem, sketch I.L. for (i) $R_a$ (ii) $R_b$ (iii) $R_c$ and (iv) $M_c$.

**Solution.** If there were no hinges at $D$ and $E$, the beam would be statically indeterminate to second degree. However, provision of hinges at $D$ and $E$ makes the beam statically determinate.
For I.L. for $R_A$: According to Muller Breslau Theorem, in order to find I.L. for $R_A$, lift the beam off the support $A$ by unity in the direction of $R_A$. The deflected shape of the beam, shown in Fig. 5.4(b) gives the I.L. for $R_A$. It should be noted that because of a hinge at $D$, only the portion $AD$ will be lifted up, and the remaining portion will remain horizontal. This suggests that when the unit load crosses $D$, $R_A$ will be zero and will continue to remain zero as the unit load moves along $DBEC$.

For I.L. for $R_B$: For $R_B$, lift the beam off the support $B$ by unity. The beam will deflect as shown in Fig. 5.4(c), which will be I.L. for $R_B$. Ordinate $DD_1$ will evidently be equal $\frac{1}{L/2} \times L=2$. When the unit load crosses $E$, $R_B$ will be zero.

For I.L. for $R_C$: Lift the beam off the support $C$ by unity. The beam will deflect as shown in Fig. 5.4(d) which will be the I.L. for $R_C$, according to the Muller Breslau Principle. Since $CC_1=1$, E will move to $E_1$ such that $EE_1=1$.

Hence, from geometry $DD_1=1$ in the negative direction. This suggests that when the unit load is between $A$ to $B$, the reaction $R_C$ will be negative i.e. it will act down wards.

For I.L. for $M_C$. Let us assume $M_C$ to be in clockwise direction. Hence in order to find I.L. for $M_C$, introduce a hinge at $C$ and rotate the beam, at $C$, by $\theta=1$ unit. The beam will deflect as shown in Fig. 5.4(e) which will evidently be the I.L. for $M_C$.

\[
\text{Ordinate } EE_1 = \frac{EC}{0} = \frac{L/2 - L/2}{1} = \frac{L}{2}
\]

Hence by geometry, $DD_1 = \frac{L}{2}$.

When the unit load is between $A$ to $B$, $M_C$ will be negative, i.e. it will act in the counterclockwise direction. For the unit load positions between $B$ and $C$, $M_C$ will act in the clockwise direction, as marked in Fig 5.4(a).

Example 5.2. A two span beam $ACDE$ has internal hinge at $B$ and an overhang DE. Using Muller-Breslau influence theorem, draw influence lines for $R_A$, $R_C$, $R_D$, $M_C$, $M_D$, $F_o$.

Solution.

For I.L. for $R_A$. For getting I.L. for $R_A$, lift the beam off the support $A$ by unity. Due to internal hinge at $B$, only portion $AB$ will be deflected, as shown in Fig. 5.5(b) which is the I.L. for $R_A$. The reaction $R_A$ remains zero for load positions between $B$ to $E$. 

Fig 5.4
**The Muller-Breslau Principle**

$$BB_1 = \frac{CC_1}{CD} \times BD = \frac{1}{L} \times \frac{3}{2}L = 1.5 \text{ (positive)}$$

Similarly $$EE_1 = CC_1 = 1 \text{ (negative)}$$

(iii) I.L. for $R_D$. For getting I.L. for $R_D$, lift the beam off the support $D$ by unity. The deflected shape of the beam, shown in Fig. 5.5 (d) will be the I.L. for $R_D$, where in

$$BB_1 = \frac{DD_1}{CD} \times BC = \frac{1}{L} \times \frac{1}{2}L = \frac{1}{2} \text{ (negative)}$$

and $$EE_1 = \frac{DD_1}{CD} \times CE = \frac{1}{L} \times 2L = 2 \text{ (positive)}$$

(iv) I.L. for $M_C$. In order to get I.L. for $M_C$, introduce a hinge at $C$ and rotate the beam by $\theta = \text{unity}$, in the anticlockwise direction, assuming that $M_C$ acts in the counter-clockwise direction. The deflected shape, shown in Fig. 5.5 (e) will be the I.L. for $M_C$, wherein, ordinate $BB_1 = BC(0) = \frac{L}{2}$ (negative). This means that when the load is in $AC$, the moment $M_C$ will be in the clockwise direction. For unit load positions between $C$ and $E$, $M_C$ will remain zero.

(v) I.L. for $M_G$. Introduce a hinge at $G$ and permit unit relative rotation of the parts on opposite sides of $G$. The deflected shape of the beam, shown in Fig. 5.5 (f) will be I.L. for $M_G$, in which ordinate $BB_1 = \frac{L}{4}$ (negative) and ordinate $EE_1 = \frac{L}{2}$ (negative) while ordinate $GG_1 = \frac{L}{4}$.

(vi) For I.L. for $F_G$, apply a cut in the beam at $G$ and give relative displacements of the two parts by unity. The deflected shape of the beam, shown in Fig. 5.5 (g) will give I.L. for $F_G$. Since $G$ is situated midway between $C$ and $D$, $GG_1 = GG_2 = 0.5$. Evidently, $BB_1$ will be $-0.5$ and $EE_1$ will be $-1$.

5.4. **Propelled Cantilevers**

1. I.L. for prop reaction

In § 5.2, we have seen the application of Muller-Breslau principle for constructing influence line for vertical reaction component of a continuous beam. We shall now apply the principle for drawing I.L. for reaction $R_a$ at the prop.

Let the unit load be at section $X$ (Fig. 5.6a). According to the Muller-Breslau principle, remove the prop at $B$. The beam will deflect...
as shown in Fig. 5'6 (b), in which $y_{BX}$ is the deflection of $B$ due to unit load at $X$. Now remove the unit load from $X$ and place it at $B$. The beam will deflect as shown in Fig. 5'6 (c), in which $y_{BS}$ is the deflection at $X$ due to unit load at $B$, and $y_{BB}$ is the deflection at $B$ due to unit load at $B$. However, since the support at $B$ is at the same level as $A$, the upward deflection at $B$ due to $R_B$ is to neutralise the downward deflection in the direction of unit load at $B$. Hence, we get from consistent deformation (Chapter 7).

$$R_B \cdot y_{BS} = y_{BX}$$

By Maxwell’s reciprocal theorem (§ 7.3)

$$y_{BX} = y_{BS}$$

\[ R_B = \frac{y_{BX}}{y_{BB}} \quad \text{(5.2)} \]

Thus, the reaction at $B$, due to unit load at any point $X$ is proportional to the deflection at point $X$ due to the unit load acting at $B$. In other words, the deflection curve shown in Fig. 5'6 (c), represents, to some scale, the influence line for $R_B$.

If the deflection $y_{BB}$ in the direction of unit load at $B$ is selected as unity, the deflection curve will directly give the I.L. for $R_B$.

2. I.L. for $M_A$

Let the unit load be at $X$. Remove the fixed support at $A$ and introduce a hinge. The beam will deflect as shown in Fig. 5'7 (b), when unit load is applied at $X$. Let $\phi'_{AX}$ be the rotation at $A$ due to unit load at $X$. Now, remove the unit load at $Y$ and apply unit

moment at $A$. The beam will deflect as shown in Fig. 5'7 (c), where $y'_{XA} = \text{deflection at section } X \text{ due to unit moment at } A \text{ and } \phi'_{AA} = \text{rotation at } A \text{ due to unit moment at } A$.

From method of consistent deformation,

$$M_A \cdot \phi_{AA} = \phi'_{AX}$$

But from reciprocal theorem,

$$\phi'_{AX} = y'_{XA}$$

\[ M_A \cdot \phi_{AA} = y'_{XA} \]

or

$$M_A = \frac{y'_{XA}}{\phi_{AA}} \quad \text{(5.3)}$$

The above relation suggests that $M_A$ is proportional to $y'_{XA}$. In other words, the deflected curve of Fig. 5'7 (c) gives, to some scale, the influence line for $M_A$. If $\phi_{AA}$ is selected as unity, the deflection curve will directly give the I.L. for $M_A$.

3. I.L. for $M_D$ (Fig. 5'8 a)

Let the unit load be at $X$. We want to plot the influence line for bending moment $M_D$ at the point $D$. According to Muller-Breslau principle, the internal stress component, for which the influence line is to be plotted is first removed. For the present case, this is accomplished by inserting a pin at $D$. The beam will then deflect under the unit load at $X$, as shown in Fig. 5'8 (b).
Let \( \phi DX \) be the rotation at \( D \), due to unit load at \( X \). Now remove the unit load from \( X \), and apply a pair of unit moments at \( D \), as shown in Fig. 5'8 (c). Let \( \phi DD \) be the resulting rotation at \( D \) and \( Y'DX \) be the resulting deflection at \( X \). Then, from method of consistent deformation,

\[
M_D \cdot \phi DD = \phi DX
\]

But from reciprocal theorem,

\[
\phi DX = Y'DX
\]

\[
\therefore \quad M_D \cdot \frac{Y'DX}{\phi DD} = Y'DX
\]

(5'4)

The above relation suggests that \( M_D \) is proportional to \( Y'DX \).

In other words, the deflected curve of Fig. 5'8 (c) gives to some scale, the influence line for \( M_D \). If \( \phi DD \) is selected as unity, the deflection curve will directly give the I.L. for \( M_D \).

4. I.L. for \( F_D \)

Let us now plot the I.L. for shear at \( D \). Let the unit load be at \( X \) (Fig. 5'9 a). In order to remove the internal stress component, i.e., shear \( F_D \) at \( D \), assume that the beam is cut at \( D \) and that a slide device is inserted in such a way that it permits relative transverse deflection between the two parts of the cut, as shown in Fig. 5'9(b), but which at the same time, maintains a common slope at both the ends of the cut. Let \( Y'DX \) be the relative linear deflection at \( D \) due to unit load at \( X \).

\[
F_D \cdot Y'DD = Y'DX
\]

But from reciprocal theorem,

\[
Y'DX = Y'DD
\]

\[
\therefore \quad F_D \cdot \frac{Y'DX}{Y'DD} = Y'DX
\]

(5'5)

Thus, \( F_D \) is proportional to \( Y'DX \). In other words, the deflection curve of Fig. 5'9 (c) gives the I.L. for \( F_D \), to some scale. If, however, \( Y'DD \) is taken as unity, the deflection at any point \( X \) of Fig. 5'9(c) gives the shear at \( D \) due to unit vertical load at \( X \).

5'5. CONTINUOUS BEAM: INFLUENCE LINE FOR BENDING MOMENT

Article 5'2 illustrates the application of Müller-Breslau principle for constructing influence line for a vertical reaction component of a continuous beam. We shall now apply the principle for drawing the influence line for bending moment at any point \( D \) of a continuous beam shown in Fig. 5'10 (a).
Let the unit load be at any point $X$. In order to remove the internal stress components, i.e., shear $F_D$ at $D$, assume that the beam is cut at $D$ and that a slide device inserted in such a way that it permits relative transverse deflection between the two parts of the cut, as shown in Fig. 5.11 (b), but which at the same time, maintains a common slope at both the ends of the cut.

Then, from compatibility of deformation at $D$, we have

$$F_D = \frac{Y_{DD}}{Y_{DX}}$$

Let us now study the applicability of the Müller-Breslau principle for plotting the influence line for the shear force at any point $D$ of a continuous beam $ABC$ shown in Fig. 5.11. Let the unit load be at any point $X$. In order to remove the internal stress components, i.e., shear $F_D$ at $D$, assume that the beam is cut at $D$ and that a slide device inserted in such a way that it permits relative transverse deflection between the two parts of the cut, as shown in Fig. 5.11 (b), but which at the same time, maintains a common slope at both the ends of the cut.

$$M_D = \frac{Y_{DD}}{Y_{DX}}$$

Eq. 5.6 suggests that $M_D$ is proportional to $Y_{DD}$. In other words, the deflection curve of Fig. 5.10 (c) gives, to some scale, the influence line for $M_D$. If however, $Y_{DD}$ is selected to be unity, the deflection at any point $X$ will give the bending moment at $D$. Eq. 5.6, therefore, further proves the validity of the Müller-Breslau principle applied to the influence line for bending moment.
But from the reciprocal theorem,
\[ Y_{DX} = Y_{XD} \]
\[ F_D = \frac{Y_{XD}}{Y_{DD}} \]  

(5.7)

Thus, \( F_D \) is proportional to \( Y_{XD} \). In other words, the deflection curve of Fig. 5.11 (c) gives the influence line for shear at \( D \), to some scale. If, however, \( Y_{DD} \) is taken as unity, the deflection at any point \( X \) of Fig. 5.11 (c) gives the shear at \( D \) due to unit vertical load at \( X \).

5.7. INFLUENCE LINE FOR HORIZONTAL REACTION

Let us now study the influence line for horizontal reaction at the hinged end \( A \) of a frame shown in Fig. 5.12(a). The unit vertical load can travel on \( BC \), or unit horizontal load can travel on \( AB \). Let us first take the case when the unit vertical load travel on \( BC \).

According to the Muller-Breslau principle, the reaction component at \( Q \) is first removed. This is accomplished by supporting end \( A \) on rollers. The end \( A \) will deflect horizontally by \( \Delta AX \) due to unit vertical load at \( X \) as shown in Fig. 5.12 (b). The unit vertical load at \( X \) is then removed and a unit horizontal load is applied at \( A \). The frame will deflect as shown in Fig. 5.12(c).

Let \( \Delta AA = \) Horizontal deflection of \( A \) due to horizontal unit load at \( A \).

\[ \Delta x_A = \text{Vertical deflection at } X \text{ due to unit horizontal load at } A \]

\[ H_A = \text{Horizontal reaction at } A \]

Then \[ H_A \Delta AA = \Delta AX \]

or \[ H_A = \frac{\Delta AX}{\Delta AA} \]

But \[ \Delta AX = \Delta x_A \], from reciprocal theorem

Hence \[ H_A = \frac{\Delta x_A}{\Delta AA} \]  

(5.8)

Eq. 5.8 shows that the deflection curve of \( BC \) [Fig. 5.12 (c)] gives the influence line, to some scale for \( H_A \) when a unit vertical load moves on \( BC \). Similarly, it can be shown that the deflection curve of \( AB \) gives the influence line for \( H_A \) when a unit horizontal load moves on \( AB \).
Example 5.3. Draw the influence lines for (i) reaction at B and (ii) moment at A for the propped cantilever shown in Fig. 5.13 (a). Compute the ordinates at intervals of 1.25 m.

Solution

\[ R_B = \frac{y_{XB}}{y_{BB}} \]  \hspace{1cm} (1)

To compute \( y_{XB} \) and \( y_{BB} \), apply a unit vertical load at B, as shown in Fig. 5.13 (b).

The ordinates of I.L. for \( R_B \) are computed in Table 5.1.

<table>
<thead>
<tr>
<th>( x ) (m)</th>
<th>0</th>
<th>1.25</th>
<th>2.50</th>
<th>3.75</th>
<th>5.00</th>
<th>6.25</th>
<th>7.50</th>
<th>8.75</th>
<th>10.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = \frac{x}{L} )</td>
<td>0</td>
<td>0.125</td>
<td>0.25</td>
<td>0.375</td>
<td>0.50</td>
<td>0.625</td>
<td>0.75</td>
<td>0.875</td>
<td>1.00</td>
</tr>
<tr>
<td>( R_B )</td>
<td>1</td>
<td>0.813</td>
<td>0.633</td>
<td>0.464</td>
<td>0.313</td>
<td>0.185</td>
<td>0.087</td>
<td>0.023</td>
<td>0.000</td>
</tr>
</tbody>
</table>

The I.L. for \( R_B \) is shown in Fig. 5.9(c).
(b) I.L. for $M_A$

In order to draw the I.L. for $M_A$, replace the fixed support at $A$ by a pin, as shown in Fig. 5'13(d). Remove the external unit load and apply a unit couple at $A$, as shown in Fig. 5'13(e). Then from Eq. 5'3:

$$M_A = \frac{y'_{X_A}}{\phi_{AA}}$$

(3) where $y'_{X_A} = $ vertical deflection at $X$ due to unit couple at $A$.

$\phi_{AA} = $ slope at $A$ due to unit couple at $A$.

Let $R'B = $ Reaction at $B$, when unit moment is acting at $A.$

$$R'B = \frac{1}{L}$$

$\therefore$ $EI\frac{dy}{dx} = -R'B \cdot x = -\frac{x}{L}$

$$EI\frac{dy}{dx} = -\frac{x^2}{2L} + C_1$$

and $EIy = -\frac{x^2}{6L} + C_1x + C_2$

At $x = 0, y = 0.$ $\therefore C_2 = 0$

$x = L, y = 0.$ $\therefore C_1 = \frac{L}{6}$

Hence $EI\frac{dy}{dx} = -\frac{x^2}{2L} + \frac{L}{6}$

(4) and $EIy = -\frac{x^2}{6L} + \frac{Lx}{6}$

At $x = L,$ $\frac{dy}{dx} = \frac{1}{EI} \left(-\frac{L^2}{2L} + \frac{Lx}{L} \right) = -\frac{L}{3}$

(5) At $x = x,$

$$y = y'_{X_A} = \frac{1}{EI} \left(-\frac{x^2}{6L} + \frac{Lx}{6} \right)$$

Substituting these values in (3), we get

$$M_A = \left(\frac{x^2}{6L} - \frac{Lx}{6}\right) \times \frac{3}{L} = \frac{1}{2} \left(\frac{x^2}{L^2} - x \right)$$

This is thus the equation of the influence line for $M_A.$ The ordinates are calculated in the tabular form in Table 5'2.

The minus sign shows that the direction of $M_A$ is in reverse direction to that of the unit moment applied at $A,$ i.e., $M_A$ act in anti-clockwise direction. The I.L. for $M_A$ is shown in Fig. 5'13(f).

---

**Example 5'4.** Determine the influence line for $R_A$ for the continuous beam shown in Fig. 5'14. Compute the ordinates at every 1 m interval.

**Solution**

Apply a unit vertical load at $A,$ as shown in Fig. 5'14(c). Then

$$R_A = \frac{y'_{X_A}}{\phi_{AA}}$$

From Fig. 5'10(c), $R_B = 2 \uparrow$ and $R_C = 1 \uparrow$

$$EI\frac{dy}{dx} = -x + R_B(x-4) = -x + 2(x-4)$$

$$\therefore EI\frac{dy}{dx} = -\frac{x^2}{2} + C_1 + (x-4)^2$$

---

**Table 5'2**

<table>
<thead>
<tr>
<th>$x$ (m)</th>
<th>0</th>
<th>1.25</th>
<th>2.5</th>
<th>3.75</th>
<th>5</th>
<th>6.25</th>
<th>7.5</th>
<th>8.75</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{x^3}{L}$</td>
<td>0</td>
<td>0.295</td>
<td>0.556</td>
<td>0.833</td>
<td>1</td>
<td>2.45</td>
<td>4.25</td>
<td>6.7</td>
<td>10</td>
</tr>
<tr>
<td>$M_A$</td>
<td>0</td>
<td>-0.615</td>
<td>-1.172</td>
<td>-1.61</td>
<td>-1.875</td>
<td>-2</td>
<td>-1.63</td>
<td>-1.025</td>
<td>0</td>
</tr>
</tbody>
</table>
and \[ E\bar{y} = -\frac{x^3}{6} + C_1 x + C_2 x + \frac{(x-4)^2}{3} \]

At \( x = 4 \), \[ y = 0, \quad \therefore 4C_1 + C_2 = \frac{64}{6} \]
At \( x = 8 \), \[ y = 0, \quad \therefore 8C_1 + C_2 = 64 \]
\[ \therefore C_1 = -\frac{40}{3} \quad \text{and} \quad C_2 = -\frac{128}{3} \]
\[ : EI \ y = -\frac{x^3}{6} + \frac{40}{3} x - \frac{128}{3} x + \frac{(x-4)^2}{3} \]

At \( x = 0 \), \[ y = y_{AA} = -\frac{1}{EI} (\frac{128}{3}) = -\frac{128}{3EI} \]
At \( x = x \), \[ y = y_{XX} = -\frac{1}{EI} \left[ \frac{x^3}{6} + \frac{40}{3} x - \frac{128}{3} x + \frac{(x-4)^2}{3} \right] \]

The values of \( y_{XX} \), for various values of \( x \) are given in Table 5.3.

The term \( \frac{1}{EI} \) has been omitted for convenience.

**Table 5.3**

<table>
<thead>
<tr>
<th>( x ) (m)</th>
<th>( y_{XX} )</th>
<th>( R_A = \frac{y_{XX}}{y_{AA}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0 + 0 - \frac{128}{6}</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>\frac{1}{6} + \frac{40}{3} - \frac{128}{3} = \frac{177}{6}</td>
<td>+0.692</td>
</tr>
<tr>
<td>3</td>
<td>\frac{8}{6} + \frac{80}{3} - \frac{128}{3} = \frac{104}{6}</td>
<td>+0.406</td>
</tr>
<tr>
<td>4</td>
<td>\frac{27}{6} + \frac{120}{3} - \frac{128}{3} = \frac{132}{6}</td>
<td>+0.168</td>
</tr>
<tr>
<td>5</td>
<td>\frac{64}{6} + \frac{160}{3} - \frac{128}{3} = 0</td>
<td>+0.00</td>
</tr>
<tr>
<td>6</td>
<td>-\frac{125}{6} + \frac{200}{3} - \frac{128}{3} + \frac{1}{3} = +\frac{21}{6}</td>
<td>-0.082</td>
</tr>
<tr>
<td>7</td>
<td>\frac{216}{6} + \frac{240}{3} - \frac{128}{3} + \frac{8}{3} = +\frac{24}{6}</td>
<td>-0.094</td>
</tr>
<tr>
<td>8</td>
<td>\frac{343}{6} + \frac{280}{3} - \frac{128}{3} + \frac{15}{6}</td>
<td>-0.059</td>
</tr>
<tr>
<td>9</td>
<td>\frac{512}{6} + \frac{320}{3} - \frac{128}{3} + \frac{64}{3} = 0</td>
<td>0</td>
</tr>
</tbody>
</table>

The I.L. for \( R_A \) is shown in Fig. 5.14(d). By inspection, the plus sign shows that \( R_A \) acts downward (↓) while minus sign shows that \( R_A \) acts upwards (↑).

**Example 5.5.** Determine the influence line for the bending moment at \( D \), the middle point of span \( BC \), of a continuous beam shown in Fig. 5.15(a). Compute the ordinates at 1 m interval.

**Solution**

In order to draw the I.L. for \( M_D \), consider a pin at \( D \). The beam will deflect under the unit load as shown in Fig. 5.15(b).

Remove the external unit load and apply a pair of unit couples at

\[ A \]

\[ B \]

\[ D \]

\[ C \]

\[ E \]

\[ F \]

\[ G \]

\[ H \]

\[ I \]

\[ J \]

\[ K \]

\[ L \]

\[ M \]

\[ N \]

\[ O \]

\[ P \]

\[ Q \]

\[ R \]

\[ S \]

\[ T \]

\[ U \]

\[ V \]

\[ W \]

\[ X \]

\[ Y \]

\[ Z \]

\[ a \]

\[ b \]

\[ c \]

\[ d \]

\[ e \]

\[ f \]

\[ g \]

\[ h \]

\[ i \]

\[ j \]

\[ k \]

\[ l \]

\[ m \]

\[ n \]

\[ o \]

\[ p \]

\[ q \]

\[ r \]

\[ s \]

\[ t \]

\[ u \]

\[ v \]

\[ w \]

\[ x \]

\[ y \]

\[ z \]

\[ A \]

\[ B \]

\[ C \]

\[ D \]

\[ E \]

\[ F \]

\[ G \]

\[ H \]

\[ I \]

\[ J \]

\[ K \]

\[ L \]

\[ M \]

\[ N \]

\[ O \]

\[ P \]

\[ Q \]

\[ R \]

\[ S \]

\[ T \]

\[ U \]

\[ V \]

\[ W \]

\[ X \]

\[ Y \]

\[ Z \]

\[ a \]

\[ b \]

\[ c \]

\[ d \]

\[ e \]

\[ f \]

\[ g \]

\[ h \]

\[ i \]

\[ j \]

\[ k \]

\[ l \]

\[ m \]

\[ n \]

\[ o \]

\[ p \]

\[ q \]

\[ r \]

\[ s \]

\[ t \]

\[ u \]

\[ v \]

\[ w \]

\[ x \]

\[ y \]

\[ z \]

\[ A \]

\[ B \]

\[ C \]

\[ D \]

\[ E \]

\[ F \]

\[ G \]

\[ H \]

\[ I \]

\[ J \]

\[ K \]

\[ L \]

\[ M \]

\[ N \]

\[ O \]

\[ P \]

\[ Q \]

\[ R \]

\[ S \]

\[ T \]

\[ U \]

\[ V \]

\[ W \]

\[ X \]

\[ Y \]

\[ Z \]

\[ a \]

\[ b \]

\[ c \]

\[ d \]

\[ e \]

\[ f \]

\[ g \]

\[ h \]

\[ i \]

\[ j \]

\[ k \]

\[ l \]

\[ m \]

\[ n \]

\[ o \]

\[ p \]

\[ q \]

\[ r \]

\[ s \]

\[ t \]

\[ u \]

\[ v \]

\[ w \]

\[ x \]

\[ y \]

\[ z \]
where \( y'_{XD} \) = Deflection at any section \( X \) due to unit couple at \( D \).

\[ \phi_{DD} = \text{Rotation at } D, \text{ due to unit couple at } D. \]

From Fig. 5.15(c) considering the equilibrium of the portion to the right of the pin,

\[ M_D = 0 = (RC \times 2) - 4 \]

or

\[ RC = \frac{1}{2} \uparrow \]

Similarly, considering the equilibrium of the left portion,

\[ M_D = 0 = Ra \times 6 \uparrow - Ra \times 2 \times 1 \]

or

\[ 6Ra = 2Rb + 1 \]

Also, for the whole beam,

\[ Ra \uparrow + Rc \uparrow = Rb \downarrow \]

\[ Ra + Rc = Rb \]

From \((i)\) and \((ii)\), we get

\[ Ra = \frac{1}{4}, Rb = \frac{1}{4}, \text{ and } Rc = 1 \downarrow \]

Knowing all the three reactions, the bending moment at any section of the beam can be determined. The B.M.D. is a triangle having a maximum ordinate of \(-2 \text{kN-m} \) at \( B \). In order to find \( y'_{XD} \) (i.e., to determine the deflection curve), we shall use the conjugate beam method. Fig. 5.15(d) shows a corresponding conjugate beam loaded with \(-M/EI\) diagram. Thus, since \( M_{max} = -2 \text{kN-m} \), the loading on the conjugate beam will be triangular, having a maximum intensity of \(+2 \text{i.e., acting downwards} \) at \( B \), \( EI \) being omitted.

Since the real beam [Fig. 5.15(c)] has a zero deflection at \( B \), the conjugate beam will have a pin at the corresponding point \( B' \) so that the B.M. there, representing deflection at \( B \), is zero. Let us first determine the reactions \( Ra', Rb', \) and \( Rc' \) of the conjugate beam. Taking moments about the pin \( B' \) and considering the equilibrium of the left portion, we get

\[ Ra' = \frac{1}{4} \left( \frac{1}{2} \times 4 \times 2 \times \frac{4}{3} \right) = \frac{4}{3} \uparrow \]

Considering the equilibrium of the right portion, we have

\[ 2Rb' + 4Rc' = \frac{1}{2} \times 4 \times 2 \times \frac{4}{3} \]

or

\[ Rb' + 2Rc' = \frac{8}{3} \]

Also, \( Ra' + Rb' + Rc' = \frac{1}{2} \times 8 \times 2 = 8 \)

or

\[ Rb' + Rc' = 8 - Ra' = 8 - \frac{4}{3} = \frac{20}{3} \]

\[ (ii) \]

From \((i)\) and \((ii)\), we get

\[ Rb' = -\frac{32}{3} \uparrow \text{ and } Rc' = -4, \text{ i.e., acting } \downarrow \]

\[ \phi_{DD} = \text{relative change in angle at } D \]

\[ = \text{sum of shears in conjugate beam to the right and left of the support } D \]

\[ = R_D' = \frac{32}{3}. \]

The calculation of \( Mx \) and \( Md \) are done in the tabular form below. (Table 5.4)

**TABLE 5.4**

<table>
<thead>
<tr>
<th>( x ) (m)</th>
<th>( Mx = y'_{XD} )</th>
<th>( Md = \frac{y'<em>{XD}}{\phi</em>{DD}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((\frac{-4}{3} \times 1) + (\frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{3})) = (-\frac{15}{12})</td>
<td>(-\frac{15}{12} \times \frac{3}{32} = -0.177)</td>
</tr>
<tr>
<td>2</td>
<td>((\frac{-4}{3} \times 2) + (\frac{1}{2} \times 2 \times \frac{1}{2} \times \frac{1}{3})) = (-2)</td>
<td>(-\frac{2}{3} \times \frac{3}{32} = -0.175)</td>
</tr>
<tr>
<td>3</td>
<td>((\frac{-4}{3} \times 3) + (\frac{1}{2} \times 3 \times \frac{3}{3} \times \frac{3}{3})) = (-\frac{7}{4})</td>
<td>(-\frac{7}{4} \times \frac{3}{32} = -0.164)</td>
</tr>
<tr>
<td>4</td>
<td>((\frac{-4}{3} \times 4) + (\frac{1}{2} \times 4 \times 2 \times \frac{4}{3})) = 0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>((4 \times 3) - (\frac{-3}{3} \times 1) + (\frac{1}{2} \times 3 \times \frac{3}{3} \times \frac{3}{3})) = (-\frac{43}{12})</td>
<td>(-\frac{43}{12} \times \frac{3}{32} = 0.336)</td>
</tr>
<tr>
<td>6</td>
<td>((4 \times 2) + (\frac{1}{2} \times 2 \times 1 \times \frac{3}{3})) = (-\frac{26}{3})</td>
<td>(-\frac{26}{3} \times \frac{3}{32} = 0.813)</td>
</tr>
<tr>
<td>7</td>
<td>((4 \times 1) + (\frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{3})) = (-\frac{49}{12})</td>
<td>(-\frac{49}{12} \times \frac{3}{32} = 0.383)</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The influence line diagram for \( Md \) is shown plotted in Fig. 5.15(d).

**Alternative Solution**

The deflection curve or the value of \( y'_{XD} \) can also be determined by the conventional double integration method used in example 5.3.

Refer Fig. 5.15(c), where \( Ra = \downarrow, Rb = \downarrow, \) and \( Rc = \uparrow \). Since there is discontinuity in the beam at the pin at \( D \), we will treat the spans \( AD \) and \( DC \) separately.
(i) For the span AD,
Measuring x from the L.H. support A, we get
\[ EI \frac{d^2y}{dx^2} = -\frac{x^3}{2} + (x-4) \]
\[ \therefore EI \frac{dy}{dx} = -\frac{x^4}{4} + C_1 + \frac{(x-4)^3}{6} \]
and
\[ Ely = -\frac{x^3}{12} + C_1 x + C_2 + \frac{(x-4)^3}{6} \]
At \( x = 0 \), \( y = 0 \) \( \therefore C_1 = 0 \)
At \( x = 4 \), \( y = 0 \) \( \therefore C_1 = \frac{64}{12} + 4C_1 \)
\( \therefore C_1 = \frac{64}{12 \times 4} = \frac{4}{3} \)
Hence the slope and deflection equations for span AD are:
\[ EI \frac{dy}{dx} = -\frac{x^4}{4} + \frac{4}{3}x + \frac{(x-4)^3}{2} \] (I)
and
\[ Ely = -\frac{x^3}{12} + \frac{4}{3}x + \frac{8}{6} = -\frac{26}{3} \] (II)
At \( x = 6 \text{ m} \), \( Ely = -\frac{216}{12} + \frac{24}{3} + \frac{8}{6} = -\frac{26}{3} \)
and
\[ \left( EI \frac{dy}{dx}\right)_{AD} = -9 + \frac{4}{3} + 2 = -\frac{17}{3} \]
(ii) For the span DC
Shear at pin D, just to its right, is equal to \( \frac{1}{2} \) (i.e. shear at D is equal and opposite to \( R_C \)).
Measuring x from D, to right
\[ EI \frac{d^2y}{dx^2} = \frac{1}{2} \cdot x - 1 \]
\[ EI \frac{dy}{dx} = \frac{x^2}{4} - x + C_1 \]
and
\[ Ely = \frac{x^3}{12} - \frac{x^2}{2} + C_1 x + C_2 \]
At \( x = 0 \), \( Ely = -\frac{26}{3} \)
\( \therefore C_2 = -\frac{26}{3} \)
At \( x = 2 \text{ m} \), \( Ely = 0 \)
\( \therefore C_1 = +5 \)

The Muller-Breslau Principle
Hence the slope and deflection equations for span DC are:
\[ EI \frac{dy}{dx} = \frac{x^4}{4} - x + 5 \] (III)
and
\[ Ely = \frac{x^3}{12} + \frac{x^2}{2} + 5x - \frac{26}{3} \] (IV)
At \( x = 0 \), \( \left( EI \frac{dy}{dx}\right)_{DC} = +5 \)
Now, \( \phi_{DD} \) = relative change in this angle at \( D \)
\[ = \left( \frac{dy}{dx}\right)_{AD} - \left( \frac{dy}{dx}\right)_{DC} \]
\[ = \frac{1}{EI} \left[ -\frac{17}{3} - 5 \right] = -\frac{1}{EI} \frac{32}{3} \]
The calculation of \( y'_{XD} \) is done in table 5.5. The term \( \frac{1}{EI} \) has been omitted for convenience.

<table>
<thead>
<tr>
<th>Distance from A (m)</th>
<th>Eq. No.</th>
<th>( y'_{XD} )</th>
<th>( M_D \cdot \frac{y'<em>{XD}}{\phi</em>{DD}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>II</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(x=0)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>II</td>
<td>15</td>
<td>-0.117</td>
</tr>
<tr>
<td>(x=1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>II</td>
<td>-2</td>
<td>-0.1875</td>
</tr>
<tr>
<td>(x=2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>II</td>
<td>-0.164</td>
<td></td>
</tr>
<tr>
<td>(x=3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>II</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(x=4)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>II</td>
<td>-0.336</td>
<td></td>
</tr>
<tr>
<td>(x=5)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>IV</td>
<td>-0.813</td>
<td></td>
</tr>
<tr>
<td>(x=6)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>IV</td>
<td>-0.383</td>
<td></td>
</tr>
<tr>
<td>(x=1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>IV</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(x=2)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 5.6. Determine the influence line for the shear force at D, the middle point of span BC, of a continuous beam shown in Fig. 5.16 (a). Compute the ordinates at 1 m interval.

Solution

In order to plot the influence line for shear force $F_D$, the shearing resistance of the beam is first removed by inserting a sliding device which permits the relative movement between the two parts but does not impair the moment resistance. Thus the slide device is such that it maintains the same slope in the distorted beam to either side of the device. The unit load (external) is removed and a pair of the unit loads (unit shear) is applied at D. The beam will then distort as shown in Fig. 5.16 (c). The S.F. at D is given by

$$F_D = \frac{Y_{ID}}{Y_{DD}} \quad \text{(1)}$$

where $Y_{ID} =$ deflection of beam at X due to unit shear at D.

$$Y_{DD} = \text{total relative movement at D due to unit shear at D.}$$

Considering the equilibrium of the portion DC to the right of the cut, we have

$$R_C = 1 \uparrow \text{ and } M_D = 1 \times 2 = 2 \text{ kN-m}$$

Similarly, considering the equilibrium of the portion to the left of the cut, and taking moments about A, we get

$$4R_B = M_D + 1 \times 6 = 2 + 6 = 8$$
$$R_B = 2 \downarrow$$

Hence $R_A = 2 - 1 = 1 \uparrow$

The bending moment diagram will be a triangle having a maximum ordinate of $-4 \text{ kN-m}$ at B.

(a) Solution by conjugate beam method

Fig. 5.16 (d) shows the conjugate beam loaded with $-M'/E$ diagram. Omitting $EI$ for convenience, the loading diagram will also be triangle having maximum ordinate of 4 at B, the load acting downwards. In addition to this loading, an unknown moment load $\mu$ will also act at D' of the conjugate beam, to satisfy the condition that the slope at both the sides of the real beam is the same. Since the slope at the real beam is represented by the shear of the conjugate beam, the shear just to the right of D' must be equal to the shear to the left of D'. This condition is satisfied by the moment $\mu$ acting at D' of the conjugate beam. There is a pin B' corresponding to the support B of the real beam.

Consider the equilibrium of the portion to the left of hinge B'.

$$\therefore \quad R_A' = \frac{1}{4} \left(\frac{1}{2} \times 4 \times 4 \times \frac{3}{4}\right) = \frac{8}{3} \uparrow$$

$$\therefore \quad R_C' = \text{total load } - R_A' = \left(\frac{1}{2} \times 8 \times 4\right) - \frac{8}{3} = \frac{40}{3} \uparrow$$

Again, considering the equilibrium of the portion to the right of pin B', and taking moment about B', we get

$$\mu + \left(\frac{1}{2} \times 4 \times 4 \times \frac{3}{4}\right) = \frac{40}{3} \times 4$$

$$\therefore \quad \mu = \frac{160}{3} - \frac{32}{3} = \frac{128}{3} \uparrow$$

Since the bending moment of the conjugate beam represents the deflection of the corresponding point of the real beam, the
moment \( \mu \) represents the relative movement of the two parts of the slide. Hence

\[ Y_{DD} = \mu = \frac{128}{3} \]

The calculations of \( M_x \) (and hence \( Y_{XD} \)) and \( F_D \) are done in the tabular form below (Table 5:6).

**Table 5:6**

<table>
<thead>
<tr>
<th>( x ) (m)</th>
<th>( M_x = Y_{XD} )</th>
<th>( F_D = Y_{XD} Y_{DD} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( \left( -\frac{8}{3} \times 1 \right) + \left( \frac{1}{2} \times 1 \times 1 \times \frac{1}{3} \right) = -\frac{5}{2} )</td>
<td>( -\frac{5}{3} \times 3 ) ( \frac{1}{128} = -0.059 )</td>
</tr>
<tr>
<td>2</td>
<td>( \left( -\frac{8}{3} \times 2 \right) + \left( \frac{1}{2} \times 2 \times 2 \times \frac{2}{3} \right) = -4 )</td>
<td>( -\frac{4}{3} \times 3 ) ( \frac{1}{128} = -0.094 )</td>
</tr>
<tr>
<td>3</td>
<td>( \left( -\frac{8}{3} \times 3 \right) + \left( \frac{1}{2} \times 3 \times 3 \times \frac{3}{3} \right) = -3.5 )</td>
<td>( -\frac{3.5}{3} \times 3 ) ( \frac{1}{128} = -0.082 )</td>
</tr>
<tr>
<td>4</td>
<td>( \left( -\frac{8}{3} \times 4 \right) + \left( \frac{1}{2} \times 4 \times 4 \times \frac{4}{3} \right) = 0 )</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>( \left( -\frac{40}{3} \times 3 \right) + \left( \frac{1}{2} \times 3 \times 3 \times \frac{3}{3} \right) + \frac{128}{3} = -43 )</td>
<td>( \frac{43}{3} \times 3 ) ( \frac{1}{128} = 0.168 )</td>
</tr>
<tr>
<td>6 (left)</td>
<td>( \left( -\frac{40}{3} \times 2 \right) + \left( \frac{1}{2} \times 2 \times 2 \times \frac{2}{3} \right) + \frac{128}{3} = -52 )</td>
<td>( \frac{52}{3} \times 3 ) ( \frac{1}{128} = 0.406 )</td>
</tr>
<tr>
<td>6(right)</td>
<td>( \left( -\frac{40}{3} \times 2 \right) + \left( \frac{1}{2} \times 2 \times 2 \times \frac{2}{3} \right) = -76 )</td>
<td>( -\frac{76}{3} \times 3 ) ( \frac{1}{128} = -0.594 )</td>
</tr>
<tr>
<td>7</td>
<td>( \left( -\frac{40}{3} \times 1 \right) + \left( \frac{1}{2} \times 1 \times 1 \times \frac{1}{3} \right) = -79 )</td>
<td>( -\frac{79}{6} \times 3 ) ( \frac{1}{120} = -0.308 )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The L.L. for \( F_D \) is shown in Fig. 5:16(d).

**Alternative Solution**

We shall now determine the values of \( Y_{XD} \) for various values of \( x \), by the double integration method. Since the beam is discontinuous at \( D \), we will write the differential equations for both the portions separately. Refer Fig. 5:16 (c). The reactions, calculated earlier, are as follows:

\( R_A = 1 \uparrow \); \( R_B = 2 \downarrow \) and \( R_C = 1 \uparrow \)

**The Muller-Breslau Principle**

(i) For portion \( AD \)

Measuring \( x \) from \( A \), towards right,

\[ EI \frac{d^2y}{dx^2} = -x^2 + 2(x-4) \]

\[ EI \frac{dy}{dx} = -\frac{x^2}{2} + C_1 + (x-4)^2 \]

and

\[ EI y = -\frac{x^8}{6} + C_1 x + C_2 + \frac{(x-4)^8}{3} \]

At \( x = 0, y = 0 \); \( C_0 = 0 \)

At \( x = 4, y = 0 = -\frac{64}{6} + 4C_1 \); \( C_1 = -\frac{64}{24} = \frac{8}{3} \)

Hence the slope and deflection equations for portion \( AD \) are

\[ EI \frac{dy}{dx} = -\frac{x^2}{2} + \frac{8}{3} + (x-4)^2 \]

(1)

and

\[ EI y = -\frac{x^8}{6} + \frac{8}{3} x + \frac{(x-4)^8}{3} \]

(II)

At \( x = 6, \) (\( EI \frac{dy}{dx} \))\(_{DA} = \frac{-36}{2} + \frac{8}{3} + 4 = -\frac{34}{3} \)

\( (Ely)_{DA} = \frac{-216}{6} + 16 + \frac{8}{3} = -\frac{52}{3} \)

(ii) For portion \( DC \)

Measuring \( x \) from \( D \), towards right, we get

\[ EI \frac{d^2y}{dx^2} = -2 + (1 \times x) \]

\[ EI \frac{dy}{dx} = -2x + \frac{x^2}{2} + C_1 \]

\[ EI y = -x^3 + \frac{x^3}{6} + C_1 x + C_2 \]

\( x = 0, \) (\( EI \frac{dy}{dx} \))\(_{DC} = (EI \frac{dy}{dx})_A = -\frac{34}{3} = C_1 \)

\( x = 2, EI y = 0 = -4 + \frac{8}{3} + \frac{34}{3} \times 2 + C_2 \)

\( C_2 = \frac{76}{3} \)

Hence the slope and deflection for portion \( DC \) are

\[ EI \frac{dy}{dx} = -2x + \frac{x^2}{2} - \frac{34}{3} \]

(III)

\[ EI y = -x^3 + \frac{x^3}{6} - \frac{34}{3} x + \frac{76}{3} \]

(IV)

At \( x = 0, (EIy)_{DC} = -\frac{76}{3} \)
The calculations of $Y_{XD}$ and $F_D$ are done in the tabular form below (Table 5-7). The term $EI$ has been omitted for convenience.

Table 5-7

<table>
<thead>
<tr>
<th>Dist. from A (m)</th>
<th>Eq. No.</th>
<th>$Y_{XD}$</th>
<th>$F_D = \frac{Y_{XD}}{Y_{DD}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>II</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>II</td>
<td>$-\frac{1}{6} + \frac{8}{3} + \frac{5}{2}$</td>
<td>$-\frac{5}{2} \times \frac{1}{128} = -0.059$</td>
</tr>
<tr>
<td>2</td>
<td>II</td>
<td>$-\frac{8}{6} + \frac{16}{3} + 4$</td>
<td>$-\frac{4}{28} \times \frac{1}{128} = -0.094$</td>
</tr>
<tr>
<td>3</td>
<td>II</td>
<td>$\frac{27}{6} + \frac{24}{3} + \frac{7}{2}$</td>
<td>$\frac{7}{2} \times \frac{1}{128} = 0.093$</td>
</tr>
<tr>
<td>4</td>
<td>II</td>
<td>$-\frac{64}{6} + \frac{32}{3} + 0$</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>II</td>
<td>$-\frac{125}{6} + \frac{40}{3} + \frac{1}{2} + \frac{3}{6}$</td>
<td>$+\frac{43}{6} \times \frac{1}{128} = +0.168$</td>
</tr>
<tr>
<td>5 (left)</td>
<td>II</td>
<td>$\frac{216}{6} + \frac{48}{3} + \frac{8}{3} - \frac{52}{3}$</td>
<td>$-\frac{52}{3} \times \frac{1}{128} = -0.406$</td>
</tr>
<tr>
<td>6 (Right)</td>
<td>IV</td>
<td>$+\frac{76}{3}$</td>
<td>$-\frac{76}{3} \times \frac{1}{128} = -0.594$</td>
</tr>
<tr>
<td>7</td>
<td>IV</td>
<td>$-\frac{14}{6} + \frac{34}{3} + \frac{76}{6} + \frac{79}{6}$</td>
<td>$-\frac{76}{6} \times \frac{1}{128} = -0.308$</td>
</tr>
<tr>
<td>8</td>
<td>IV</td>
<td>$-\frac{4}{6} + \frac{8}{3} + \frac{76}{3} = 0$</td>
<td>0</td>
</tr>
</tbody>
</table>

5.8. FIXED BEAMS

1. I.L. for support moment

Let us now take a fixed beam $AB$, and plot the I.L. for support moment $M_A$ at $A$. Let the unit load be at section $X$, distant $x$ from $A$ [Fig. 5.17(a)]. As per Müller-Breslau principle, introduce a hinge at $A$, thus getting a basic determinate structure, which will deflect under the unit load at $X$, as shown in Fig. 5.17(b). Let $\phi_{AA}$ be the resulting rotation at $A$. Now remove the unit load and apply a unit moment at $A$, due to which the beam will deflect as shown in...
deflection at \( x \), due to unit moment applied at \( A \). Then
\[
M_A \cdot \phi_{AA} = \phi_{44}
\]
But \( \phi_{44} = y''x_A \), by reciprocal theorem.
Hence \( M_A \cdot \phi_{AA} = y''x_A \)
\[
M_A = \frac{y''x_A}{\phi_{AA}} \quad (5'9)
\]

Thus, the deflection of Fig. 5'17 (c), to some scale, gives the I.L. for \( M_A \). If \( \phi_{AA} \) is taken as unity, the deflection at any point \( X \) gives the I.L. for \( M_A \).

Thus, the basic problem is shown in Fig. 5'17 (d), wherein we have to find the value of deflection \( y''x_A \) at \( X \), due to unit moment applied at \( A \). We will solve the problem by the conjugate beam method. Let the reactive moment at end \( B \) be \( M \), due to unit moment applied at end \( A \). The component B.M.D. is shown in Fig. 5'17 (e). The conjugate beam \( A'B' \) along with the elastic loading (equal to \( M/EI \) diagram) is shown in Fig 5'17 (f).

For the conjugate beam, \( M_a' = 0 \), because of the hinge at \( A' \). Hence
\[
M_a' = 0 = \frac{1}{2} M \cdot L \left[ \frac{2}{3} - \frac{1}{2} \frac{L}{EI} \times \frac{1}{3} L \right]
\]
which gives \( M = \frac{1}{2} \).

For reaction \( R' a \) at \( A \), take moments at \( B' \) and equate to zero, since end \( B' \) of the conjugate beam is free.
\[
\therefore \ R' a \times L + \frac{1}{2} \times \frac{L}{EI} \times \frac{1}{2} L \times \frac{1}{3} L = 0
\]
which gives
\[
R' a = \frac{L}{4EI} \quad (\dagger)
\]
i.e.
\[
R' a = \frac{L}{4EI} \quad (\dagger)
\]

Now \( M_{x} \) of the conjugate beam will give \( y''x_A \) of the real beam.
\[
\therefore \ M_x = \frac{L}{4EI} x + \frac{1}{2} \left( \frac{x}{2EI} \right) x \times \frac{x}{3}
\]
\[
= \frac{x^3}{6} \left[ \frac{1}{2} \left( \frac{x}{L} \right) + \frac{2}{EI} \right] \]
\[
\text{or} \quad M_x = \frac{1}{12 EI} (3L^3 x - 6 x^2 L - 3x^3)
\]
\[
\therefore \ y''x_A = M_x = \frac{1}{12 EI} (3L^3 x - 6 x^2 L - 3x^3)
\]
\[
\text{Also,} \quad \phi_{AA} \text{ of real beam} = R' a = \frac{L}{4EI}
\]

THE MULLER-BRESLAU PRINCIPLE

Hence from Fig. 5'9,
\[
M_A = \frac{y''x_A}{\phi_{AA}} = \frac{1}{12 EI} \left[ 3L^3 x - 6 x^2 L - 3x^3 \right] \left[ \frac{4 EI}{L} \right]
\]
\[
\text{or} \quad M_A = -\frac{x}{L} (L-x)^2 \quad (5'10)
\]

The minus sign shows that \( M_A \) is opposite to the direction of unit moment applied at \( A \).
\[
\therefore \quad M_A = -\frac{x}{L} (L-x)^2
\]

Let us check this result by taking \( x = a \) and \((L-x) = b \) and by taking a point load \( W \) in place of unit load. In that case,
\[
M_A = \frac{W}{L^3} a b^3
\]
\[
\text{which matches with the well known result.}
\]

The I.L. for \( M_A \) is shown in Fig. 5'17 (g). For finding the maximum value of \( M_A \), we have
\[
\frac{dM_A}{dx} = 0 = \frac{1}{L^3} [L^2 + 3x^3 - 4Lx]
\]
\[
\text{or} \quad (L-x)(L-3x) = 0
\]

From which, we get \( x = L/3 \).
\[
\therefore \quad M_A = \frac{1}{L^3} \left( L - \frac{L}{3} \right)^3 = \frac{4}{27} L
\]

2. I.L. for Support reaction

Let us now plot I.L. for support reaction \( R_A \), for a fixed beam shown in Fig. 5'18 (a). Fig. 5'18 (b) shows the basic determinate structure by removing the support reaction \( R_A \), but by keeping fixidity intact at end \( A \) through an induced moment. The end \( A \) will deflect by an amount \( y_{AA} \), due to unit load placed at \( X \).

Now remove the unit load from \( X \), and place it at end \( A \), due to which the beam will deflect by \( y_{AA} \) at \( A \) and \( y_{44} \) at \( X \), as shown in Fig. 5'18 (c). From the method of consistent deformation,
\[
R_A, y_{AA} \text{ at } A \text{ and } y_{44} \text{ at } X
\]
\[
\therefore \quad R_A = \frac{y_{AA}}{y_{44}} \quad (5'11)
\]

Thus, the deflection curve of Fig 5'18 (c), gives, to same scale, the I.L. for \( R_A \). If \( y_{44} \) is selected as unity, the deflection curve gives the I.L. for \( R_A \), in which the ordinate \( y_{44} \) at any point \( X \), due to unit load placed at \( A \), gives the ordinate of I.L. diagram. Thus the basic problem, shown in Fig. 5'18 (d) lies in finding the value of deflection \( y_{44} \), due to unit load and a reactive moment \( M \) at end \( A \).
THE MULLER-BRESLAU PRINCIPLE

the real beam, represented by shear at $A'$ of the conjugate beam, is zero, we have $R'_{A}=0$ for the conjugate beam.

\[ R'_{A}=0 = \frac{1}{2} \cdot \frac{L}{E} \cdot \frac{1}{2} \cdot \frac{M}{E} \cdot L \]

From which $M=L$.

Again deflection $y_{xA}$ of the real beam is given by B.M. $M'x$ of the conjugate beam.

\[ y_{xA}=M'x = \frac{(L-x)^2}{6} \left( \frac{x}{E} + \frac{2L}{E} \right) \]

or

\[ y_{xA}=\frac{(L-x)^2}{6E} \left( x+2L \right) - \frac{1}{6E} (L-x)^2 \]

or

\[ y_{xA}=\frac{(L-x)^2}{6E} \frac{1}{(L+2x)} \]

At $x=0$, $y_{xA}=M''_{A} = \frac{L^2}{6E}$.

Now $R_{A} = \frac{y_{xA}}{y_{AA}} = \frac{(L-x)^2}{6E} \frac{1}{(L+2x)} \frac{6E}{L^2}$

or

\[ R_{A} = \frac{(L-x)^2}{L^2} \frac{1}{(L+2x)} \]

which gives the equation of I.L. for $R_{A}$.

At $x=0$, $R_{A}=1$ (as expected).

Check. For a single point load $W$ acting at $a$ from $A$ and $b$ from $B$, we have the well known expression

\[ R_{A} = \frac{Wb^2}{L^2} \cdot (3a+b) \]

Putting $W=1$, $a=x$ and $b=L-x$, we get

\[ R_{A} = \frac{(L-x)^2}{L^2} \frac{3x+L-x}{L^2} = \frac{(L-x)^2}{L^2} \frac{(L+2x)}{L^2} \]

which is the same as Eq. 5'12.

Fig. 5'18 (g) shows the I.L. for $R_{A}$, which is a third degree curve.

PROBLEMS

1. A beam $ABC$ of uniform section, length $2L$, is hinged at the collinear supports at its centre and ends. Derive the equation to the influence lines for bending moment at the central support. Taking $L=4$ m, plot the influence lines to scale indicating values at every quarter of each span.

2. A continuous beam $ABC$ is shown in Fig. 5'19. Compute the ordinate of the influence line for the reaction at $C$, at every quarter point of each span.
STRENGTH OF MATERIALS AND THEORY OF STRUCTURES

For the continuous beam shown in Fig. 5-20, draw the influence lines for reaction at A, B and C. Indicate the values at every quarter of each span.

Answers
1. $O_2 = 0; O_3 = +0.236; O_4 = +0.376; O_5 = +0.328; O_6 = 0$
   $O_4 = +0.328; O_5 = +0.376; O_6 = +0.236; O_7 = 0.$
2. $O_2 = 0; O_3 = -0.028; O_4 = -0.045; O_5 = -0.039; O_6 = 0$
   $O_4 = +0.149; O_5 = +0.386; O_6 = +0.680; O_7 = 1.$

3.

<table>
<thead>
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<th>Ordinate</th>
<th>$O_2$</th>
<th>$O_3$</th>
<th>$O_4$</th>
<th>$O_5$</th>
<th>$O_6$</th>
<th>$O_7$</th>
<th>$O_8$</th>
</tr>
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<tr>
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<td>+0.375</td>
<td>+0.1406</td>
<td>0</td>
<td>-0.0275</td>
<td>-0.0312</td>
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<tr>
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<td>+0.8750</td>
<td>+1.0781</td>
<td>+1.00</td>
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<td>+0.5938</td>
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<tr>
<td>$R_C$</td>
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<td>-0.2500</td>
<td>-0.2188</td>
<td>0</td>
<td>+0.1954</td>
<td>+0.4375</td>
</tr>
</tbody>
</table>

SECTION 2

STATICALLY INDETERMINATE STRUCTURES

Chapters
6. STATICALLY INDETERMINATE BEAMS AND FRAMES
7. THE GENERAL METHOD
8. THE THREE MOMENT EQUATION METHOD
9. SLOPE DEFLECTION METHOD
10. MOMENT DISTRIBUTION METHOD
11. COLUMN ANALOGY METHOD
12. METHOD OF STRAIN ENERGY
13. DEFORMATION OF PERFECT FRAMES
14. REDUNDANT FRAMES
15. CABLES AND SUSPENSION BRIDGES
16. ARCHES
61. INTRODUCTION

A structural system, generally called a structure, may be defined as an assembly of members such as bars, cables, arches, etc., with the purpose of transmitting external loads of the surrounding environment to the foundation. A structural system is in state of equilibrium if the constraints permit no rigid-body movement upon application of loads. The displacements or deformations are negligible in comparison with the dimensions of the structure. A structure is said to be stable when it deforms elastically and immediate elastic restraint is developed under the action of externally applied loads. The stability of a structure depends upon the number and arrangement of internal members and external reaction components. If a system does not have a sufficient number of internal, or external constraints it will undergo a rigid-body movement upon the application of a small displacement. Such a system is said to be statically unstable, and is usually referred to as a mechanism.

If a structure is stable under the action of forces acting in a plane, three conditions of equilibrium must be satisfied: \( \Sigma H = 0 \); \( \Sigma V = 0 \) and \( \Sigma M = 0 \), where \( \Sigma H \) is the algebraic sum of forces in horizontal or \( x \)-direction, \( \Sigma V \) is the algebraic sum of forces in vertical or \( y \)-direction and \( \Sigma M \) is the algebraic sum of moments of all the forces about a point. If the unknown forces (reactions and stress components) in the system can be determined by the equation of equilibrium alone, the system is said to be statically determinate structure. When the number of unknown reaction or stress (force) components exceeds the number of conditions of equilibrium, the system is said to be statically indeterminate or redundant structure and the excess restraints or members are described as redundants. In such cases, the
equations of static equilibrium alone cannot provide solution; they must be supplemented by the equations of compatibility of deformation. The degree of indeterminateness or redundancy is the number of unknown reactive restraints or stresses over and above the number of condition equations available for solution.

A redundant structure can further be classified into two categories: externally redundant system and internally redundant system. Externally redundant structures are those which have redundant restraints. Internally redundant structures are those which have redundant members and are overstiff. However, a redundant structure may have both external as well as internal redundancies.

The main difference between the redundant structures and the statically determinate ones resides in the fact that the stress distribution depends for the first ones not only on the loading but also on the relative dimensions of their members and on the properties of materials of which the members are made. Statically indeterminate structures are very sensitive to such factors as the settlement of their supports, temperature variation, lack of fitness of members which give rise to additional stresses, while the same factors would have no influence whatsoever on statically determinate structures. However, statically indeterminate structures are most widely used in various engineering activities.

6.2. TYPES OF SUPPORTS: REACTION COMPONENTS

There are three types of supports which may be encountered in plane structures (i) a roller support, (ii) a hinge support and (iii) a built-in or fixed supports. These are shown in Fig. 6.1 (a), (b) and (c) respectively.

A roller support consists of two rockers—the upper rocker and the lower rocker, with a pin in between permitting the rotation of the upper rocker with respect to the lower one. Both the rockers can move together on rollers along the bearing plate. Such a support supplies a reactive force which acts normal to the surface of rolling and is directed through the centre of the hinged pin. Thus, only one parameter of the reaction, i.e. its magnitude, has to be known in order to determine the reactions completely. Such support is also known as free end support or simple support.

A hinged support [Fig. 6.1 (b)] differs from the roller support by the fact that the lower rocker is fixed and cannot move. The reaction passes through the centre of the pin but its magnitude and direction is unknown. In other words, this support has two reaction components—the horizontal and the vertical. Schematically, a hinged support is represented by two hars connected by pin, or sometimes simply by a pin.

![Fig. 6.1](image)

External Stability

The built-in-support, shown in Fig. 6.1 (c), has zero degree of freedom. The determination of the reactions developed by this support requires the knowledge of three parameters—the direction and magnitude of a force passing through any chosen point (or its horizontal and vertical components) and the magnitude of the moment about the same point. Thus, a fixed support provides three reaction components.

6.3. EXTERNAL REDUNDANCY

For any structure, supported on external supports, the total reaction components can be easily found. The stability of a structure depends on the number and arrangement of the reaction components and component parts, rather than on the strength of the supports and parts of the structure. In general, three reaction components are necessary for the external stability of plane structures. This condition of three reaction components is necessary but not always sufficient. The arrangement of the three reaction components is very important from stability point of view. For example, if the lines of action of the three components are concurrent, the structure is externally unstable because the point of concurrency becomes the instantaneous centre of rotation giving a critical configuration. Similarly, a structure will also be the unstable if the three reaction components have parallel lines of action, since the structure does not have any resistance to horizontal motion.
For a plane structure, three equations of static equilibrium are available. In addition to this, extra condition equation may sometimes be available by special features of construction, such as internal pins or links. A pin [Fig. 6.2 (d)] provided anywhere in the structure cannot transmit moment from one part of the structure to the other part, and thus provides one additional condition equation: \( \Sigma M = 0 \) at pin. Similarly, a link [consisting of a short bar with a pin at each end, as shown in Fig. 6.2 (e)] provided anywhere in the structure is incapable of transmitting moment as well as horizontal force from one part to the other and thus provides two additional condition equations: \( \Sigma M = 0 \) and \( \Sigma H = 0 \) at the link. Thus total number of condition equations of statical equilibrium for any structure are equal to the three equations of statical equilibrium plus additional condition equations because of a pin or a link anywhere in the structure.

A structure is unstable if the total number of reaction components \( R \) are less than the total number of condition equations available. If the number of reaction components are equal to the condition equations, the structure is externally determinate. If, however, the number of reaction components are more than the condition equations the structure is statically indeterminate externally, the degree of indeterminacy or redundancy being equal to the number by which the reaction components exceed the condition equations, and is represented by the equation

\[
E = R - r
\]

where \( E \) = Degree of external redundancy
\( r \) = total number of condition equations available.

6.4. STATICALLY INDETERMINATE BEAMS

A continuous beam is a typical example of externally indeterminate structure. Fig. 6.2 shows some statically indeterminate beams.

Fig. 6.2 (a) shows a propped cantilever. For general loading the total reaction components \( R \) are equal to \( (3 + 2) = 5 \), while the total number of condition equations \( r \) are equal to 3. Hence the beam is statically indeterminate, externally, to second degree. However, for vertical loading, only two reaction components \( (M \text{ and } V) \) are available at the fixed end and one reaction component \( V \) available at the propped end, making the total reaction components \( R \) equal to 3, while the number of condition equations of statical equilibrium are only two \( (\Sigma M = 0 \text{ and } \Sigma V = 0) \), and the beam is statically indeterminate to single degree.

Fig. 6.2 (b) shows a fixed beam with 6 reaction components, and three condition equations. For the general system of loading, therefore, a fixed beam is statically indeterminate to third degree. However, for vertical loading on the beam, the total reaction components are four only (two at each joint), and only two condition equations are available making the beam statically indeterminate to second degree only.

In Fig. 6.2 (c), the total reaction components are equal to \( (3 + 1 + 1 + 1) = 6 \) while the condition equations are three. Hence the beam is statically indeterminate to third degree, for the general system of loading.

In Fig. 6.2 (d), there is a pin at \( A \). The total number of condition equations are equal to three equations of statical equilibrium plus one condition equation (i.e. \( \Sigma M = 0 \)) at the pin at \( A \), making a total of 4. The reaction components are equal to \((1 + 2 + 2) = 5\). The beam is statically indeterminate to single degree only.

Similarly, the beam of Fig. 6.2 (e) has a link \( BC \), giving two additional condition equations. The total number of condition equations are, therefore, equal to \( 3 + 2 = 5 \), while the reaction components are equal to \( 2 + 1 + 2 = 5 \). The beam is, therefore, statically determinate.

It should be noted that in the case of continuous beams (or statically indeterminate beams), the shear and moment at any point in the beam are readily known once the reaction components are determined. Thus, these beams are statically determinate internally. The degree of indeterminacy of a beam is therefore equal to its external redundancy.

6.5. DEGREE OF REDUNDANCY OF ARTICULATED STRUCTURES

A pin jointed frame or articulated structure is composed of a number of bars or straight members connected by frictionless pins,
forming geometrical figures which are usually triangles. A stable and determinate frame can be built up as an assemblage of triangles. The first triangle is made up of three joints and three members, and each successive triangle require two additional members and one additional joint. If \( j \) is the total number of joints and \( m \) is the total number of members, we get

\[
m = 3 + 2(j - 3)
\]

or

\[
m = 2j - 3 \tag{6.2}
\]

If, however, the total number of reaction components absolutely necessary for stability (and hence the total number of condition equations available) are \( r \) and not 3, the above equation can be written as

\[
m = 2j - r \tag{6.3}
\]

The above equation gives the criterion for finding the degree of internal indeterminacy or internal redundancy. A truss or frame is said to be statically determinate internally if it has members given by Eq. 6.3. If, however, the number of members in a frame are more than given by Eq. 6.3, the frame is said to have internal redundancy. If it has fewer members, it is unstable. The degree of internal redundancy \( I \) is therefore given by

\[
I = m - (2j - r) \tag{6.4}
\]

The degree of external indeterminacy \( E \) is given by Eq. 6.1. Hence the total redundancy or indeterminateness which is equal to the sum of external indeterminateness and internal indeterminateness, is given by

\[
T = E + I = (R - r) + (m - (2j - r))
\]

or

\[
T = m + R - 2j \tag{6.5}
\]

where

\[
R = \text{degree of redundancy (total)}
\]

\[
m = \text{total number of reaction components.}
\]

Fig. 6.3 shows some typical articulated structures.

(i) In Fig. 6.3(a),

\[ j = 6 \quad m = 9 \; \quad R = 2 + 1 = 3 \; \quad r = 3 \]

Thus the frame is statically determinate both externally as well as internally and is stable.

(ii) In Fig. 6.3(b),

\[ j = 8 \quad m = 15 \quad R = 2 + 2 = 4 \; \quad r = 3 \]

\[ E = 3 - 3 = 0 \]

\[ I = m - (2j - r) = 9 - (2 \times 6 - 3) = 0 \]

\[ T = E + I = 0 \]

(iii) In Fig. 6.3(c),

\[ j = 5 \; \quad m = 6 \]

\[ R = 4 \; \quad r = 3 + 1 \; \text{(due to hinge at A)} = 4 \]

\[ E = 4 - 4 = 0 \]

\[ I = m - (2j - r) = 6 - (2 \times 5 - 4) = 0 \]

\[ T = E + I = 0 \]

(Also, \( T = m + R - 2j = 6 + 4 - 10 = 0 \).)

The frame is thus statically determinate, both externally as well as internally.
6. DEGREE OF REDUNDANCY OF RIGIDLY JOINTED FRAMES

In the case of pin-jointed frames, the members carry axial forces only and hence two equations are available at each joint. The members of stiff-jointed frames, on the other hand, resist thrust, shear and bending moment. Therefore, three equations are available at each joint.

Fig. 6:4 shows a stiff-jointed frame. Assuming that the reaction components are known for the purposes of determining internal indeterminacy, and treating column AB as free body, the thrust, shear and moment in AB are known. Consider joint B at which there are total nine unknowns (i.e. thrust, shear and moment each for BA, BC and BE), out of which three unknowns of BA are known and three conditions of static equilibrium at joint B can be arbitrarily assigned to the three unknowns of BC. Thus, there remain three unknowns in the member BE at joint B. Once the internal stresses in BE are determined, the total number of knowns at joint E are six (three for ED and three for EG), out of which three equations of statical equilibrium of joint E can be arbitrarily assigned to the unknowns of the member ED. Thus there remain three unknowns in the member EG at the joint E. Knowing the stress components in BC and ED can be easily determined. Thus, on the whole, there are 6 unknowns (3 for BE and 3 for EG) to be determined and the frame is internally indeterminate to 6th degree.

In general, therefore, the degree of indeterminacy I can be represented by the formula : 

\[ I = 3a \]

(6.6)

where \( a \) is the number of areas completely enclosed by members of the frame. In Fig. 6:4, \( a = 2 \), and hence \( I = 3 \times 2 = 6 \) as determined above. The above formula is also applicable to continuous beams (Fig. 6:2), where \( a = 0 \) and hence \( I = 0 \). i.e., a continuous beam is statically determinate internally since the moment and shear at any point on the beam can be readily determined once the external redundant reactions are determined.

The external indeterminateness is given by Eq. 6.1.

\[ E = R - r \]

(6.7)

Hence the total indeterminateness or redundancy is given by :

\[ T = E + I = (R - r) + 3a = (R - 3) + 3a \]

(6.8)

Fig. 6:5 shows some stiff-jointed structures.

(i) In Fig. 6:5(a),

\[ R = 3 \times 3 = 9 \quad a = 2 \]

\[ E = R - 3 = 9 - 3 = 6 \]

\[ I = 3a = 3 \times 2 = 6 \]

\[ T = E + I = 6 + 6 = 12 \]

Thus the structure is statically indeterminate to twelfth degree.

(ii) In Fig. 6:5(b),

\[ R = 2 + 2 + 2 = 6 \quad a = 3 \]

\[ E = R - 3 = 6 - 3 = 3 \]

\[ I = 3a = 3 \times 3 = 9 \]

\[ T = E + I = 3 + 9 = 12 \]

(iii) In Fig. 6:5(c),

\[ R = 2 + 2 + 4 = 8 \quad a = 0 \]

\[ E = R - 3 = 8 - 3 = 5 \]

\[ I = 3 \times 0 = 0 \] (i.e., the structure is statically determinate internally)

\[ T = 1 \]
There are two basic methods available for analysing statically indeterminate structures: (i) compatibility method, and (ii) equilibrium method.

Compatibility method. This method is also sometimes known as flexibility coefficient method or force method. In this method the redundant forces are chosen as unknowns and additional equations are obtained by considering the geometrical conditions imposed on the deformations of the structures. The common methods that fall under this category are: the method of consistent deformation (or the general method), three moment theorem, column analogy method, elastic centre method, Maxwell-Mohr equations, Castigliano's theorem of minimum strain energy, etc.

Equilibrium method. This method is also known as deformation method or stiffness coefficient method. In this method, displacements of joints are taken as unknowns. The equilibrium equations are expressed in terms of these displacements and the external loads are solved to give the actual joint displacements from which redundant forces can be computed. The common methods that fall under this category are: Slope-deflection method, moment distribution method, minimum potential energy method, etc.

The choice between compatibility method and equilibrium method largely depends upon the type of structure and the manner in which it is supported. For example, consider a continuous beam loaded as shown in Fig. 6.6(a). For the case of vertical loads shown, the reactions at A, B, C and D will be vertical—making a total of 4 reaction components while the only two equations (i.e., $\Sigma V = 0$ and $\Sigma M = 0$) are available from statical equilibrium. Hence the structure is statically indeterminate to second degree and any two reactions can be taken as unknowns for the compatibility method, and only two compatibility equations will be required. On the other hand, if equilibrium methods were to be used, there are four unknown joint rotations ($\theta_A$, $\theta_B$, $\theta_C$, and $\theta_D$) and four equilibrium equations will have to be formulated and solved. The compatibility method will therefore be preferred for this beam. Now take the case of the same beam but fixed at A and D as shown in Fig. 6.6(b). There will be two additional moments (unknowns), making total reaction components equal to $4 + 2 = 6$. Since only two equations ($\Sigma V = 0$ and $\Sigma M = 0$) are available from statical equilibrium, the beam is statically indeterminate to the fourth degree. In the compatibility method four equations will have to be formulated and solved for any four unknown reactions. On the other hand, if equilibrium method were to be used, only two equations in terms of joints rotations $\theta_A$ and $\theta_C$ need be solved since $\theta_B$ and $\theta_D$ are each zero. Though the fixidity of joints A and D have increased the redundants to four, the joint displa-
ments are reduced by two. The equilibrium method will be more suitable to this case.

PROBLEMS

1. Find the degree of indeterminateness of the beams shown in Fig. 6-7 for the general case of loading.

![Diagram of beams](image)

Fig. 6-7

2. Find the degree of internal, external and total indeterminateness of the pin-pointed frames shown in Fig. 6-8.

![Diagram of frames](image)

Fig. 6-8

3. Find the degree of redundancy of the stiff-jointed frames shown in Fig. 6-9.

![Diagram of frames](image)

Fig. 6-9

Answers

1. (a) $E=3$  
   (b) $E=0$  
   (c) $E=4$

2. (a) $E=0; I=0; T=1$  
   (b) $E=0; I=0; T=0$  
   (c) $E=1; I=0; T=1$.

3. (a) $T=12$  
   (b) $T=0$  
   (c) $T=13$.

Answers

1. (a) $E=3$  
   (b) $E=0$  
   (c) $E=4$

2. (a) $E=0; I=0; T=1$  
   (b) $E=0; I=0; T=0$  
   (c) $E=1; I=0; T=1$.

3. (a) $T=12$  
   (b) $T=0$  
   (c) $T=13$. 
7

The General Method
(METHOD OF CONSISTENT DEFORMATION)

7.1. INTRODUCTION

The general method, or the method of consistent deformation, as is sometimes known, is credited to Clerk Maxwell (1864), Otto Mohr (1874) and Muller-Breslau (1886). This chapter, however, deals with the Muller-Breslau version of the general method in which the condition equations of geometrical coherence of a structure are obtained by superposition of displacements as caused by the applied loads and individual redundant stresses and reactions. The degree of redundancy of a structure is equal to the number of excess reaction components over those required for statical equilibrium. The method essentially consists in replacing the redundant reaction components by unknown force or moment reactions and then writing the condition equations for geometrical coherence of the structure by the superpositions of the displacements as caused by the applied loads and the unknown redundant force and/or moments. The structure obtained by replacing the redundant reactions by unknown forces or moments is known as a basic determinate structure. A number of such basic determinate structures can be obtained out of the original redundant structure, depending upon the choice of the redundant reaction component(s) to be replaced.

7.2. STATICALLY INDETERMINATE BEAMS AND FRAMES

The statically indeterminate beam or frame is analysed by the method of consistent deformation by first obtaining a basic determinate structure. The condition equations for geometrical coherence of a structure with various types of supports are as follows:

1. If a roller support is removed, the deflection in the direction perpendicular to the plane of rolling must be zero.

2. If a hinged support is removed, the deflection in the vertical and horizontal directions at the point must be zero.

3. If a fixed support is removed, the rotation as well as the vertical and horizontal deflections at the point must be zero.

There are, thus as many physical conditions of geometry as there are redundant reaction components.

Example 7.1: A cantilever of uniform flexural stiffness is proposed at the remote end. Find the load on the prop when a force \( W \) is applied at the centre of the cantilever.

Solution

Let \( V_B = \) Redundant reaction (vertical) at \( B \).

The basic determinate structure is obtained by replacing the prop at \( B \) by an unknown vertical reaction \( V_B \) as shown in Fig. 7.1(b).

Let \( \Delta_B = \) Deflection of the end \( B \) of the basic determinate structure due to external loading [Fig. 7.1(c)].
\[ \delta_{BB} = \text{Deflection of the end } B \text{ of the basic determinate structure due to unit load at } B \text{ (The first suffix denotes the point where the deflection is reckoned and the second suffix denotes the position of the unit point load) (Fig. 7'1(e)).} \]

From conditions of geometry at \( B \), we get
\[ \Delta_B + V_B \cdot \delta_{BB} = 0 \quad (7'1) \]

The deflection \( \Delta_B \) and \( \delta_{BB} \) can be obtained either by the area moment method or the conjugate beam method. Fig. 7'1(e) shows the load \( W \) acting on the beam, with \( \Delta_B \) as the deflection of the end \( B \); the corresponding bending moment diagram is shown in Fig. 7'1(d).

\[
\Delta_B = \frac{1}{EI} \sum A \delta = \frac{1}{EI} \left( \frac{1}{2} \cdot \frac{WL}{2} \cdot \frac{L}{2} \right) \left( \frac{L}{2} + \frac{2}{3} \cdot \frac{L}{2} \right)
\]

\[
= \frac{5}{48} \frac{WL^3}{EI}
\]

Fig. 7'1(e) and (f) shows the unit load acting at \( B \) and the corresponding B.M.D. respectively.

\[ \therefore \delta_{BB} = \frac{1}{EI} \sum A \delta = \frac{1}{EI} \left( -\frac{1}{2} \cdot L \cdot L \right) \left( \frac{2}{3} L \right) = -\frac{L^3}{3EI} \]

Substituting in Eq. 7'1, we get
\[
\frac{5}{48} \frac{WL^3}{EI} - V_B \cdot \frac{L^3}{3EI} = 0
\]

or
\[
V_B = \frac{5}{16} W
\]

**Alternative Solution**

An alternative basic determinate structure can be obtained by treating moment at \( A \) as redundant. The basic determinate structure, thus, has end \( A \) as simply supported with an unknown moment \( M_A \) acting, as shown in Fig. 7'2(b).

Let
\[ \theta_A = \text{slope at end } A \text{ of the basic determinate structure, due to the external loading as shown in Fig. 7'2(c).} \]
\[ \phi_{AA} = \text{slope at end } A \text{ of the basic determinate structure, due to unit moment acting at } A \text{ as shown in Fig. 7'2(e).} \]

Then by conditions of geometry at \( A \),
\[ \theta_A + M_A \cdot \phi_{AA} = 0 \quad (7'2) \]

\[ \phi_{AA} = -\frac{2}{3} \left[ \frac{1}{2} \cdot L \cdot \frac{L}{4} \right] \frac{1}{EI} = -\frac{L}{3EI} \]

Substituting in Eq. 7'2, we get
\[
\frac{W}{16 EI} - M_A \cdot \frac{L}{3EI} = 0
\]

\[ M_A = \frac{3WL}{16} \]
Knowing $M_A, V_A$ can be found by taking moments at $B$

\[ A \quad W - \frac{L}{2} + M_A - V_A L = 0 \]

\[ \therefore \quad V_A = \frac{W}{2} + \frac{M_A}{L} = \frac{W}{2} + \frac{3W}{16} = \frac{11W}{16} \]

\[ \therefore \quad V_B = \frac{W - V_A}{16} W = \frac{5W}{16} \]

**Example 7.2.** A beam $AB$ of span 4 m is fixed at $A$ and $B$ and carries a point load of 5 kN at a distance of 1 m from end $A$. Calculate the support moments by the method of consistent deformation.

**Solution**

From condition of geometry at $A$ and $B$, we get

\[ \theta_A + M_A \cdot \phi_A + M_B \cdot \phi_{AB} = 0 \quad [7.3(a)] \]

\[ \theta_B + M_A \cdot \phi_B + M_B \cdot \phi_{BB} = 0 \quad [7.3(b)] \]

Let us use conjugate beam method for the calculations of $\theta_A, \phi_A, \phi_{AB}, \phi_{BB}, \phi_{BA}$. Fig. 7.3 (d), (f) and (h) show the bending moment diagrams for the external load, unit couple at $A$ and unit couple at $B$ respectively. These B.M. diagrams become the loading for the conjugate beam. The slope at any point of real beam is equal to the shear force at the corresponding point of the conjugate beam. The sign convention for positive and negative shear force is shown in Fig. 7.4.

**Fig. 7.3**

**Fig. 7.4**

From Fig. 7.3(d), we get

\[ \theta_A = \text{shear at end } A \text{ of the conjugate beam} \]

\[ = \frac{1}{4} \left( \frac{1}{2} \times \frac{15}{4} \times 4 \right) \left( \frac{4+1}{3} \right) \frac{1}{EI} = \frac{35}{8} \frac{1}{EI} \]

\[ \theta_B = \text{shear at end } B \text{ of the conjugate beam} \]

\[ = -\frac{1}{4} \left( \frac{1}{2} \times \frac{15}{4} \times 4 \right) \left( \frac{4+1}{3} \right) \frac{1}{EI} = -\frac{25}{8} \frac{1}{EI} \]

Similarly from Fig. 7.3 (f)

\[ \phi_{AA} = -\frac{1}{4} \left( \frac{1}{2} \times 4 \times 1 \right) \frac{2}{3} \frac{1}{EI} = -\frac{4}{3} \frac{1}{EI} \]

\[ \phi_{BA} = +\frac{1}{4} \left( \frac{1}{4} \times 4 \times 1 \right) \frac{1}{3} \frac{1}{EI} = \frac{2}{3} \frac{1}{EI} \]
And, from Fig. 7.3(b) -
\[
\phi_{AB} = \frac{-1}{4} \left( \frac{1}{2} \times 4 \times 1 \right) \frac{4}{3} \frac{1}{EI} = -\frac{2}{3} \frac{1}{EI}
\]
\[
\phi_{BB} = \frac{1}{4} \left( \frac{1}{2} \times 4 \times 1 \right) \frac{2}{1} \frac{4}{3} \frac{1}{EI} = +\frac{1}{3} \frac{1}{EI}
\]
Substituting these values in Eq. 7.3(a) and 7.3(b), we get
\[
\frac{35}{8} \frac{1}{EI} - M_A \left( \frac{4}{3} \frac{1}{EI} \right) - M_B \left( \frac{2}{3} \frac{1}{EI} \right) = 0
\]
and
\[
\frac{25}{8} \frac{1}{EI} + M_A \left( \frac{2}{3} \frac{1}{EI} \right) + M_B \left( \frac{4}{3} \frac{1}{EI} \right) = 0
\]
Solving Eqs. (1) and (2), we get
\[
M_A = +\frac{45}{16} \text{kN-m} \quad \text{and} \quad M_B = +\frac{15}{16} \text{kN-m}.
\]
The final B.M.D. for the beam is shown in Fig. 7.3(i)(a).

7.3. MAXWELL’S LAW OF RECIPROCAL DEFLECTION

As applied to beam deflections and rotations, Maxwell’s theorem of reciprocal deflections has the following three versions:

(1) The deflection at \( A \) due to unit force at \( B \) is equal to deflection at \( B \) due to unit force at \( A \) [Fig. 7.5(a)].

Thus,
\[
\delta_{BA} = \delta_{AB} \quad (7.4)
\]

(2) The slope at \( A \) due to unit couple at \( B \) is equal to the slope at \( B \) due to unit couple at \( A \) [Fig. 7.5(b)].

Thus,
\[
\phi_{BA} = \phi_{AB} \quad (7.5)
\]

(3) The slope at \( A \) due to unit load at \( B \) is equal to deflection at \( B \) due to unit couple at \( A \) [Fig. 7.5(c)].

Thus,
\[
\phi_{BA} = \delta_{BA} \quad (7.6)
\]

Fig. 7.5

7.4. GENERALISED MAXWELL’S THEOREM : BETTI’S RECIPROCAL THEOREM

Generalised Statement. If an elastic system is in equilibrium under one set of forces with their corresponding displacements and if the same system is also in equilibrium under second set of forces acting through the same points with their corresponding displacements then the product of first group of forces and the corresponding displacements caused by second group is equal to the product of the second group of forces and the corresponding displacements caused by the first group.

\[
P_A \Delta A + P_B \Delta B = P'_A \cdot \Delta A + P'_B \Delta B \quad (7.7)
\]

where \( P \) and \( \Delta \) constitute first group of forces and their corresponding displacements, and \( P' \) and \( \Delta' \) constitute second group of forces and displacements.

Proof

By unit load method, \( \delta = \int \frac{M_{mx} dx}{EI} \) in general (See Vol. 1).

where

- \( M \) = bending moment at any point \( X \) due to external load.
- \( m \) = bending moment at any point \( X \) due to unit load applied at the point where deflection is required.

Let \( m_{xA} \) = bending moment at any point \( X \) due to unit load at \( A \).

\( m_{xB} \) = bending moment at any point \( X \) due to unit load at \( B \).

When unit load (external load) is applied at \( A \), \( M = m_{xA} \).

To find deflection at \( B \) due to unit load at \( A \), apply unit load at \( B \). Then \( m = m_{xB} \).

Hence
\[
\frac{M_{mx} dx}{EI} = \int \frac{m_{xA} \cdot m_{xB}}{EI} \quad (1)
\]

Similarly, when unit load (external load) is applied at \( B \), \( M = m_{xB} \).

To find the deflection at \( A \) due to unit load at \( B \), apply unit load at \( A \). Then \( m = m_{xA} \).

Hence
\[
\frac{M_{mx} dx}{EI} = \int \frac{m_{xB} \cdot m_{xA}}{EI} \quad (2)
\]

Comparing (1) and (2), we get
\[
\delta_{BA} = \delta_{AB}.
\]

Similarly, other versions of the reciprocal theorem can also be proved.
That is, the virtual work done by the first set of forces acting through the second set of displacements is equal to the virtual work done by the second set of forces acting through the first set of displacements.

In Betti's theorem, the symbols $P$ and $\Delta$ can also denote couples and rotations respectively, as well as forces and linear deflections, i.e., $M_A \Delta_A + M_B \Delta_B = M_A \Delta_B + M_B \Delta_A$ (7.8)

Thus, according to Betti's law, we have, in general

$$\Sigma P \cdot \Delta^t = \Sigma M \cdot \Delta^t = \Sigma P \cdot \Delta + \Sigma M \cdot \Delta$$ (7.9)

**Example 7.3.** A continuous beam $ABC$ is loaded as shown in Fig. 7.6 (a). Determine all reactions and draw B.M. and S.F. diagrams.

**Solution**

A basic determinate structure is obtained by replacing the central support by an upward force $V_B$ [Fig. 7.6 (b)].

Since there are three unknowns, i.e., $V_A$, $V_B$ and $V_C$, the beam is indeterminate to the first degree and the following condition equation will be used.

$$\Delta_B = V_B, \delta_{BB} \quad \text{(Numerically)} \quad (1)$$

Now

$$\Delta_B = W, \delta_{BD}$$

But from reciprocal deflections, $\delta_{BD} = \delta_{DB}$

Hence

$$\Delta_B = W, \delta_{BD}$$

Substituting in (1), we get the modified condition equation

$$W, \delta_{BD} = V_B, \delta_{BB} \quad (2)$$

When the unit load acts at $B$ [Fig. 7.6 (e)], the B.M. diagram will be a triangle having a maximum ordinate of $+\frac{W}{L}$ at $B$. Hence the conjugate beam [Fig. 7.6 (f)] loaded with $-\frac{M}{EI}$ diagram will be acted upon by a triangular load acting upwards.

From conjugate beam method, [Fig. 7.6 (e)] and [Fig. 7.6 (f)],

$$EI \delta_{BB} = \text{B.M. at } B \text{ due to loading of Fig. 7.6 (f)}$$

$$= \left( \frac{L^3}{4} \times L \right) - \left( \frac{1}{2} \times L \times \frac{L}{2} \right) \left( \frac{1}{3} \times L \right) = \frac{L^3}{6}$$

and

$$EI \delta_{DB} = \text{B.M. at } D \text{ due to loading of Fig. 7.6 (f)}$$

$$= \left( \frac{L^3}{4} \times \frac{L}{2} \right) - \left( \frac{1}{2} \times \frac{L}{2} \times \frac{L}{4} \right) \left( \frac{1}{3} \times 2 \right) = \frac{11L^3}{96}$$

Substituting in (2), we get

$$W \cdot \frac{11L^3}{96} = V_B \cdot \frac{L^3}{6}$$

Taking moment about $C$, we get

$$V_B = \frac{11}{16} \cdot W$$

or

$$V_B = \frac{11}{16} \cdot W$$

The general method

$$V_A = \frac{11}{16} \cdot \frac{W}{L} \cdot \frac{3}{2}$$

or

$$V_A = \frac{WL}{2L} \left( \frac{3}{2} \cdot \frac{11}{16} \right) = \frac{13}{32} \cdot W$$

Fig. 7.6.
Three Moment Equation Method

8.1. CLAPEYRON'S THEOREM OF THREE MOMENTS

General Statement

Let us consider two consecutive spans $AB$ and $BC$ of a continuous beam, loaded with any system of loading. The $\mu$ and $\mu'$ diagrams can be constructed as usual.

Fig. 8.1 (a) shows two consecutive spans $AB-BC$ of a continuous beam with any type of loading. Let suffix 1 (i.e. $L_1, \frac{E_1}{L_1}, I_1$, etc.) stand for span $AB$, and suffix 2 (i.e. $L_2, \frac{E_2}{L_2}, I_2$, etc.) stand for span $BC$.

\[
\begin{align*}
V_C &= W - \left( \frac{11}{16} W + \frac{13}{32} W \right) = \frac{32}{32} - \frac{22}{32} - \frac{13}{32} W \\
&= -\frac{3}{32} W = \frac{3}{32} W \\
\end{align*}
\]

B.M.D.

\[
M_A = 0, \quad M_B = -\frac{13}{32} W, \quad \frac{L}{2} = -\frac{13}{64} WL
\]

\[
M_B = -\frac{3}{32} WL
\]

The B.M.D. and S.F.D. have been shown in Fig. 7.6 (g) and 7.6 (h) respectively.

PROBLEMS

1. A beam is fixed at both the ends and carries a central point load. Find the support moments.

2. A cantilever of span $L$ is propped at the free end. Calculate the prop reaction if it carries a uniformly distributed load of $w$ per unit length.

3. A fixed beam of length $L$ is loaded at third points by two point loads of $W$ each. Calculate the fixing moments and plot the B.M. and S.F. diagrams.

4. A beam $AB$, of flexural rigidity $EI$ and span $L$ carries a uniformly distributed load of intensity $w$ per unit length. It is encastre at $A$ and $B$ but support $B$ settled during the application of the load by an amount $\delta$. Show that if $\delta = \frac{wL^4}{72EI}$, there is no fixing moment at $B$.

Answers

1. $M_A = M_B = \frac{WL}{8}$
2. $V_B = \frac{3}{8} wL$
3. $\frac{2}{9} wL$
Fig. 8.1 (b) shows the deflected shape of the two spans after loading, in which the three supports A, B, and C have settled to position A' B' C' by the amounts \( \delta_A, \delta_B, \delta_C \) respectively, below the original centre line.

Then \( y_B^A = \delta_B - \delta_A = \delta_1 \) (say)

where \( y_B^A \) is the deflection of B with respect to A.

Similarly, \( y_B^C = \delta_B - \delta_C = \delta_2 \) (say)

where \( y_B^C \) is the deflection of B with respect to C.

Fig. 8.1 (c) shows the fixing moment diagram, and Fig. 8.1 (d) shows the free bending moment diagram for the two spans.

Taking span AB first, and measuring x positive to the right,

\[
xL_1 y'' + y = \delta_1 \\
xL_2 y'' + y = \delta_2
\]

With usual notations.

Multiplying both sides by \( x \) and integrating over the range \( x=0 \) to \( x=L_1 \), we get

\[
E_1 I_1 \int_0^{L_1} x \, dy \, dx = \mu_1 + \mu_1', \text{ with usual notations.}
\]

At \( x=L_1 \), \( \frac{dy}{dx} = i_B \) and \( y_B^A = \delta_1 \)

Similarly, considering span BC, taking C as origin, and \( x \) positive to the left, we can obtain,

\[
i_B = \frac{1}{E_2 I_2 L_2} \left( A_2 \bar{v}_x + A'_2 \bar{v}_x' \right) + \frac{\delta_2}{L_2}
\]

From which \( i_B = \frac{1}{E_2 I_2 L_1} \left( A_2 \bar{v}_x + A'_2 \bar{v}_x' \right) + \frac{\delta_2}{L_1} \) (1)

Similarly, considering span BC, taking C as origin, and \( x \) positive to the left, we can obtain,

\[
i_B = \frac{1}{E_2 I_2 L_2} \left( A_2 \bar{v}_x + A'_2 \bar{v}_x' \right) + \frac{\delta_2}{L_2}
\]

(2)

Due to the continuity of the beam, \( i_B = -i_B' \)

Hence adding (1) and (2), we get

\[
\frac{1}{E_1 I_1 L_1} \left( A_1 \bar{v}_x + A'_1 \bar{v}_x' \right) + \frac{\delta_1}{L_1} + \frac{1}{E_2 I_2 L_2} \left( A_2 \bar{v}_x + A'_2 \bar{v}_x' \right) + \frac{\delta_2}{L_2} = 0
\]

Substituting \( A'_1 \bar{v}_x' = (M_A + 2M_B) \frac{L_1^2}{6} \), we get

\[
\frac{A_1 \bar{v}_x}{E_1 I_1 L_1} + \frac{L_1}{6 E_1 I_1} \left( M_A + 2M_B \right) + \frac{\delta_1}{L_1} + \frac{A_2 \bar{v}_x}{E_2 I_2 L_2} + \frac{L_2}{6 E_2 I_2} \left( M_A + 2M_B \right) + \frac{\delta_2}{L_2} = 0
\]

This is the generalised theorem of three moments. While substituting the numerical values of \( A_1 \) and \( A_2 \) for a given loading system, proper care of the sign must be taken. For usual downward loading, \( A_1 \) and \( A_2 \) will be negative. Let us use the above equation for some special cases.

### 8.2. EI CONSTANT : GENERAL LOADING

If \( E_1 I_1 = E_2 I_2 = EI \), then, from Eq. 8.1, we get

\[
M_A L_1 + 2M_B (L_1 + L_2) + Mc L_2 + \frac{6 A_1 \bar{v}_x}{L_1} + \frac{6 A_2 \bar{v}_x}{L_2} + 6 E I \left( \frac{\delta_1}{L_1} + \frac{\delta_2}{L_2} \right) = 0
\]

(8.1)

### 8.3. EI CONSTANT : NO SETTLEMENT

If \( E_1 I_1 = E_2 I_2 = EI \) and \( \delta_1 = 0, \delta_2 = 0 \), we have

\[
M_A L_1 + 2M_B (L_1 + L_2) + Mc L_2 + \frac{6 A_1 \bar{v}_x}{L_1} + \frac{6 A_2 \bar{v}_x}{L_2} = 0
\]

(8.3)

### 8.4. EI CONSTANT : U.D.L. ON BOTH SPANS

If the beams have constant EI, and if the supports do not yield Eq. 8.3 will be applicable. Let \( w_1 \) and \( w_2 \) be the U.D.L. on the two spans respectively.

Then \( A_1 = -\frac{2}{3} \left( \frac{w_1 L_1^3}{8} \right) \)

\[
A_1 \bar{v}_x = -\frac{w_1 L_1^2}{12}
\]

Similarly, \( A_2 \bar{v}_x = -\frac{w_2 L_2^2}{24} \)

Substituting these in Eq. 8.3, we get the special form

\[
M_A L_1 + 2M_B (L_1 + L_2) + Mc L_2 + \frac{6}{L_1} \left( -\frac{w_2 L_2^4}{24} \right) + \frac{6}{L_2} \left( -\frac{w_2 L_2^4}{24} \right) = 0
\]

(8.4)

If however, the supports settle, the above equation can be modified as follows:
8.5. FIXED BEAM

If the beam is fixed at both the ends, the three moment theorem can be used for finding out the support moments by imagining a zero span to the left of A and zero span to the right of B as shown in Fig. 8.2.

$$M_0L + 2M_y(L_1 + L_2) + McL_1 + 6EI\left(\frac{5}{L_1} + \frac{5}{L_2}\right) = \frac{w_1L_1^3}{4} + \frac{w_2L_2^3}{4}$$

(8.5)

Thus, applying three moments theorem for the spans A'A-AB, we get

$$0 + 2M_A(0 + L) + M_BL + 0 + \frac{6A\xi_2}{L} = 0$$

or

$$2M_A + M_B + \frac{6A\xi_2}{L} = 0$$

Similarly, for span AB-BB'

$$MA + 2M_B(L + 0) + 0 + \frac{6A\xi_2}{L} = 0$$

or

$$M_A + 2M_B + \frac{6A\xi_2}{L} = 0$$

(1)

Solving (1) and (2), $M_A$ and $M_B$ can be easily found.

Example 8.1. A beam ABC of length 2L rests on three supports equally spaced and is loaded with U.D.L. unit length throughout the length of the beam as shown in Fig. 8.3. Plot the B.M. and S.F. diagrams.

Solution.

Applying the three moment theorem for U.D.L. (Eq. 8.4), we get

$$M_A + 2M_B(L + 0) + McL = \frac{wL^3}{4} + \frac{wL^3}{4}$$

For the reaction $R_A$, write the equation of B.M. at B. Thus,

$$-R_A + \frac{wL}{2} = M_B = \frac{wL}{8}$$

$$R_A = \frac{wL}{2} - \frac{wL}{8} = \frac{3}{8} wL$$

$c=\frac{3}{8} L$ by symmetry.

For point of inflexion,

$$-\frac{3}{8} wL + \frac{wx^2}{2} = 0$$

which gives $x = \frac{3}{4} L$.

The B.M.D. can be drawn by superimposing $\mu$-diagram over $\mu$-diagram as shown in Fig. 8.3 (b).
Example 8.2. A cantilever beam ABCD covers three spans, \( IB = 6 \text{ m}, BC = 12 \text{ m} \) and \( CD = 4 \text{ m} \). It carries uniformly spread loads of \( 2 \text{ kN}, 1 \text{ kN} \) and \( 3 \text{ kN} \) per metre run on \( AB, BC \) and \( CD \) respectively. If the girder is of same cross-section throughout, find the bending moment at the supports \( B \) and \( C \) and the pressure on each support. Plot the B.M. and S.F. diagrams.

Solution

The free B.M. diagrams for \( AB, BC \) and \( CD \) can be constructed as usual.

For \( AB \), \( M_{\text{max}} = \frac{WL^2}{8} = \frac{2 \times 6 \times 6}{8} = 9 \text{ kN-m} \)

For \( BC \), \( M_{\text{max}} = \frac{1 \times 12 \times 12}{8} = 18 \text{ kN-m} \)

For \( CD \), \( M_{\text{max}} = \frac{3 \times 4 \times 4}{8} = 6 \text{ kN-m} \)

Fig. 8.4.

Applying the three moment theorem (Eq. 8.4) for spans \( AB \) and \( BC \),

\[ M_A \times 6 + 2M_B(6+12) + M_C 12 = \frac{2 \times 6^4}{4} + \frac{1 \times 12^4}{4} \]

Similarly, for spans \( BC-CD \) :

\[ M_B \times 12 + 2M_C(12+4) + M_D 4 = \frac{1 \times 12^4}{4} + \frac{3 \times 4^4}{4} \]

\[ 12M_B + 32M_C = 432 + 48 = 480, \text{ since } M_D = 0 \]

\[ MB + 0^333 MC = 15 \]

From (1) and (2), we get

\[ MC = 10^71 + 11^43 \text{ kN-m} \]

For reaction at \( A \), write expression for B.M. at \( A \).

Thus

\[ -(R_A \times 6) + (6 \times 2 \times 3) = M_B = +11^43 \]

\[ R_A = \frac{36 - 11^43}{6} = 24^57 = 4^09 \text{ kN} \]

For reaction at \( B \), write expression for B.M. at \( B \).

Thus

\[ -R_A \times 18 + R_B \times 12 + 6 \times 2(12 + 3) + 1 \times 12 \times 6 = M_C = +10^71 \]

\[ 12R_B = -(4^09 \times 18) + (12 \times 15) + 72 - 10^71 \]

\[ R_B = \frac{167^66}{12} = 13^79 \text{ kN} \]

Similarly for \( R_D \), write equation for B.M. at \( C \):

\[ -R_D \times 4 + (3 \times 4 \times 2) = M_C = +10^71 \]

\[ R_D = \frac{24 - 10^71}{4} = 3^32 \text{ kN} \]

Here,

\[ RC = (2 \times 6) + (1 \times 12) + (3 \times 4) - (R_A + R_B + R_D) = 36 - (4^09 + 13^79 + 3^32) = 36 - 21^38 = 14^62 \text{ kN} \]

The B.M. and S.F. diagrams are shown in Fig. 8.4.

Example 8.3. A continuous beam ABCD, 20 m long is carried on supports at its end and is propped at the same level at points 5 m and 12 m from left end \( A \). It carries two concentrated loads of \( 80 \text{ kN} \) and \( 50 \text{ kN} \) at \( 2 \text{ m} \) and \( 9 \text{ m} \) respectively from \( A \) and uniformly distributed load of \( 10 \text{ kN/m} \) run over the span \( CD \). Find the B.M. at the reactions at the four supports.

Solution

The free B.M. diagrams for three spans can be drawn as usual.

For span \( AB \)

\[ M_{\text{max}} = \frac{WAB}{L} = \frac{80 \times 2 \times 3}{5} = 96 \text{ kN-m} \]
\[ A = \frac{1}{2} \times 5 \times 96 = -240; \quad \bar{x} = \frac{1}{3}(5+2) = \frac{7}{3} \]

\[ A \bar{x} = -240 \times \frac{7}{3} = -560, \text{ with } A \text{ as origin.} \]

For span \( BC \)
\[ M_{max} = \frac{50 \times 4 \times 2}{7} = 85.7 \text{ kN-m} \]
with \( C \) as origin, \( A \bar{x} = -\left(\frac{4}{7} \times 85.7\right) \left(\frac{7+3}{4}\right) = -1000 \]
with \( B \) as origin, \( A \bar{x} = -\left(\frac{4}{7} \times 85.7\right) \left(\frac{7+4}{4}\right) = -1100 \]

For span \( CD \)
\[ M_{max} = \frac{10 \times 8^2}{8} = 80 \text{ kN-m} \]
\[ A \bar{x} = -\frac{3}{2} \times 80 \times 8 \times \frac{8}{2} = -1706.7, \text{ with } D \text{ as origin.} \]
\[ M_D = 0 \]

Applying three moments theorem for span \( AB-BC \),
\[ 5M_A + 2MB(5+7) + 7MC + \frac{6A_2\bar{x}_2}{7} + \frac{6A_2\bar{x}_3}{8} = 0 \]
\[ 5 \times 0 + 24MB + 7MC = \frac{6 \times 560}{5} + \frac{6 \times 1000}{7} = 673 + 857 = 1530 \]
\[ \text{or} \quad M_C + 0.29MC = 63.8 \text{ (since } M_A = 0) \quad (1) \]

Similarly, for span \( BC-CD \)
\[ 7M_B + 30MC + 0 = \frac{6 \times 1100}{6} + \frac{6 \times 1706}{7} = 943 + 1280 = 2223 \]
\[ \text{or} \quad M_B + 4.28MC = 317.6 \text{ (since } M_D = 0) \]

Solving (1) and (2), we get
\[ M_C = \frac{317.6 - 63.8}{4.28 - 0.29} = 63.5 \text{ kN-m} \]
and \[ M_B = 63.8 - 0.29 \times 63.5 = 45.5 \text{ kN-m} \]

For reaction at \( A \), write equation for \( M_B \):
\[ (-R_A \times 5) + (80 \times 3) = M_B = 45.5 \]
\[ \text{or} \quad R_A = \frac{240 - 45.4}{5} = 38.4 \text{ kN} \]

For reaction at \( B \), write equation for \( M_C \):
\[ -R_A(5+7) - R_B \times 7 + 80(3+7) + (50 \times 3) = M_C = 63.5 \]
\[ \text{or} \quad R_B = 60 \text{ kN} \]

For reaction at \( D \), write expression for \( M_C \):
\[ -R_D \times 8 + (10 \times 4) = M_C = 63.5 \]
\[ \text{or} \quad R_D = \frac{320 - 63.5}{8} = 32.1 \text{ kN} \]

For reaction at \( C \),
\[ R_C = (80 + 50 + 80) - (R_A + R_B + R_D) \]
\[ = 210 - (38.9 + 60 + 32.1) = 79 \text{ kN}. \]

The B.M. and S.F. diagrams are shown in Fig. 8.6.

Example 8.4. Solve example 8.3 if the support \( B \) sinks by 10 mm below \( A \) and \( C \). Moment of inertia for the whole beam = 85 \times 10^6 \text{ mm}^4, and \( E = 2.1 \times 10^6 \text{ N/mm}^2 \).

Solution
Applying three moment theorem for span \( AB-BC \),
\[ 24M_B + 7MC - 1530 + 6EI \left( \frac{\bar{x}_1}{L_1^2} + \frac{\bar{x}_2}{L_2^2} \right) = 0 \]
in which \( \frac{6A_2\bar{x}_1}{L_1^2} + \frac{6A_2\bar{x}_2}{L_2^2} = -1530 \) from example 8.3.

While substituting the numerical values of \( E, I, \bar{x}_1 \) and \( \bar{x}_2 \), proper care of units must be taken.
\[ E = 2.1 \times 10^6 \text{ N/mm}^2, I = 85 \times 10^6 \text{ mm}^4 \]
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Thus, substituting all the values in kN and m units,

\[ 24M_B + 7MC - 1530 + 6 \times 17850 \left( \frac{1}{100 \times 5} + \frac{1}{100 \times 7} \right) = 0 \]

or

\[ M_B + 0.29MC = 48.5 \]  \hspace{0.5cm} (1)

For span BC-CD,

\[ \delta_2 = \text{movement of } C \text{ with respect to } D = 0 \]

\[ 7M_B + 30MC - 2223 + 6 \times 17850 \left( \frac{1}{100 \times 7} + 0 \right) \]

or

\[ M_B + 4.28MC = 339.6 \]  \hspace{0.5cm} (2)

Solving (1) and (2), we get

\[ MC = 73 \text{ kN-m} \]

\[ MB = 27.4 \text{ kN-m} \]

The reactions at various supports can now be found in the same manner as illustrated in the previous examples.

Example 8.5. Solve example 8.3 if the end A is fixed and D is simply supported.

Solution

Imagine a point A' to the left of A such that AA' = 0

\[ 0 + 2MA(5+0) + MB \times 5 - \left( \frac{1}{2} \times 5 \times 96 \right) \left\{ \frac{1}{3} (5+3) \right\} = 0 \]

or

\[ 10MA + 5MB - 640 = 0 \]

or

\[ MA + 0.5MB = 64 \]  \hspace{0.5cm} (1)

For span AB-BC, we have

\[ 5MA + 24MB + 7MC = 1530 \] (as in example 8.3)

or

\[ MA + 4.8MB + 14MC = 306 \]  \hspace{0.5cm} (2)

For span BC-CD, we have

\[ M_B + 4.28MC = 317.6 \] (as in example 8.3)

From (1) and (2),

\[ 4.3M_B + 14MC = 242 \]

or

\[ M_B + 3.25MC = 56.2 \]  \hspace{0.5cm} (4)

From (3) and (4), we have

\[ MC = \frac{261.4}{3.955} = 66 \text{ kN-m} \]

\[ MB = 317.6 - 283 = 34.6 \text{ kN-m} \]

\[ MA = 64 - 17.3 = 46.7 \text{ kN-m} \]

For reaction at D,

\[ -RD \times 8 + (8 \times 10 \times 4) = MC = 66 \]

or

\[ RD = \frac{320 - 66}{8} = \frac{254}{8} = 31.8 \text{ kN} \]

For reaction at C,

\[ (-RD \times 15) - (RC \times 7) + (80 \times 11) + (50 \times 4) = MB = 34.6 \]

or

\[ RC = 81.4 \]

For reaction at A,

\[ -RA \times 5 + MA + 80 \times 3 = MB \]

or

\[ RA = \frac{MA - MB + 240}{5} = \frac{46.7 - 34.6 + 240}{5} = 50.4 \]

For reaction at B,

\[ RB = (80 + 50 + 80) - (50.4 + 81.4 + 31.8) = 46.4 \]

The B.M. and S.F. diagrams are shown in Fig. 8.6.

Example 8.6. A straight elastic beam of uniform section rests on four similar elastic supports which are placed L metres apart. The supports are such that they are compressed by d for each unit of load.
Substituting the values of $M_B$ and $R_A$ in (2), we get
\[
L \left\{ \frac{W}{6} - \left( \frac{W}{2} - R_B \right) \right\} = \frac{1}{5} \left\{ \frac{WL}{6} - \frac{6ELd}{L^2} \left( R_B - \left( \frac{W}{2} - R_B \right) \right) \right\}
\]
or
\[
\frac{5WL}{6} - \frac{5}{2} WL - WR_BL = \frac{WL}{6} - \frac{12ELd}{L^2} R_B + \frac{3ELdW}{L^2}
\]
or
\[
R_B \left( \frac{12ELd}{L^2} + SL \right) = \frac{3ELdW}{L^2} + \frac{11}{6} WL
\]
or
\[
R_B = \left[ \frac{3ELdW + \frac{11}{6} WL}{\frac{12ELd}{L^2} + SL} \right] = \frac{W}{3L} \left( \frac{3ELdW + \frac{11}{6} WL}{\frac{12ELd}{L^2} + SL} \right)
\]
Hence proved.

Example 8.7. A bridge of uniform cross-section rests on rigid abutments at the ends and three equal pontoons as shown in Fig. 8.8 and has a concentrated load $W$, at the middle. When the bridge is unloaded the pontoons just touch it without exerting any force. With the load $W$ at the middle and the two end pontoons removed the central deflection is one half what it would be with no pontoons.

Find the reactions and draw the bending moment diagram for the bridge due to central load with three pontoons in position.

Solution. (Fig. 8.8)
Due to symmetry, $R_A = R_E$; $R_B = R_D$.
Let the settlement of any pontoon be $Rk$, where $R$ is the reaction and $k$ is a constant.
The value of \( k \) can be determined from the data given in the problem when there is only a central pontoon. In such a case, let \( P \) = reaction on the pontoon.

\[ \text{Central deflection} = \frac{(W-P)(4L)^3}{48EI} = \frac{4(W-P)L^3}{3EI} \]

With no pontoon, deflection = \( \frac{1W(4L)^3}{48EI} \)

As per given condition, \( \frac{4(W-P)L^3}{3EI} = \frac{1W_L^3}{2} \),

from which, \( P = \frac{W}{2} \).

Central deflection with the pontoon = \( \frac{W_k}{3} \).

But it is equal to \( \frac{W}{2} \).

\[ \therefore \quad \frac{W}{2} = k = \frac{4L^3}{3EI} \quad \text{Also} \quad \frac{6EIk}{L^2} = 8L. \]

Applying three moments theorem for span \( AB-BC \).

\[ M_A L + 2M_B (L + L) + M_C L + 6EI \left( \frac{\delta_1}{L} + \frac{\delta_2}{L} \right) + \frac{6A_1 \bar{x}_1}{L} + \frac{6A_2 \bar{x}_2}{L} = 0 \]

Here, \( M_A = 0, \quad A_1 = 0, \quad A_2 = 0 \) (since there is no loading on \( AB \) and \( BC \))

\[ \delta_1 = \delta_2 - \delta_4 = (R_b k - 0) = R_b k, \quad \text{since} \quad \delta_4 = 0 \]

\[ \delta_2 = \delta_2 - \delta_6 = (R_b - R_c) k \]

\[ \therefore \quad 4M_B L + M_C L + \frac{6EIk}{L} \left( 2R_b - R_c \right) = 0 \]

or

\[ 4M_B + M_C + \frac{6EIk}{L^2} \left( 2R_b - R_c \right) = 0 \]

But

\[ \frac{6EIk}{L^2} = 8L. \]

Hence

\[ 4M_B + M_C + 8L(2R_b - R_c) = 0 \quad \text{(1)} \]

For the span \( BC-CD \),

\[ M_B L + 2M_C (2L) + M_D L + \frac{6EIk}{L} \left( R_c - R_b + R_c - R_d \right) = 0 \]

But \( R_b = R_d \) and \( M_b = M_d \)

\[ \therefore \quad M_B + 2M_C = \frac{6EIk}{L^2} \left( R_c - R_b \right) = 0 \quad \text{(2)} \]

### Example 8.8

A continuous beam \( ABCD \) has a weight \( w \) per unit length and rests on four knife edge supports \( A, B, C \) and \( D \). The middle span carries a load \( W \) as shown in Fig. 8.9. Find (a) bending moments at \( B \) and \( C \), (b) reactions at \( A \) and \( D \).

**Solution**

For span \( AB \),

\[ M_{max} = \frac{wa^2}{8} \]

\[ A \bar{x} = -\frac{2}{3} \frac{wa^2}{8} \times \frac{a}{2} = -\frac{wa^2}{24} \]

For span \( BC \),

\[ M_{max} = -\frac{w(2a)^2}{8} \text{ due to U.D.L.} = -\frac{wa^2}{8} \]

and

\[ M_{max} = -\frac{W}{2} \left( \frac{2a - z}{2a} \right) \text{ due to point load.} \]
\[ \begin{align*}
A\bar{\tau} \text{ (with } B \text{ as origin)} &= -\left[ \left\{ \frac{2}{3} \times \frac{wa^2}{2} \times 2a \times a \right\} 
+ \left\{ \frac{1}{2} \times \frac{Wz}{2a} \times 2a \times \frac{1}{3} (2a+z) \right\} \right] \\
&= -\frac{2}{3} wa^4 - \frac{Wz}{6} (2a-z) (2a+z)
\end{align*} \]

\[ \begin{align*}
A\bar{\tau} \text{ (with } C \text{ as origin)} &= -\left[ \left\{ \frac{2}{3} \times \frac{wa^2}{2} \times 2a \times a \right\} 
+ \left\{ \frac{1}{2} \times \frac{Mc(2a-z)}{2a} \times 2a \times \frac{1}{3} (2a+2a-z) \right\} \right] \\
&= -\frac{2}{3} wa^4 - \frac{Wz}{6} (2a-z) (4a-z)
\end{align*} \]

Applying three moment equation for span AB BC,

\[ M_B, 2M_B (a+2a) + Mc, 2a = \left[ \frac{6}{a} \times \frac{wa^4}{24} \right] + \frac{2}{3} wa^4 + \frac{Wz}{6} (2a-z)(4a-z) \]

or

\[ 6M_B + 2Mc = \frac{9}{4} wa^4 + \frac{Wz}{2a^2} (2a-z)(4a-z) \] \hspace{1cm} (1)

Similarly applying three moment equation for spans BC CD

\[ M_B, 2a + 2Mc (2a+a) + M_D, a = \frac{6}{2a} \left[ \frac{2}{3} wa^4 \right] - \frac{Wz}{6} (2a-z)(2a+z) \]

\[ + \left[ \frac{6}{a} \times \frac{wa^4}{24} \right] \]

or

\[ 2M_B + 6Mc = \frac{9}{4} wa^4 + \frac{Wz}{2a^2} (2a-z)(2a+z) \] \hspace{1cm} (2)

From (1) and (2), we get

\[ M_B = \frac{9}{32} wa^4 \frac{Wz}{16a^2} (2a-z)(5a-2z) \]

and

\[ Mc = \frac{9}{32} wa^4 \frac{Wz}{16a^2} (2a-z)(a+2z) \]

For reaction at A, write equation for \( M_B \)

\[ -(R_A \times a) + \frac{wa^2}{2} = M_B \]

\[ R_A = \frac{1}{2} wa - \frac{M_B}{a} = \frac{1}{2} wa - \frac{9}{32} wa 
- \frac{Wz}{16a^2} (2a-z)(5a-2z) \]

or

\[ R_A = \frac{7}{32} wa - \frac{Wz}{16a^2} (2a-z)(5a-2z) \]

Example 8'9. For a three span beam shown in Fig. 8'10(a) find the reactions and support moments, and draw the B.M. and S.F. diagrams.

Solution

\[
\begin{array}{c}
\text{Fig. 8'10.} \\
\text{Here the moment of inertia is variable for each span, but } E \text{ is the same. Hence from Eq. 8'1} \\
M_A \frac{L_1}{l_1} + 2M_B \left( \frac{L_1}{l_1} + \frac{L_2}{l_2} \right) + Mc \frac{L_2}{l_2} + \frac{6A_1 \bar{x}_1}{l_1 l_1} + \frac{6A_2 \bar{x}_2}{l_2 l_2} = 0 \\
The free B.M.D. is shown in Fig. 8'10 (b). \\
For span AB} \\
A\bar{\tau} = -\frac{1}{2} \times 4 \times 100 \times 2 = -400 \end{array}
\]

\[
\begin{align*}
\text{THREE MOMENT EQUATION METHOD} \\
R_B = \frac{1}{2} wa - \frac{Mc}{a} = \frac{7}{32} wa - \frac{Wz}{16a^2} (2a-z)(a+2z). \\
\end{align*}
\]
For span BC

\[ A\bar{x} = \text{zero, since there is no loading on BC.} \]

Now \( M_A = \frac{L_1}{I_1} + 2M_B \left( \frac{L_1}{I_1} + \frac{L_2}{I_2} \right) + M_C \frac{L_2}{I_2} + \frac{6A_4 \bar{x}_2}{I_2L_2} \]

Here \( M_A = 0 \) and \( M_B = M_C \), due to symmetry

\[ 2M_B \left( \frac{4}{2I} + \frac{5}{I} \right) + M_B \left( \frac{5}{I} \right) = \frac{6 \times 400}{2I(4)} + 0 \]

or

\[ 14M_B + 5M_B = 300 \]

From which \( M_B = 15.789 \text{ kN-m} = M_C \)

For reaction \( R_A \), take moments about \( B \)

\(-R_A(4) + 100 \quad (2) = M_B \)

or

\[ R_A = \frac{200}{4} - \frac{M_B}{4} = 50 - \frac{15.789}{4} = 46.05 \text{ kN} = R_B \]

Hence \( R_B = 100 - R_A = 100 - 46.05 = 53.95 \text{ kN} = R_C \)

The B.M. and S.F. diagrams are shown in Fig. 8'10 (c) and (d) respectively.

Example 8'10. A two span continuous beam, fixed at the ends is loaded as shown in Fig. 8'11(a). Find the reactions and support moments and draw the B.M. and S.F. diagrams.

Solution. The free B.M. diagrams for each span are shown in Fig. 8'11 (b).

For span \( AB \)

\[ M_{max} = -\frac{6(10)^2}{8} = -75 \text{ kN-m} \]

\[ A\bar{x} = -\frac{2}{3} \times 75 \times 10(5) = -2500 \text{ (with A or B as origin)} \]

For span \( BC \)

\[ A\bar{x} \text{ (with B as origin)} = \left( \frac{1}{2} \times 4 \times 68.571 \times \frac{8}{3} \right) \]

\[ -\left( \frac{1}{2} \times 3 \times 51.429 \right) \times (4+1) = -20 \]

\[ A\bar{x} \text{ (with C as origin)} = \left( \frac{1}{2} \times 4 \times 68.571 \times \left( 3 + \frac{4}{3} \right) \right) \]

\[ -\left( \frac{1}{2} \times 3 \times 51.429 \right) \times 2 \]

\[ = +440 \]

Imagine a point \( A' \) to the left of \( A \), such that \( AA' = 0 \)

Hence for spans \( A'A \) and \( AB \)

\[ 0 + 2M_A(0 + 10) + M_B(10) + 20 + \frac{6 \times 2500}{10} = 0 \]

or

\[ 20M_A + 10M_B = 1500 \]

or

\[ 2M_A + M_B = 150 \] (1)

For spans \( AB-BC \), we have

\[ M_A(10) + 2M_B(10 + 7) + Mc(7) = \frac{6 \times 2500}{10} + \frac{6 \times 440}{7} = 0 \]

or

\[ 10M_A + 34M_B + 7Mc = 1122.86 \]

Imagine a point \( C' \) to the right of \( C \), such that \( CC' = 0 \)

Hence for spans \( BC-C'C' \),

\[ M_B(7) + 2Mc(7) + 0 + \frac{6 \times 20}{7} + 0 = 0 \]

or

\[ 7M_B + 14Mc = 17.14? \] (2)

\[ M_B = 14.286 \text{ kN-m} \]

From (1) and (2)

\[ 29M_B + 7Mc = 372.86 \]

From (3) and (4), we get

\[ M_B = 14.286 \text{ kN-m} \]

Hence from (1), \( M_A = 67.857 \text{ kN-m} \)

and from (3) \( M_C = -5.918 \text{ (i.e. 5.918)} \)
For reaction at $A$, take moments about $B$ and equate to it to $M_B$

$$-RA(10) + M_A + (10 \times 6 \times 5) = M_B$$

or

$$-10RA + 67'857 + 300 = 14'286$$

From which $R_A = 35'357$ kN ($\uparrow$)

Similarly, for reaction at $C$, take moments about $B$ and equate it to $M_B$.

$$-RC \times 7 - Mc + 120 = M_B$$

or

$$-7RC - 5'918 + 120 = 14'286$$

From which $R_C = 14'257$ kN ($\uparrow$)

$$R_A = (6 \times 10) - (RA + RC) - 60 - (35'357 + 14'257)$$

$$= 10'386$$ ($\uparrow$)

The final B.M. and shear force diagrams are shown in Fig. 8'1 (c) and (d) respectively.

PROBLEMS

1. A fixed beam $AB$ of span $L$ carries a uniformly distributed load of $w$ per unit length and is propped at a distance $\frac{L}{3}$ from $A$. If the deflection of the beam at this point is $kR$, where $R=$ load on the prop, determine the magnitude of $R$.

2. A fixed beam carries a load which varies uniformly in intensity from zero at $A$ to $2w$ at $B$. A prop is placed at mid-span which removes all the deflection at this point. Calculate the load carried by the prop.

3. A beam $ABCD$, 16 m long is continuous over three spans: $AB=6$ m; $BC=5$ m; and $CD=5$ m, the supports being at the same level. There is a uniformly distributed load of 20 kN/m over $BC$. On $AB$, is a point load of 80 kN at 2 m from $A$ and $CD$, there is a point load of 60 kN at 3 m from $D$. Calculate the moments and reactions at the supports.

4. Solve question 3 if the support $B$ sinks by 0.5 cm. $I$ for the section is 9300 cm$^4$ and $E=2.10 \times 10^5$ N/mm$^2$.

5. Solve question 3 if the end $A$ is fixed.

6. $ABCD$ is a straight uniform beam of length $4L$. It is freely supported at its ends $A$ and $D$, and at two intermediate supports $B$ and $C$ distant $L$ from either end. The supports at $A$ and $D$ are rigid but those at $B$ and $C$ are such that they deflect by an amount $\lambda$ for each unit of load which is placed upon them. The beam carries a uniformly distributed load $w$ per unit length along its entire length.

Show that the reactions at the supports are

$$wL \left[ \frac{91L^3 + 48EIh}{8} \right]$$

and

$$\frac{19L^3}{8} \left[ \frac{19L^3 + 3EIh}{8} \right]$$

(Cambridge)

7. A uniformly continuous girder $ABC$ rests upon three similar floating supports, situated at each end and at the middle point $B$. The buoyancy of each float is such that every additional tonne of load increases its immersion by $h$. Initially, all the floats are equally immersed. If a load $W$ tonnes is placed on the girder at $B$, show that the proportion carried by the central float is

$$\frac{W \left( 1 + \frac{3hEI}{a^2} \right)}{\left( 1 + \frac{9hEI}{a^8} \right)}$$

where $2a$ is the length of the girder.

8. A beam of length $2a$ and flexural rigidity $EI$ carries a uniformly distributed load $w$ unit length and rests on three supports one at each end and one in the middle. Assuming that the beam was straight before loading, show that, for the greatest bending moment to be as small as possible, the central support must be $(8\sqrt[4]{2} - 11) w a^4$ lower than the end supports which are at the same level.

9. A beam rests on three supports $A$, $B$ and $C$ at the same level. The spans $AB$ and $BC$ are of equal length $L$. The span $AB$ is loaded with a load $W$ concentrated at the middle, and the span $BC$ has an equal load uniformly distributed. Find the reaction at the supports.

If the middle support sinks an amount $\frac{5}{96} \frac{WL^3}{EI}$ below the end supports, show that there will be no bending moment at $B$.

ANSWERS

1. $4374 \frac{wL^4}{4EI} + 16L^3$

2. $\frac{wL}{2}$

3. $M_B = 56'8$ kN-m; $M_C = 45'8$ kN-m; $R_A = 43'5$ kN; $R_C = 88'3$ kN; $R_D = 53'6$ kN; $R_B = 93'0$ kN

4. $M_A = 44'8$ kN-m; $M_C = 54'9$ kN-m; $R_C = 44'2$ kN; $R_B = 87'8$ kN; $R_D = 95'0$ kN; $R_A = 13$ kN

5. $M_A = 70'6$ kN-m; $M_B = 36'3$ kN-m; $M_C = 51'0$ kN-m

6. $\frac{11}{3} W$; $\frac{21}{2} W$; $\frac{11}{2} W$.
The Slope Deflection Method

1. INTRODUCTION: SIGN CONVENTIONS

The slope deflection method, in its present form, was first presented by Professor G.A. Maney (1915) of the University of Minnesota. In this method, the joints are considered to be rigid, i.e., the joints rotate as a whole and the angles between the tangents to the elastic curve meeting at the joint do not change due to deformation. The rotations of the joints are treated as unknowns. A series of simultaneous equations, each expressing the relation between the moments acting at the ends of the members are written in terms of slope and deflection. The solution of the slope-deflection equation along with the equilibrium equations, gives the values of the unknown rotations of the joints. Knowing these rotations, the end moments are calculated using the slope deflection equations.

During the decade just prior to the introduction of the moment distribution method, nearly all continuous frames were analyzed by the slope deflection method. The sign convention used in the case of bending of simple beams, etc., becomes clumsy if used for the case of more complex beams and frames where more than two members meet at a joint. In our earlier sign convention for simple beams, a moment is considered to be positive if it bends the beam convex upwards and negative if it bends the beam concave upwards. Thus, for the case of structure shown in Fig. 9.1, the three moments acting at the rigid joint B, where the three members BA, BC and BD meet are all positive according to the previous sign convention since all the three moments tend to bend the three corresponding beams convex upwards. Hence the equilibrium equation \( \Sigma M_B = 0 \) at the joint B cannot be conveniently applied if the previous sign convention is used, though the joint B is in equilibrium.

However, the examination of joint B (Fig. 9.1) reveals that the moments \( M_{BA} \) and \( M_{BD} \) are clockwise while the moment \( M_{BC} \) is anti-clockwise. If the new sign convention is based on the direction of the moment, we get

\[ M_{BA} + M_{BD} - M_{BC} = 0. \]

Hence in the new sign convention that will be used in this method, a support moment acting in the clockwise direction will be taken as positive and that in the anti-clockwise direction as negative. A corresponding change will have to be made while plotting the support moment diagram. For any span of a beam or member, with rigid joints a positive support moment (or end moment) at the right hand end will be plotted above the base line and negative support moment below the base. Similarly, for the left hand end, the negative end moment is plotted above the base and positive end moment is plotted below the base line, as shown in Fig. 9.2.
the above sign convention for the end moments. The sign convention for the rotation and settlement is adopted in the same way. Rotation (or slope) will be taken as positive and settlement as negative. (2) If one end of a beam settle, the rotation (or deflection) will be taken as positive, if it rotates as a whole in the clockwise direction and negative if it rotates as a whole in the anti-clockwise direction.

**EQUATIONS**

The fundamental slope deflection equations, consider the beam AB, fixed at end A and B, and subjected to external loading. Due to the external loading, the beam will deflect, and the ends will rotate. Let us now apply end moments at ends A and B respectively, of such magnitude that the beam will rotate. In other words, the applied moments are additional moments and therefore, a suffix 'a' will show these end moments. Such moment, which will cause the ends to zero, will hereafter be called the fixed end moments and therefore, a suffix 'f' will show these fixed end moments. The applied moments are denoted by $M_A$ and $M_B$ respectively.

But $Y = L \theta_A - \frac{L}{3} \theta_B^2$ [From Fig. 9.3 (b)].

The slope deflection method.

$$\theta = \frac{1}{EI} \sum A \xi$$

$$\theta = \frac{1}{EI} \left( \frac{1}{2} m_{BA} - m_{BA} \right)$$

The deflection of $B$ is in upward direction with respect to the tangent of $A$.

Fig. 9.3.

Derivation of slope-deflection equations.
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\[ \frac{L^3}{6EI}(2m_{AB} - m_{BA}) = \left( L\theta_A - \delta \right) \]

\[ 2m_{AB} - m_{BA} = \frac{6EI}{L} \left( \theta_A - \frac{\delta}{L} \right) \]  \tag{1}

Similarly, \( \gamma_A = \text{deviation of } A \) with respect to the tangent at \( B \)

\[ \frac{1}{EI} \sum_{A} A \gamma_A = \frac{1}{EI} \left( \frac{1}{2}m_{BA}, L \right) \left( \frac{2L}{3} - \frac{1}{2}m_{AB}, L \right) \left( \frac{L}{3} \right) \]

\[ = \frac{1}{6EI} \left( 2m_{BA} - m_{AB} \right) \]

But \( \gamma_A = L\theta_A - \delta \) From Fig. 9*3 (b)

\[ \frac{L^3}{6EI} \left( 2m_{BA} - m_{AB} \right) = L\theta_A - \delta \]

or

\[ 2m_{BA} - m_{AB} = \frac{6EI}{L} \left( \theta_B - \frac{\delta}{L} \right) \]  \tag{2}

Solving (1) and (2) for \( m_{AB} \) and \( m_{BA} \), we get

\[ m_{AB} = \frac{2EI}{L} \left( 2\theta_A + \theta_B - \frac{3\delta}{L} \right) \]  \tag{1}

and

\[ m_{BA} = \frac{2EI}{L} \left( 2\theta_B + \theta_A - \frac{3\delta}{L} \right) \]  \tag{2}

Thus, the values of the additional moments \( m_{AB} \) and \( m_{BA} \) are known. Superimposing the effects of Fig. 9*3 (a) and 9*3 (b), we get the final deflected shape of the beam as shown in Fig. 9*3 (d), where in \( \theta_A \) and \( \theta_B \) are the final rotations of the ends \( A \) and \( B \), and \( \delta \) is the deflection or settlement of the end \( B \) under the external loading. The final moments \( M_{AB} \) and \( M_{BA} \) at the ends \( A \) and \( B \) are respectively given by:

\[ M_{AB} = m_{AB} + M_{FAB} = \frac{2EI}{L} \left( 2\theta_A + \theta_B - \frac{3\delta}{L} \right) + M_{FAB} \]  \tag{9*1}

\[ M_{BA} = m_{BA} + M_{FBA} = \frac{2EI}{L} \left( 2\theta_B + \theta_A - \frac{3\delta}{L} \right) + M_{FBA} \]  \tag{9*2}

These are the fundamental slope deflection equations for the span \( AB \). Writing \( \frac{L}{L} = K \) and \( \frac{\delta}{L} = R \), the slope-deflection equations are sometimes written in the following form:

\[ M_{AB} = 2EK(2\theta_A + \theta_B - 3R) + M_{FAB} \]  \tag{9*3}

\[ M_{BA} = 2EK(2\theta_B + \theta_A - 3R) + M_{FBA} \]  \tag{9*4}

9*3. CONTINUOUS BEAMS AND FRAMES WITHOUT JOINTS

**TRANSLATION**

A continuous beam is essentially a statically indeterminate structure which must satisfy both the conditions of geometry as well as statical equilibrium. In the method of slope-deflection, the conditions of geometry are satisfied at the very outset of the slope deflection equations. In addition to this, the algebraic sum of the moments acting at a joint must be equal to zero, This condition furnishes as many equations as the number of joints. For example, for the continuous beam of Fig. 9*4, we get from the equilibrium of the joint \( B \),

\[ M_{BA} + M_{BC} = 0 \]  \tag{9*6}

Thus, we have two slope deflections equations (Eqs. 9*3 and 9*4), and one equilibrium equation (Eq. 9*5). The simultaneous solution of these equations give the unknown rotations. The end moments can then be calculated by substituting the values of these rotations in the slope-deflection equations. The procedure for the solution of a problem of continuous beam or frame without joint translation is summarised below:

1. Treat each span as a fixed beam and calculate the fixed end moments.
2. Write down slope deflection equations in terms of end moments, fixing moments, joint rotations and joint translation for each span.
3. Write down equilibrium equations for the individual joints.
4. Substitute the rotations back into the slope deflection equations and solve for the end moments.

**Example 9*1.** A beam \( ABC \), 10 m long, fixed at ends \( A \) and \( B \) is continuous over joint \( B \) and is loaded as shown in Fig. 9*4 (a). Using the slope deflection method, compute the end moments and plot the bending moment diagram. Also, sketch the deflected shape of the beam.

The beam has constant \( EI \) for both the spans.

**Solution.**

(a) Fixed end moments

Treating each span as a fixed beam, the fixed end moments are as follows:

\[ M_{FAB} = -\frac{5 \times 3 \times 2^3}{5^2} = -2.4 \text{ kN-m} \]

\[ M_{FBA} = +\frac{5 \times 2 \times 3^3}{5^2} = +3.6 \text{ kN-m} \]
The slope deflection method

\[ M_{BA} + M_{BC} = 0 \]
\[ \therefore (0.8\, EI\, \theta_B + 3.6) + (0.8\, EI\, \theta_B - 5.0) = 0 \]
or
\[ 1.6\, EI\, \theta_B = 1.4 \]
or
\[ EI\, \theta_B = +1.4 \]

The plus sign indicates that \( \theta_B \) is positive (i.e., rotation of tangent at \( B \) is clockwise).

\( (d) \text{ Final moments} \)

Substituting the value of \( EI\, \theta_B \) in Eqs. (1) to (4), we get:
\[ M_{AB} = 0.4 \left(\frac{1.4}{1.6}\right) - 2.4 = -1.05 \, \text{kN-m} \]
\[ M_{BA} = 0.8 \left(\frac{1.4}{1.6}\right) + 3.6 = +4.30 \, \text{kN-m} \]
\[ M_{BC} = 0.8 \left(\frac{1.4}{1.6}\right) - 5.0 = -4.30 \, \text{kN-m} \]
and
\[ M_{CB} = 0.4 \left(\frac{1.4}{1.6}\right) + 5.0 = +5.35 \, \text{kN-m} \]

Fig. 9.4 (b) shows the bending moment diagram. The deflected shape of the beam is shown in Fig. 9.4 (c).

Example 9.2. Solve example 9.1 if ends \( A \) and \( C \) are simply supported (or hinged).

\( \text{Solution} \)

\( (a) \text{ Fixed end moment} \)

These are the same as calculated in the previous example:
\[ M_{FAB} = -2.4 \, \text{kN-m} \]; \[ M_{FBA} = +3.6 \, \text{kN-m} \]
\[ M_{FBC} = -5.0 \, \text{kN-m} \]; \[ M_{FCB} = +5.0 \, \text{kN-m} \]
(b) Slope deflection equations.

For span AB,

\[
M_{AB} = \frac{2EI}{5} (2\theta_a + \theta_2) - 2\theta_b = 0 \quad \text{at} \quad B
\]

\[
M_{BA} = \frac{2EI}{5} (2\theta_b + \theta_2) + 3\theta_a = 0 \quad \text{at} \quad A
\]

For span BC,

\[
M_{BC} = \frac{2EI}{5} (2\theta_c + \theta_2) - 5\theta_a = 0 \quad \text{at} \quad B
\]

and

\[
M_{CB} = \frac{2EI}{5} (2\theta_a + \theta_2) + 5\theta_a = 0 \quad \text{at} \quad C
\]

(c) Equilibrium equations

Since end A is freely supported, \( M_{AB} = 0 \)

\[
E I \theta_A = \frac{22\theta}{12}
\]

Also end C is freely supported, \( M_{CB} = 0 \)

\[
E I \theta_C = \frac{-8\theta}{12}
\]

Solving Eqs. I, II, and III, we get

\[
E I \theta_A = \frac{22\theta}{12}
\]

\[
E I \theta_B = \frac{27\theta}{12}
\]

\[
E I \theta_C = \frac{-8\theta}{12}
\]

(d) Final moments

Substituting the values of \( E I \theta_A \) and \( E I \theta_B \) in Eq. (2), we get

\[
M_{BA} = 0.4 \left[ \frac{2 \times 27 + 22\theta}{12} \right] + 3\theta_a = +6.15 \text{ kN-m}
\]

As a check, substituting in Eq. (3)

\[
M_{BC} = 0.4 \left[ \frac{2 \times 27 + 8\theta}{12} \right] - 5\theta_a = -6.15 \text{ kN-m}
\]

\[
M_{AB} + M_{BC} = +6.15 - 6.15 = 0
\]

The bending moment diagram and the deflected shape of the beam are shown in Fig. 9-5 (b) and (c) respectively.

Note. The beam is statically indeterminate to single degree only. This problem has also been solved by the moment distribution method (Example 10.2) treating the moment at B as unknown.

However, in the slope-deflection method, the slope or rotations are taken as unknowns, and due to this the problem involves three unknown rotations \( \theta_1, \theta_2 \) and \( \theta_3 \). Hence the method of slope deflection is not recommended for such a problem.

Example 9.3. A continuous beam ABCD consists of three span and is loaded as shown in Fig. 9-6 (a). Ends A and D are fixed. Determine the bending moments at the supports and plot the bending moment diagram.

Solution

(a) Fixed end moments

\[
M_{FAB} = \frac{-2 \times 6^2}{12} = -6 \text{ kN-m} \quad ; \quad M_{FBA} = \frac{+2 \times 6^2}{12} = +6 \text{ kN-m}
\]

\[
M_{FBC} = \frac{-5 \times 3 \times 2^2}{5^2} = -2.4 \text{ kN-m}
\]

\[
M_{FCD} = \frac{-8 \times 5}{8} = -5 \text{ kN-m} \quad ; \quad M_{FDC} = \frac{+8 \times 5}{8} = +5 \text{ kN-m}
\]

(b) Slope deflection equations

\[
\theta_A = \theta_0 = 0 \quad \text{since ends A and D are fixed.}
\]

\[
M_{AB} = \frac{2EI}{6} \left[ \theta_a \right] - 6 = \frac{EI}{3} \theta_A - 6
\]

\[
M_{BA} = \frac{2EI}{6} \left[ 2\theta_a + 6 \right] = \frac{2EI}{3} \theta_B + 6
\]

\[
M_{BC} = \frac{2EI(2\theta)}{5} \left[ 2\theta_b + 0 \right] - 2\theta_a = \frac{4EI}{5} \left( 2\theta_b + 0 \right) - 2\theta_a
\]

\[
M_{CB} = \frac{2EI(2\theta)}{5} \left[ 2\theta_a + 0 \right] + 3\theta_a = \frac{4EI}{5} (0 \theta_a + 2\theta_2) + 3\theta_a
\]
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\[ M_{CD} = \frac{2EI}{5} [20c] - 5 = \frac{4EI}{5} \theta_c - 5 \]  \hspace{1cm} (5)

\[ M_{BC} = \frac{2EI}{5} [0c] + 5 = \frac{2EI}{5} \theta_c + 5 \]  \hspace{1cm} (6)

(c) **Equilibrium equations**

At joint \( B \),

\[ M_{BA} + M_{BC} = 0 \]

or

\[ \left[ \frac{2EI}{3} \theta_B + 6 \right] + \left[ \frac{4EI}{5} (20\sigma + \theta_c) - 2.4 \right] = 0 \]  \hspace{1cm} (I)

At joint \( C \),

\[ M_{CB} + M_{CD} = 0 \]

or

\[ \left[ \frac{4EI}{5} (\theta_B + 2\sigma) + 3'6 \right] + \left[ -\frac{4EI}{5} \theta_c - 5 \right] = 0 \]  \hspace{1cm} (II)

From (I) and (II), we get

\[ EI\theta_B = -2'03 \]  \hspace{1cm} (i)

\[ EI\theta_c = +1'26 \]  \hspace{1cm} (ii)

(d) **Final moments**

Substituting these values in Eqs. (1) to (6), we get

\[ M_{AB} = \frac{1}{3} (-2'03) - 6 = -6'68 \text{ kN-m} \]

\[ M_{BA} = \frac{2}{3} (-2'03) + 6 = +4'65 \text{ kN-m} \]

\[ M_{BC} = \frac{4}{5} [(-2 \times 2'03) + 1'26] - 2'4 = -4'65 \text{ kN-m} \]

\[ M_{CB} = \frac{4}{5} [(-2'03) + (2 \times 1'26)] 	imes 3'6 = +3'99 \text{ kN-m} \]

\[ M_{CD} = \frac{4}{5} (1'26) - 5 = -3'99 \text{ kN-m} \]

\[ M_{DC} = \frac{2}{5} (1'26) + 5 = +5'50 \text{ kN-m} \]

The bending moment diagram and the deflected shape are shown in Fig. 9-6 (b) and (c) respectively.

**Example 9.4.** A continuous beam \( ABCD \), 12 m long is fixed at \( A \) and \( D \), and is loaded as shown in Fig. 9-7 (a). Analyse the beam completely if the following movements take place simultaneously:

(i) The end \( A \) yields, turning through \( \frac{1}{250} \) radians in a clockwise direction.

(ii) End \( B \) sinks 30 mm in downward direction.

(iii) End \( C \) sinks 20 mm in downward direction.

The beam has constant \( I = 38'20 \times 10^6 \text{ mm}^4 \) and \( E = 2 \times 10^6 \text{ N/mm}^2 \).

**Solution**

![Figure 9.7](image)

(a) **Fixed end moments**

\[ M_{FAB} = -\frac{8 \times 5}{8} = -5'0 \text{ kN-m} \]

\[ M_{FBA} = +\frac{8 \times 5}{8} = +5'0 \text{ kN-m} \]

\[ M_{FBC} = -\frac{4 \times 32}{12} = -3'0 \text{ kN-m} \]

\[ M_{FCB} = +\frac{4 \times 32}{12} = +3'0 \text{ kN-m} \]

\[ M_{FCD} = -\frac{6 \times 4}{8} = -3'0 \text{ kN-m} \]

\[ M_{FDC} = +\frac{5 \times 4}{8} = +3'0 \text{ kN-m} \]

(b) **Slope deflection equations**

All the unknowns are assumed to be positive.

For \( AB \),

\[ K = \frac{I}{5} \quad ; \quad R = +\frac{30}{500} = \frac{3}{500} \quad ; \quad \theta_A = +\frac{1}{250} \]

\[ M_{AB} = \frac{2EI}{5} \left( \frac{2}{250} + \theta_B - \frac{9}{500} \right) - 5'0 \]  \hspace{1cm} (I)
For BC,

\[ M_{BC} = \frac{2EI}{3} \left( 20B + 3 \cdot 2 \cdot 400 \right) - 30 \]

(3)

\[ M_{CB} = \frac{2EI}{3} \left( 20C + 3 \cdot 2 \cdot 400 \right) + 30 \]

(4)

For CB,

\[ K = \frac{1}{3} ; R = -\frac{20 - 20}{400} = \frac{2}{400} \text{ and } \theta_B = 0 \]

\[ M_{CD} = \frac{2EI}{4} \left( 20C + 3 \cdot 2 \cdot 400 \right) - 30 \]

(5)

\[ M_{DC} = \frac{2EI}{4} \left( 0C + 3 \cdot 2 \cdot 400 \right) + 30 \]

(6)

(c) Equilibrium equations:

There are two unknowns \( \theta_B \) and \( \theta_C \). Thus two simultaneous equations will be required which will be provided by the conditions of equilibrium at joints B and C.

At joint B,

\[ M_{BA} + M_{BC} = 0 \]

\[ \therefore \frac{2EI}{5} \left( 20B + \frac{1}{250} - \frac{9}{300} \right) + 5 + \frac{2EI}{3} \left( 20B + 3 \cdot 1 \cdot 300 \right) - 30 = 0 \]

or

\[ \frac{32}{15} \frac{EI}{3} \theta_B + \frac{2}{3} \frac{EI}{2} \theta_C + \frac{4}{3750} E I + 20 = 0 \]

or

\[ \frac{EI}{16} \theta_B + \frac{15}{16} \frac{EI}{20} \theta_C + \frac{15}{16} = 0 \]

(7)

At joint C,

\[ M_{CB} + M_{CD} = 0 \]

\[ \therefore \frac{2EI}{3} \left( 20C + 3 \cdot 1 \cdot 300 \right) + 3 + \frac{2EI}{4} \left( 20C + 3 \cdot 2 \cdot 400 \right) - 30 = 0 \]

or

\[ \frac{7}{3} \frac{EI}{3} \theta_C + \frac{2}{3} \frac{EI}{1200} \theta_B + \frac{17}{800} EI = 0 \]

or

\[ \frac{7}{2} \frac{EI}{2} \theta_C + \frac{17}{800} EI = 0 \]

(8)

Subtracting Eq. (8) from Eq. (7),

\[ \frac{5}{16} \frac{EI}{2000} \theta_C + \frac{15}{16} - \frac{7}{2} \frac{EI}{800} \theta_C - 17 \frac{EI}{800} = 0 \]

The slope deflection method

or

\[ \frac{51}{16} \frac{EI}{4000} \theta_C = \frac{83}{16} \frac{EI}{15} \]

But \( EI = \frac{2 \times 10^6 \times 1000 \times 38.2 \times 10^6}{(1000)^2} = 764 \text{ kN-m}^2 \)

\[ \therefore \frac{51}{16} \times 764 \theta_C = \frac{83}{16} \times 764 \times \frac{15}{16} \]

or

\[ \theta_C = -6.124 \times 10^{-3} \text{ radians} \]

Substituting the value of \( \theta_C \) in Eq. (8),

\[ \frac{6}{2} \frac{EI}{6} \times 10^{-3} + \frac{17}{800} EI = 0 \]

or

\[ \theta_B = 0.184 \times 10^{-3} \text{ radians} \]

(d) Final moments

Substituting the values of \( \theta_B \) and \( \theta_C \) in Eqs. (1) to (6), we get the values of moments at the supports:

\[ M_{AB} = \frac{2 \times 764}{5} \left( \frac{2 \times 0.184 \times 10^{-3}}{250} + 0.184 \times 10^{-3} - \frac{9}{300} \right) - 50 = -8.0 \text{ kN-m} \]

\[ M_{BA} = \frac{2 \times 764}{5} \left( 2 \times \frac{0.184 \times 10^{-3}}{250} + \frac{1.9}{500} \right) + 50 = +0.83 \text{ kN-m} \]

\[ M_{BC} = \frac{2 \times 764}{3} \left( 2 \times 0.184 \times 10^{-3} - 0.124 \times 10^{-3} - \frac{3}{300} \right) - 30 = -0.83 \text{ kN-m} \]

\[ M_{CB} = \frac{2 \times 764}{3} \left( 2 \times 0.124 \times 10^{-3} + 0.184 \times 10^{-3} + \frac{3}{300} \right) + 30 = +1.95 \text{ kN-m} \]

\[ M_{CD} = \frac{2 \times 764}{4} \left( -2 \times 0.124 \times 10^{-3} + \frac{6}{400} \right) + 30 = +6.39 \text{ kN-m} \]

The bending moment diagram and the deflected shape of the beam are shown in Fig. 9.7 (b) and (c) respectively.

Example 9.5. A continuous beam \( \text{ABC} \) is supported on an elastic column \( \text{BD} \) and is loaded as shown in Fig. 9.8. Treating joint B as rigid, analyse the frame and plot the bending moment diagram and the deflected shape of the structure.

Solution.

(a) Fixed end moments

\[ M_{FAB} = -10 \times 2 \times 3^2 = -72 \text{ kN-m} \]

\[ M_{FBA} = +10 \times 3 \times 2^2 = +48 \text{ kN-m} \]
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$M_{FBC} = \frac{2 \times 3^2}{12} = -1.5 \text{kN-m}$; $M_{FCB} = \frac{2 \times 3^2}{12} = +1.5 \text{kN-m}$

$M_{FBDB} = M_{FBDB} = 0$

Fig. 9.8.

Slope deflection equations

The slopes $\theta_A$ and $\theta_D$ are zero since ends $A$ and $D$ are fixed.

For span $AB$:

$M_{AB} = \frac{2EI}{3} [\theta_B] + \frac{4E}{3} \theta_B - 7.2$  \hspace{1cm} (1)

$M_{AD} = \frac{2EI}{3} [\theta_B] + \frac{4E}{3} \theta_B + 4.8$  \hspace{1cm} (2)

For span $BC$:

$M_{BC} = \frac{2EI}{3} [\theta_B + \theta_C] + \frac{4E}{3} \theta_B + \frac{2E}{3} \theta_C - 1.5$  \hspace{1cm} (3)

$M_{CB} = \frac{2EI}{3} [\theta_B + \theta_C] + \frac{4E}{3} \theta_B + \frac{2E}{3} \theta_C + 1.5$  \hspace{1cm} (4)

For span $BD$:

$M_{DB} = \frac{2EI}{3} [\theta_B] + \frac{4E}{3} \theta_B$  \hspace{1cm} (5)

$M_{DB} = \frac{2EI}{3} [\theta_B] + \frac{2E}{3} \theta_B$  \hspace{1cm} (6)

(a) Equilibrium equations

At joint $B$, $M_{FAB} + M_{FBC} + M_{FBD} = 0$

\[ \frac{8}{3} EI \theta_B + 4.8 + \left( \frac{4E}{3} \theta_B + \frac{2E}{3} \theta_C - 1.5 \right) + \left( \frac{4E}{3} \theta_B - 1.5 \right) = 0 \]

or

\[ \frac{64}{15} EI \theta_B + \frac{2E}{3} \theta_C + 3.3 = 0 \]  \hspace{1cm} (I)

At joint $C$, $M_{CB} = 0$

\[ \frac{4}{3} EI \theta_B + \frac{2E}{3} \theta_B + 1.5 = 0 \]  \hspace{1cm} (II)

Solving Eq. (I) and (II) for $\theta_B$ and $\theta_C$, we get

$E I \theta_B = -0.648$  \hspace{1cm} (i)

$E I \theta_C = -0.801$  \hspace{1cm} (ii)

(d) Final moments

Substituting these values in Eq. (1) to (6), we get

$M_{AB} = \frac{4}{5} (-0.648) - 7.2 = -7.72 \text{kN-m}$

Fig. 9.9.

$M_{BA} = \frac{8}{5} (-0.648) + 4.8 = +3.76 \text{kN-m}$

$M_{BC} = \frac{4}{5} (-0.648) + \frac{2}{3} (-0.801) - 1.5 = -2.90 \text{kN-m}$

$M_{CB} = \frac{4}{3} (-0.801) + \frac{2}{3} (-0.648) + 1.5 = 0$

$M_{DB} = \frac{4}{3} (-0.648) = -0.86 \text{kN-m}$

The bending moment diagram and the deflected shape of the structure are shown in Fig. 9.9.

Example 9.6. Analyse the rigid frame shown in Fig. 9.10.

Solution.

(a) Fixed end moments

$M_{FBA} = \frac{2 \times 4^3}{12} = -2.67 \text{kN-m}$; $M_{FBA} = \frac{2 \times 4^3}{12} = +2.67 \text{kN-m}$
\[ M_{FBD} = \frac{-4 \times 4}{8} = -2 \text{ kN-m} \]
\[ M_{FDB} = \frac{+4 \times 4}{8} = +2 \text{ kN-m} \]

(b) Slope deflection equations

\[ \theta_A \text{ and } \theta_B \text{ are zero.} \]
\[
M_{AB} = \frac{2EI(2l)}{4} [2\theta_B] - 2 \cdot \theta_B = EI\theta_B - 2 \cdot \theta_B \\
M_{BA} = \frac{2EI(2l)}{4} [2\theta_B] + 2 \cdot \theta_B = 2EI\theta_B + 2 \cdot \theta_B \\
M_{BD} = \frac{2EI(2l)}{4} [2\theta_B] - 2 = EI\theta_B - 2 \\
M_{DB} = \frac{2EI(2l)}{4} [2\theta_B] + 2 = \frac{1}{2} EI\theta_B + 2 \\
M_{BC} = -2 \times 2 = -4 \text{ kN-m.} \]

(b) Equilibrium equation

For the equilibrium of the joint B

\[ M_{BA} + M_{BD} + M_{BC} = 0 \]

\[
\therefore \quad (2EI\theta_B + 2 \cdot \theta_B) + (EI\theta_B - 2) + (-4) = 0 \\
\text{or} \quad 3EI\theta_B = 3 \cdot 33 \\
\text{or} \quad EI\theta_B = 1 \cdot 11 \\
\]

(d) Final moments

Substituting the value of \( EI\theta_B \) in Eqs. (1) to (4), we get

\[ M_{AB} = 1 \cdot 11 - 2 \cdot \theta_B = -1 \cdot 56 \text{ kN-m} \]
\[ M_{BA} = 2(1 \cdot 11) + 2 \cdot \theta_B = +4 \cdot 89 \text{ kN-m} \]
\[ M_{BD} = 1 \cdot 11 - 2 = -0 \cdot 89 \text{ kN-m} \]
\[ M_{DB} = \frac{1}{2}(1 \cdot 11) + 2 = +2 \cdot 56 \text{ kN-m} \]
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(a) Fixed end moments

\[ M_{FB} = \frac{w(2L)^3}{12} = -\frac{wL^3}{3}; \quad M_{FCB} = \frac{w(2L)^3}{12} = +\frac{wL^3}{3} \]

(b) Slope deflection equations

\[ \theta_A \text{ and } \theta_B \text{ are zero since ends } A \text{ and } D \text{ are fixed. Also, since } \]

\[ \text{the frame is symmetrical and the loading is also symmetrical, there is no side-sway or deflection of the frame. The unknowns are, therefore, } \theta_B \text{ and } \theta_C. \]

For \( AB \),

\[ M_{AB} = \frac{2EI}{L} (0_B) = \frac{2}{L} EI\theta_B \tag{1} \]

\[ M_{BD} = \frac{2EI}{L} (2\theta_B) = \frac{4}{L} EI\theta_B \tag{2} \]

For \( BC \),

\[ M_{BC} = \frac{2EI}{L} \left[ 2\theta_B + \theta_C \right] - \frac{wL^2}{3} = \frac{2}{L} EI\theta_B + \frac{1}{L} EI\theta_C - \frac{wL^2}{3} \tag{3} \]

\[ M_{CD} = \frac{2EI}{L} \left[ 2\theta_C + \theta_B \right] + \frac{wL^2}{3} = \frac{2}{L} EI\theta_C + \frac{1}{L} EI\theta_B + \frac{wL^2}{3} \tag{4} \]

For \( CD \),

\[ M_{CD} = \frac{2EI}{L} \left[ 2\theta_C \right] = \frac{4}{L} EI\theta_C \tag{5} \]

\[ M_{DC} = \frac{2EI}{L} \left[ \theta_C \right] = \frac{2}{L} EI\theta_C \tag{6} \]

(c) Equilibrium equations

At the joint \( B \),

\[ M_{AB} + M_{BC} = 0 \]

\[ \therefore \quad \frac{4}{L} EI\theta_B + \frac{2}{L} EI\theta_B + \frac{1}{L} EI\theta_C - \frac{wL^2}{3} = 0 \]

or

\[ \frac{6}{L} EI\theta_B + \frac{1}{L} EI\theta_C - \frac{wL^2}{3} = 0 \]

But by symmetry, \( \theta_C = -\theta_B \)

\[ \therefore \quad \frac{5}{L} EI\theta_B = \frac{wL^2}{3} \tag{i} \]

or

\[ \frac{EI}{L} \theta_B = \frac{wL^2}{3} \times \frac{1}{5} = \frac{wL^3}{15} \]

and

\[ \frac{EI}{L} \theta_C = -\frac{EI}{L} \theta_B = -\frac{wL^3}{15} \tag{ii} \]

(d) Final moments

Substituting these values in Eqs. (1) to (6), we get

\[ M_{AB} = 2 \left( \frac{wL^2}{15} \right) = +\frac{2}{15} wL^2 \]

\[ M_{BD} = 4 \left( \frac{wL^2}{15} \right) = +\frac{4}{15} wL^2 \]

\[ M_{FB} = 2 \left( \frac{wL^2}{15} \right) = -\frac{2}{15} wL^2 \]

\[ M_{FCB} = 4 \left( \frac{wL^2}{15} \right) = -\frac{4}{15} wL^2 \]

The bending moment diagram and the deflected shape of the frame are shown in Fig. 9.13.

**Fig. 9.13.**

9.4 PORTAL FRAMES WITH SIDE SWAY

In the case of continuous beams, etc., the effect of yielding or settlement of support is taken into account by introducing initial fixed end moments. In the case of portal frames, however, the amount of the joint moment or "sway" is not known and form an additional unknown. The portal frames may sway due to one of the following reasons:

1. Eccentric or unsymmetrical loading on the portal frame.
2. Unsymmetrical outline of portal frame.
3. Different end conditions of the columns of the portal frame.
4. Non-uniform section of the members of the frame.
5. Horizontal loading on the columns of the frame.
6. Settlement of the supports of the frame.
7. A combination of the above.
In such cases, the joint translations become additional unknown quantities. Some additional conditions will, therefore, be required for analysing the frame. The additional conditions of equilibrium are obtained from the consideration of the shear force exerted on the structure by the external loading. The horizontal shear exerted by a member is equal to the algebraic sum of the moments at the ends divided by the length of the member. Thus the horizontal shear resistance of all such members can be found and the algebraic sum of all such forces must balance the external horizontal loading, if any.

In Fig. 9.14, the horizontal reactions are given by,

\[ H_a = \frac{M_{ab} + M_{ba} - P h}{L_1} \]  

and

\[ H_d = \frac{M_{cd} + M_{dc} + \frac{1}{2} w L_2^2}{L_2} \]

The above reactions have been calculated on the assumption that all the end moments are clockwise.

---

**Example 9.8.** Analyse the portal frame shown in Fig. 9.15. Also sketch the deflected shape of the frame. The end A is fixed and end D is hinged.

**Solution**

(a) **Fixed end moments:**

There will be no fixed end moments for any of the spans of the frame as 10 kN load is acting on the joint.
(b) Slope deflection equations:
The unknowns in this case are $\theta_b$, $\phi_c$, $\phi_d$ and the joint translation $\delta$. Because $E$, $I$ and $L$ are same for all members, $K$ is constant. Assuming no change in the length of $BC$, the horizontal movement of $B$ and $C$ will equal to $\delta$. For $AB$ and $CD$, $R = \frac{\delta}{4}$.

\[
M_{AB} = 2EK(\theta_b - 3R) \quad (1)
\]

\[
M_{BA} = 2EK(2\theta_b - 3R) \quad (2)
\]

\[
M_{BC} = 2EK(\theta_b + \phi_c) \quad (3)
\]

\[
M_{CB} = 2EK(\theta_b + \phi_d) \quad (4)
\]

\[
M_{CD} = 2EK(2\theta_c + \phi_d - 3R) \quad (5)
\]

\[
M_{DC} = 2EK(2\phi_c + \phi_d - 3R) \quad (6)
\]

Since $M_{DC}$ is zero, $\phi_d$ can be expressed in terms of $\phi_c$, thereby reducing the number of unknowns to three.

\[
\phi_d = \frac{3R - \phi_c}{2}
\]

(c) Equilibrium equations:
At joint $B$,

\[
M_{BA} + M_{BC} = 0
\]

or

\[
2EK(2\theta_b - 3R) + 2EK(\theta_b + \phi_c) = 0
\]

At joint $C$,

\[
M_{CB} + M_{CD} = 0
\]

or

\[
8EK\theta_b - 6EK\phi_c + 2EK\phi_d = 0
\]

or

\[
4\theta_b - 3R + \phi_d = 0
\]

Substituting the values of $\phi_d = \frac{3R - \phi_c}{2}$, we get

\[
4\theta_b + \frac{3R - \phi_c}{2} - 3R = 0
\]

or

\[
7\phi_b + 2\phi_c - 3R = 0
\]

(d) Shear equations:

\[
\frac{M_{BA} + M_{AB}}{4} + \frac{M_{CD}}{4} + P = 0
\]

or

\[
\frac{2EK(2\theta_b - 3R) + 2EK(\theta_b - 3R)}{4} + \frac{2EK(2\theta_b + \theta_d - 3R)}{4} + 10 = 0
\]

or

\[
30\theta_b - 6R + \frac{20}{EK} + 2\phi_c + \frac{3R - \phi_c}{2} - 3R = 0
\]

From equation 7, $\phi_c = 3R - 4\phi_b$

Substituting in equation (8)

\[
21R - 28\phi_b + 2\phi_b - 3R = 0
\]

or

\[
18R = 28\phi_b
\]

or

\[
R = \frac{13}{9} \phi_b
\]

Hence

\[
\phi_c = \frac{3 \times 13}{9} \phi_b - 4\phi_b = \frac{8\phi_b}{3}
\]

and

\[
\phi_d = \frac{3\times 13}{2} \times \phi_c - \frac{\phi_b}{3} - 2\phi_b
\]

(e) Final moments:
Substituting in equations 1 to 6, we get

\[
M_{AB} = 2EK\left(\frac{30}{11 \cdot EK} - \frac{3 \times 130}{33 \cdot EK}\right) = -18.18 \text{ kN-m}
\]

\[
M_{BA} = 2EK\left(\frac{2 \times 30}{11 \cdot EK} - \frac{3 \times 130}{33 \cdot EK}\right) = -12.73 \text{ kN-m}
\]

\[
M_{BC} = 2EK\left(\frac{2 \times 30}{11 \cdot EK} + \frac{10}{11 \cdot EK}\right) = +12.73 \text{ kN-m}
\]

\[
M_{CB} = 2EK\left(\frac{20}{11 \cdot EK} + \frac{30}{11 \cdot EK}\right) = 9.09 \text{ kN-m}
\]

and

\[
M_{CD} = 2EK\left(\frac{20}{11 \cdot EK} + \frac{60}{11 \cdot EK} - \frac{3 \times 130}{33 \cdot EK}\right) = -9.09 \text{ kN-m}
\]

\[
6\phi_b - 15R + \frac{40}{EK} + 36\phi_c = 0
\]

or

\[
0\phi_b = \frac{40}{EK}
\]

or

\[
\phi_b = \frac{30}{11 \cdot EK}
\]

and hence

\[
R = -\frac{130}{33 \cdot EK}, \phi_c = \frac{10}{11 \cdot EK} \text{ and } \phi_d = \frac{60}{11 \cdot EK}
\]
The bending moment diagram and the deflected shape of the beam has been shown in Fig. 9.16.

**Example 9.16.** A portal frame ABCD is fixed at A and D, and has rigid joints at B and C. The column AB is 3 m long and column CD, 2 m long. The beam BC is 2 m long and is loaded with uniformly distributed load of intensity 6 kN/m. The moment of inertia of AB is 2I and that of BC and CD is I (Fig. 9.17). Plot B.M. diagram and sketch the deflected shape of the frame.

**Solution**

(a) **Fixed-end moments**

\[ M_{FBC} = +\frac{6 \times 2^3}{12} = -2 \text{ kN-m} \]

\[ M_{FCD} = +2 \text{ kN-m} \]

\[ M_{FAB} = M_{FBA} = M_{FDC} = M_{FDC} = 0. \]

Let the joints B and C move horizontally by \( \delta \).

(b) **Slope deflection equations**

\[ M_{AB} = \frac{2EI(0 - \frac{38}{3})}{3} = \frac{4}{3} EI(0 - \delta) \]  
\[ M_{BA} = \frac{2EI(20\delta - \frac{38}{3})}{3} = \frac{4}{3} EI(20\delta - \delta) \]  
\[ M_{BC} = \frac{2EI(20\delta + 6\delta)}{2} = -2 \]  
\[ M_{CB} = \frac{2EI(20\delta + 6\delta)}{2} = +2 \]  
\[ M_{CD} = \frac{2EI(20\delta - \frac{38}{2})}{2} = EI(20\delta - 1.58) \]

and \( M_{DC} = EI(0\delta - 1.58) \)

(c) **Equilibrium equations**

At joint B,

\[ M_{BA} + M_{BC} = 0 \]

or \( \frac{4}{3} EI(20\delta - \delta) + EI(20\delta + 0\delta) - 2 \]

or \( \frac{8}{3} \delta - \frac{4}{3} \delta + 20\delta + 0\delta = \frac{2}{EI} \]

or \( 140\delta + 30\delta - 48 = 6 EI \)  

or \( 0 = 0 + 0 - 1.58 + \frac{2}{EI} \) = 0

At joint C,

\[ M_{CB} + M_{CD} = 0 \]

or \( EI(20\delta + 0\delta) + 2 EI(20\delta - 1.58) \]

or \( 0 = 0 + 0 - 1.58 + \frac{2}{EI} \) = 0

(d) **Shear equation**

\[ \frac{M_{AB} + M_{BA} + M_{CD} + M_{DC}}{2} = 0 \]

or \( \frac{4}{3} EI(0\delta - \delta) + \frac{4}{3} EI(20\delta - \delta) \]

or \( \frac{4}{3} EI(0\delta - 1.58) + EI(20\delta - 1.58) = 0 \)

or \( 80\delta - 88 + 160\delta - 88 + 90\delta - 13.58 + 180\delta - 13.58 = 0 \)

or \( 240\delta + 270\delta = 438 \)

From equation (7), \( 0 = \frac{2}{EI} + \frac{48}{3} - \frac{14}{3} 9\delta \)

Substituting the value of \( 0 \delta \) in equation (8),

\[ \frac{8}{3} EI + \frac{16}{3} 8 - \frac{56}{3} 9\delta - 9\delta + 0\delta - 1.58 + \frac{2}{EI} = 0 \]

or \( \frac{53}{3} 9\delta = \frac{10}{3} + \frac{23}{8} \)

or \( \frac{53}{3} 9\delta = \frac{30}{8} + \frac{23}{106} \)

Substituting the value of \( 0 \delta \) in equation (9), we get

\[ 240\delta + \frac{54}{EI} + 368 - 1266\delta = 438 \]

or \( 102\delta = \frac{54}{102EI} - \frac{7}{102} \delta \)

or \( \delta = \frac{54}{102EI} - \frac{7}{102} \delta \)
Equating the value of $\theta_B$ given by equations 10 and 11,
\[
\frac{30}{53EI} + \frac{23}{106} = \frac{54}{102EI} - \frac{7}{102}
\]
or
\[
0.286 = -\frac{0.035}{EI}
\]

or
\[
\frac{\delta}{EI} = -\frac{0.123}{EI}
\]

Substituting in equation 10,
\[
\frac{0.538}{E} = \frac{30}{53EI} - \frac{23}{106} \times \frac{0.123}{EI} = -0.88 kN-m
\]

Similarly,
\[
\theta_C = \frac{2}{EI} - \frac{4}{3} \times \frac{0.123}{EI} - \frac{14}{3} \times \frac{0.538}{EI}
\]
\[
= \frac{2}{EI} - \frac{0.164}{EI} - \frac{2.51}{EI} = -0.674
\]

(c) Final moments

Substituting in equations 1 to 6, we get the values of end moments. Thus,
\[
M_{AB} = \frac{4}{3} EI \left( \frac{0.538}{EI} + \frac{0.123}{EI} \right) = +0.88 kN-m
\]
\[
M_{BA} = \frac{4}{3} EI \left( \frac{2 \times 0.538}{EI} - \frac{2 \times 0.123}{EI} \right) = +1.60 kN-m
\]
\[
M_{BC} = EI \left( \frac{2 \times 0.538}{EI} - \frac{0.674}{EI} \right) - 2 = -1.60 kN-m
\]

The bending moment diagram and the deflected shape of the frame have been shown in Fig. 9'18.

Example 9'10. A column AB fixed at the ends carries a load of 8 kN on the bracket as shown in Fig. 9'19. Plot the bending moment diagram and the deflected shape of the column.

Solution

The load of 8 kN will give rise to a clockwise couple of 4 kN-m at C. The point C will be displaced by an amount $\delta$.

(a) Slope deflection equations

There are two unknowns $\theta_C$ and $\delta$. We shall assume $\theta_C$ to be positive. $\delta$ for CA is assumed positive and that for CB, is assumed negative.

\[
M_{CA} = \frac{2EI}{3} \left( \theta_C - \frac{38}{3} \right) \quad (1)
\]
\[
M_{CB} = \frac{2EI}{3} \left( 2\theta_C - \frac{38}{3} \right) \quad (2)
\]
\[
M_{BC} = \frac{2EI}{2} \left( 2\theta_C + \frac{38}{2} \right) \quad (3)
\]
\[
M_{CB} = \frac{2EI}{2} \left( \theta_C + \frac{38}{2} \right) \quad (4)
\]

(b) Equilibrium equation

As there are two unknowns, two equations will be required for finding out the values of unknowns. One equation will be provided by the fact that the clockwise couple at C causes clockwise moments in CA and CB.

\[
M_{CA} + M_{CB} = 4
\]
\[
\frac{2EI}{3} \left( 2\theta_C - \frac{38}{3} \right) + \frac{2EI}{2} \left( 2\theta_C + \frac{38}{2} \right) = 4
\]
\[
\text{or} \quad 4 \frac{3}{EI} \theta_C - \frac{2EI}{3} + 2EI \theta_C + \frac{3}{2} EI \delta = 4
\]
\[
\text{or} \quad 20EI \theta C + 5EI \delta = 24
\]
\[
\text{or} \quad 20EI \theta C + 5EI \delta = 24
\]
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(c) *Shear equation*

The couple acting at C also gives rise to horizontal reaction at A and B, the two being equal in magnitude but opposite in direction.

Now horizontal reaction at \( A = \frac{M_{ac} + M_{ca}}{3} \) and, horizontal reaction at \( B = \frac{M_{bc} + M_{cb}}{2} \)

As the two are equal so,

\[
\frac{2EI}{3} (\theta_c - 8) + \frac{2EI}{3} (2\theta_c - 8) = \frac{2EI}{2} \left( \frac{20\theta_c + 3\theta_c}{2} + 8 \right) + \frac{2EI}{2} \left( \frac{\theta_c + \frac{3}{2} - 8}{2} \right)
\]

or

\[
\frac{4}{3} EI\theta_c - \frac{4}{3} EI\theta + \frac{8}{3} EI\theta_c - \frac{4}{3} EI\theta = 6EI\theta_c + \frac{9}{2} EI\theta + 3EI\theta_c + \frac{9}{2} EI\theta
\]

or

\[
30EI\theta_c = -70EI\theta
\]

or

\[
\theta_c = -\frac{7}{3} \theta
\]

The B.M. diagram and the deflected shape have been shown in Fig. 9.20.

Example 9.11. A portal frame \( ABCD \) is hinged at \( A \) and fixed at \( D \) and has stiff joints at \( B \) and \( C \). The loading is as shown in Fig. 9.21. Draw the bending moment diagram and deflected shape of the frame.

Solution

(a) **Fixed end moments**

\[
M_{ABC} = -\frac{6 \times 8}{8} = -1.5 \text{ kN-m}
\]

\[
M_{FBC} = +1.5 \text{ kN-m}
\]

\[
M_{FCD} = -\frac{2 \times 4^2}{12} = -\frac{8}{3} \text{ kN-m}
\]

\[
M_{FDC} = +\frac{8}{3} \text{ kN-m}
\]

(b) **Slope deflection equations**

Let joints \( B \) and \( C \) move horizontally by \( \theta \). There are four unknowns: \( \theta_a, \theta_b, \theta_c \) and \( \theta \).
Assume all unknowns to be positive.

\[ M_{AB} = \frac{2EI}{2} \left( 20a + 0b - \frac{3b}{3} \right) = EI (20a + 0b - \delta) \] (1)

\[ M_{AB} = \frac{2EI}{2} \left( 20a + 0b - \frac{3b}{3} \right) = EI (20a + 0b - \delta) \] (2)

\[ M_{BC} = \frac{2EI}{2} (20c + 0b) - 15 = EI (20c + 0b) - 15 \] (3)

\[ M_{CB} = \frac{2EI}{2} (20c + 0b) + 13 = EI (20c + 0b) + 13 \] (4)

\[ M_{CD} = \frac{2EI}{4} (20c - \frac{3b}{4}) - \frac{8}{3} = \frac{EI}{2} (20c - \frac{3b}{4}) - \frac{8}{3} \] (5)

\[ M_{DC} = \frac{2EI}{4} (20c - \frac{3b}{4}) + \frac{8}{3} = \frac{EI}{2} (20c - \frac{3b}{4}) + \frac{8}{3} \] (6)

(c) Equilibrium equations

At joint B,

\[ M_{BA} + M_{BC} = 0 \]

or

\[ EI (20a + 0b - 8) + EI (20a + 0b) = 3 \] 2 = 0 \]

or

\[ 40a + 0b + 0b - 8 = 3 \frac{2}{2} = 0. \] (7)

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At joint C,

\[ M_{CB} + M_{CD} = 0 \]

or

\[ EI (20c + 0b - 3) + EI (20c + 0b) = 3 \frac{2}{2} = 0 \]

or

\[ 30c + 0b - 3 = -7 \frac{6}{6} = 0 \] (8)

(d) Shear equation

\[ \frac{M_{AB} + M_{BA}}{L_1} + \frac{M_{CD} + M_{DC}}{L_2} = \frac{wL}{2} \]

or

\[ EI (20a + 0b - 3) + EI (20b + 0a - 3) \]

\[ \frac{EI}{2} (20c - \frac{3b}{4}) - \frac{8}{3} + EI \left( \frac{8c - \frac{3b}{4}}{4} \right) + \frac{8}{3} = \frac{2 \times 4}{2} \]

or

\[ 80a + 40b - 38 + 80a + 40b - 38 + 36c - \frac{9}{8} - \frac{36c}{2} - \frac{9}{8} = \frac{48}{EI} \]

or

\[ 120a + 120b - 9 - 0c - \frac{41}{4} = \frac{48}{EI} \] (9)

The end A is hinged. So \( M_{AB} = 0 \)

i.e., \( EI (20a + 0b - 3) = 0 \)

or \( 0b = 8 - 20a \)

Substituting the value of \( 0b \) in equation 7,

\[ 48 - 80a + 0c - 8 = \frac{3}{2} = 0 \]

or

\[ 0c - 70a + 38 = \frac{3}{2} = 0 \] (10)

Substituting the value of \( 0b \) in equation 8,

\[ 36c + 8 - 29a - \frac{3}{8} = \frac{7}{8} = 0 \]

or

\[ 36c - 29a + \frac{5}{8} - \frac{7}{6} = 0 \] (11)

Substituting the value of \( 0b \) and \( 0c \) in equation 9, we get

\[ 120a + 120b - 240a + 27 \frac{4}{4} = 63 \frac{2}{2} = 0 \]

or

\[ 339 - 156a + 47 \frac{72}{6} = 8 \]
Substituting the values of $\delta c$ and $0_A$ in equation 12, we get

$$\delta = \frac{9}{2E_1} \left( -\frac{199}{156E_1} + \frac{47 \times 9}{78} \right) = -\frac{98 + 5}{8} = -\frac{6E_1}{7}$$

or

$$\frac{959}{312} \delta = -\frac{3395}{78E_1}$$

or

$$\delta = -\frac{3395}{78E_1} \times \frac{312}{959} = -\frac{13580}{959E_1}$$

Hence

$$0_A = \frac{330}{156E_1} - \frac{47 \times 13580}{959E_1} = -\frac{642}{E_1}$$

(ii)

and

$$0_B = \frac{3}{2E_1} - \frac{7 \times 642}{E_1} - \frac{3 \times 13580}{959E_1} = -\frac{0.94}{E_1}$$

(iii)

The bending moment diagram and the deflected shape of the frame are shown in Fig. 9.22.

Example 9.12. Analyse the frame shown in Fig. 9.23. $E_1$ is constant for the whole frame.

Solution

(a) Fixed end moments:

$$M_{FB} = -\frac{10 \times 4}{8} = -5 \text{ kN-m} \quad M_{FCB} = +5 \text{ kN-m}$$

(b) Slope deflection equations:

The unknown quantities are $\theta_A, \theta_B, \theta_C$ and $R$. But as $M_{DC}$ is zero, $\theta_D$ can be expressed in terms of $\theta_C$ thus leaving only three unknowns. Treating all these unknowns as positive, we get

$$M_{cA} = 2E_1 \left[ \theta_A - \frac{38}{2} \right]$$

(1)

$$M_{BA} = 2E_1 \left[ \theta_B - \frac{38}{2} \right]$$

(2)

$$M_{BC} = 2E_1 (20\theta_D + \theta_C) - 5$$

(3)

$$M_{CB} = 2E_1 (20\theta_C + \theta_D) + 5$$

(4)

$$M_{CB} = 2E_1 \left[ \frac{20\theta_C + \theta_D - \frac{38}{3}}{} \right]$$

(5)
and
\[ M_{DC} = 2EK \left( 2\theta_b + 0c - \frac{38}{3} \right) = 0 \]
and
\[ 0 = \frac{5 - 0c}{2} \quad (6) \]

(c) Equilibrium equations:
At joint B,
\[ M_{BA} + M_{BC} = 0 \]
\[ 2EK \left( 2\theta_b - \frac{3}{2} \right) + 2EK (2\theta_b + 0_c) - 5 = 0 \]
or
\[ 8\theta_b + 28c - 38 = \frac{5}{EK} \quad (7) \]
At joint C,
\[ M_{CB} + M_{CD} = 0 \]
\[ 2EK (2\theta_b + 0_c) + 5 + 2EK \left( 2\theta_c + 0_d - \frac{38}{3} \right) = 0 \]
or
\[ 8\theta_c + 2\theta_b + 2\theta_b - 28 + \frac{5}{EK} = 0 \quad (8) \]

(d) Shear equations:
\[ \frac{M_{BA} + M_{BC}}{2} + \frac{M_{CD}}{3} = 0 \]
\[ 2EK \left( \theta_b - \frac{3}{2} \right) + 2EK \left( 2\theta_b - \frac{3}{2} \right) \]
\[ + \frac{2EK(2\theta_c + 0_d - 8)}{3} = 0 \]
or
\[ 9\theta_b + 40c + 2\theta_b - 118 = 0 \quad (9) \]
From equation 7, \( \theta_b = \frac{-5}{8EK} + \frac{3}{8} \delta - \frac{0c}{4} \).
Substituting the values of \( \theta_b \) and \( \theta_b \) in equation 8, we get
\[ \frac{13}{2} \quad \theta_c - \frac{8}{4} + \frac{25}{4EK} = 0 \]
or
\[ \delta = 26 \theta_c + \frac{25}{EK} \quad (10) \]
Substituting the values of \( \theta_b \) and \( \theta_b \) in equation 9, we get
\[ \frac{-53}{8} \delta + \frac{3}{4} \theta_c + \frac{45}{8EK} = 0 \]
or
\[ \delta = \frac{6}{53} \theta_c + \frac{45}{53EK} \quad (11) \]
EQUATING the values of \( \delta \) in equations 10 and 11, we get
\[ 26 \theta_c + \frac{25}{EK} = \frac{6}{53} \theta_c + \frac{45}{53EK} \]

(e) Final moments
Substituting the values of \( \theta_a, \theta_c, \theta_d \) and \( \delta \) in equations 1 to 5, we get
\[ M_{AB} = 2EK \left( \frac{1.139}{EK} - \frac{3.75}{EK} \right) = +0.028 \text{ kN-m} \]
\[ M_{BA} = 2EK \left( \frac{2 \times 1.139}{EK} - \frac{3}{2} \times \frac{0.75}{EK} \right) = +2.31 \text{ kN-m} \]
\[ M_{BC} = 2EK \left( \frac{2 \times 1.139}{EK} - \frac{0.933}{EK} \right) = -2.31 \text{ kN-m} \]
\[ M_{CB} = 2EK \left( \frac{2 \times 1.139}{EK} + \frac{0.933}{EK} \right) + 5 = +3.55 \text{ kN-m} \]
and
\[ M_{CD} = 2EK \left( \frac{-2 \times 0.933}{EK} + \frac{0.842}{EK} - \frac{0.75}{EK} \right) = -3.55 \text{ kN-m} \]

The bending moment diagram and the deflected shape of the frame have been shown in Fig. 9.24.

Example 9.13. The frame shown in Fig. 9.25 has fixed ends at A and D. The end A rotates clockwise through \( \frac{0.20}{EK} \) radians and the
end D slips to the right through $0.4\frac{E}{K}$ units. Find the moments induced in the members of the frame and sketch the deflected shape. Take $E/K$ constant.

**Solution.**

Since there is no external loading, there will be no fixed end moments.

When $D$ moves to the right through a known distance $\Delta$ the joint $B$ and $C$ will move to the right through some unknown distance $\delta$. The movement $\delta$ causes rotation of $AB$ and $DC$ and $\Delta$ causes negative rotation of $CD$. So, the net rotation of $DC$ with respect to $AB$ is the algebraic sum of rotations caused by $\delta$ and $\Delta$. There are thus three unknowns: $\delta$, $\theta_c$, and $\delta$.

(a) **Slope deflection equations**:

$$M_{AB} = 2EK \left( \frac{2 \times 0.2}{E/K} + 0.6 - \frac{3\delta}{3} \right)$$  

(b) **Equilibrium equations**:

At joint $B$:

$$M_{BA} + M_{BC} = 0$$

or

$$2EK \left( 2\theta_B + \frac{0.2}{E/K} - \frac{3\delta}{3} \right) + 2EK (2\theta_B + 6\delta) = 0$$

or

$$40\theta_B + 6\delta - 8 + \frac{0.2}{E/K} = 0$$

At joint $C$,

$$M_{CB} + M_{DC} = 0$$

or

$$2EK (2\theta_C + 0.6) + 2EK \left\{ 2\theta_C + 3 \left( \frac{0.4}{E/K} - \delta \right) \right\} = 0$$

or

$$8\theta_C + 2\theta_B + \frac{1.2}{E/K} - 3\delta = 0$$

(c) **Shear equation**:

$$M_{AB} + M_{CB} + M_{CD} = 0$$

or

$$2EK \left( \frac{0.4}{E/K} + \theta_B + \delta \right) + 2EK \left( 2\theta_B + \frac{0.2}{E/K} - \delta \right)$$

or

$$2EK \left\{ \frac{2\theta_C + \frac{3}{2} \left( \frac{0.4}{E/K} - \delta \right)}{2} \right\}$$

or

$$2EK \left\{ \theta_C + \frac{3}{2} \left( \frac{0.4}{E/K} - \delta \right) \right\} = 0$$

This reduces to,

$$\frac{48}{E/K} + 6\theta_B + 9\theta_C - 13\delta = 0$$

Substituting the value of $\theta_C$ from equation 7 in equation 8, we get

$$88 - \frac{1.6}{E/K} - 32\theta_B + 26\theta_B + \frac{1.2}{E/K} - 3\delta = 0$$

or

$$5 \delta - \frac{0.4}{E/K} = 30 \theta_B$$

or

$$\delta_B = \frac{5}{6} - \frac{0.4}{30 E/K}$$

Substituting the values of $\theta_C$ and $\theta_B$ in equation 9, we get

$$\frac{48}{E/K} + 3\delta - \frac{0.4}{E/K} + 9\delta - \frac{1.8}{E/K} - 68 + \frac{0.48}{E/K} - 13\delta = 0$$

or

$$\delta = \frac{3\delta^3}{9E/K}$$
Hence \( \theta_b = \frac{3 \cdot 4}{9 \times 6E} - \frac{0 \cdot 4}{30E} = \frac{0 \cdot 0497}{E} \) radians \( \ldots (ii) \)

and \( \theta_c = \frac{3 \cdot 4}{9E} - \frac{0 \cdot 021}{E} = - \frac{0 \cdot 021}{E} \) radians \( \ldots (iii) \)

(d) Final moments:

Substituting in equations 1 to 6, we get the value of moments as follows:

\[
\begin{align*}
M_{ab} &= +0 \cdot 145 \text{ units; } M_{ba} = -0 \cdot 156 \text{ units} \\
M_{bc} &= +0 \cdot 156 \text{ units; } M_{cb} = +0 \cdot 0154 \text{ units} \\
M_{cd} &= -0 \cdot 0154 \text{ units; } M_{dc} = +0 \cdot 025 \text{ units}
\end{align*}
\]

It is to be noted that the quantity \( EK \) has the units of moment. The units of the above moments will, therefore, be the same as the units of \( EK \).

The bending moment diagram and the deflected shape of the frame have been shown in Fig. 9-26.

Example 9.14. The portal frame shown in Fig. 9-27 has fixed ends. If the end \( D \) sinks by \( \Delta \), find the moment induced in the frame. The members have the same uniform cross-section.
(c) Shear equation

\[
\frac{M_{AB} + M_{BA}}{3L} + \frac{M_{CD} + M_{DC}}{2L} = 0
\]

\[
2EI\theta_B - \frac{2EI\theta_B}{3L^2} + \frac{4EI\theta_B}{3L} - \frac{2EI\theta_B}{3L^2} - \frac{2EI\theta_B}{3L^2} + \frac{EI\theta_C}{L} - \frac{3EI\theta}{2L^2} = 0
\]

It reduces to,

\[
\frac{12\theta_B + 270\theta_C - 35}{L} = 0
\]

Substituting the values of \(\theta_B\) in equation 7,

\[
\frac{58}{L} + \frac{5\Delta}{L} - \frac{40}{3} \theta_C + \frac{2}{3} \frac{8}{L} - \frac{3}{4} \frac{\Delta}{L} = 0
\]

or

\[
\theta_C = \frac{13}{37} \cdot \frac{8}{L} - \frac{21}{74} \Delta
\]

Substituting the value of \(\theta_B\) in equation 9,

\[
\frac{188}{L} + \frac{18\Delta}{L} - \frac{480\theta_C + 27}{3} \frac{\Delta}{L} - \frac{358}{L} = 0
\]

or

\[
\theta_C = \frac{6\Delta}{72} - \frac{17}{21} \Delta
\]

Equating the two values of \(\theta_C\),

\[
\frac{6}{7L} \Delta - \frac{17}{21L} \Delta = \frac{13}{37L} \Delta + \frac{21}{74L} \Delta
\]

\[
\theta_C = \frac{13}{37L} \Delta + \frac{21}{74L} \Delta
\]

\[
\theta_B = \frac{3}{2L} \times 0.4935 \Delta + \frac{3}{2L} \Delta - 4 \times 0.457 \Delta = 0.412 \Delta
\]

(d) Final moments:

By substituting the values of \(\theta_B\), \(\theta_C\) and \(\Delta\) in equations 1 to 6 we get the end-moments as follows:

\[
M_{AB} = \frac{0.06EI\Delta}{L^2}; \quad M_{BA} = \frac{0.22EI\Delta}{L^2}
\]

\[
M_{BC} = \frac{0.22EI\Delta}{L^2}; \quad M_{CB} = \frac{0.17EI\Delta}{L^2}
\]

\[
M_{CD} = \frac{0.17EI\Delta}{L^2}; \quad M_{DC} = \frac{0.28EI\Delta}{L^2}
\]

The bending moment diagram and the deflected shape have been given in Fig 9.28. The values marked in Fig 9.28 (a) are to be multiplied by the factor \(\frac{EI\Delta}{L^3}\).

PROBLEMS

1. A beam ABC, 12 m long, fixed at A and C and continuous over support B, is loaded as shown in Fig. 9.29. Calculate the end moments and plot the bending moment diagram.

2. A continuous beam ABCD is fixed at ends A and D, and is loaded as shown in Fig. 9.30. Spans AB, BC and CD have moments of inertia of I, 1.5I and 0.5I respectively and are of the same material. Determine the moments at the supports and plot the bending moment diagram.
3. Solve problem 2 if there is no support at $D$.

4. Using the slope deflection method, calculate the moments at the support of the beam loaded as shown in Fig. 9-31.

5. Draw the bending moment diagram and sketch the deflected shape of the frame shown in Fig. 9-32. All members are of the same material.

6. Draw the bending moment diagram and sketch the deflected shape of the frame shown in Fig. 9-33.

7. The frame $ABCDEF$ shown in Fig. 9-34 has rigid joints throughout and is rigidly held at $A$, $E$ and $F$. It carries a uniformly distributed load of $w$ per unit length along $BD$. The stiffness ratios of the members are shown in the diagram and all the members are of equal length. Determine the bending moment throughout the frame and sketch the bending moment diagram.
8. A portal frame $ABCD$, fixed at ends $A$ and $D$ carries a point load 2.5 kN as shown in Fig. 9.35. Draw the bending moment diagram and sketch the deflected shape of the beam.

9. Analyse completely the portal frame shown in Fig. 9.36.

Answers:

1. $M_{AB} = -5.25; M_{BA} = +7.5; M_{BC} = -7.5; M_{CD} = +9.75$ kN-m.
2. $M_{AB} = -4.6$ kN-m; $M_{BA} = +2.98; M_{BC} = -2.98; M_{CB} = +5.7$ kN-m.
3. $M_{AB} = -5.83; M_{BA} = +0.55; M_{BC} = -0.55; M_{CB} = +16$ kN-m.
4. $M_{AB} = -7.06$ kN-m; $M_{BA} = +3.63; M_{BC} = -3.63; M_{CB} = +5.14$; $M_{CD} = +5.10$ kN-m.
5. $M_{AB} = -0.82$ kN-m; $M_{BA} = +1.35; M_{BC} = -1.51; M_{CB} = +0.99$ kN-m.
6. $M_{AB} = +0.91; M_{BA} = +1.82; M_{BC} = -1.82; M_{CB} = +1.71$ kN-m.
7. $M_{AB} = + wL^3/12; M_{BA} = - wL^3/12; M_{BC} = - wL^3/36; M_{CB} = + wL^3/36$ kN-m.
8. $M_{AB} = + wL^3/9; M_{BA} = + wL^3/9; M_{BC} = - wL^3/36; M_{CB} = + wL^3/36$ kN-m.
9. $M_{AB} = +0.137$ kN-m; $M_{BA} = +0.647; M_{BC} = -0.647; M_{CB} = +0.461$ kN-m.

10.1. INTRODUCTION: SIGN CONVENTIONS

The method of moment distribution belongs to the group of approximate methods. Essentially, it consists of solving the simultaneous equations in the slope deflection method by successive approximations using a series of cycles, each converging towards the precise final result. The series may, therefore, be terminated whenever one reaches the degree of precision required by the particular problem under consideration. It leads to a very substantial reduction in the number of equations and in the case of structures, where joints can sustain angular twists alone but cannot be deflected, the method permits to avoid completely the solution of simultaneous equations with several unknowns. The method was first introduced by Prof. Hardy Cross in 1930. The moment distribution method could be used for the analysis of all types of statically indeterminate beams or rigid frames.

The sign convention used in the case of bending of simple beams, etc., becomes clumsy if used for the case of more complex beams.
beams and frames where more than two members meet at a joint. In our earlier sign convention for simple beams, a moment is considered to be positive if it bends the beam convex upwards, and negative if it bends the beam concave upwards. Thus, for the case of structure shown in Fig. 10.1, the three moments acting at the rigid joint B, where the three members BA, BC and BD meet, are all positive according to the previous convention since all the three moments tend to bend the three corresponding beams convex upwards. Hence the equilibrium equation \( \Sigma M_B = 0 \) at the joint B cannot be conveniently applied if the previous sign convention is used, though the joint B is in equilibrium.

However, examination of joint E (Fig. 10.1) reveals that the moments, \( M_{BA} \) and \( M_{BD} \) are clockwise while the moment \( M_{BC} \) is anticlockwise. If the new sign convention is based on the direction of the moment, we get

\[ M_{BA} + M_{BD} - M_{BC} = 0. \]

Hence in the new sign convention that will be used in this method, a support moment acting in the clockwise direction will be taken as positive and that in the anti-clockwise direction as negative. A corresponding change will have to be made while plotting the support moment diagram. For any span of a beam or member with rigid joints, a positive support moment (or end moment) at the right hand end will be plotted above the base line and negative support moment below the base. Similarly, for the left hand end, the negative end moment is plotted above the base line and positive end moment is plotted below the base line, as shown in Fig. 10.2.

In addition to the above sign convention for the end moments, the following sign convention for the rotation and settlement is adopted: (1) A clockwise rotation will be taken as positive and anti-clockwise rotation as negative. (2) If one end of the beam settles, the settlement will be taken as positive if it rotates the beam as a whole in the clockwise direction, and negative if it rotates the beam as a whole in the anticlockwise direction.

10.2. FUNDAMENTAL PROPOSITIONS

For the better understanding of the theory and the mechanism of moment distribution, the following fundamental relations and deductions for prismatic beams are useful.

1. Beam hinged at both ends

Fig. 10.3 (a) shows a beam hinged at both ends, and subjected to a moment \( \mu \) at the end A. Due to this, the rotations at the ends A

![Fig. 10.2: Sign convention.](image)

![Fig. 10.3: Beam with both ends hinged.](image)
and $B$ are $\theta_{AB}$ and $\theta_{BA}$ respectively. Fig. 10.3 (b) shows the corresponding bending moment diagram. Since the beam is hinged at both the ends, the fixed end moments $M_{FAB}$ and $M_{FBA}$ are zero. Hence the slope-deflection equations for the span $AB$ are:

$$M_{AB} = \frac{2EI}{L} [2\theta_{AB} + 0_{BA}] \quad (1)$$

and

$$M_{B} = \frac{2EI}{L} [0_{AB} + 29_{BA}] \quad (2)$$

For the joint $B$, the equilibrium equation is

$$M_{BA} = 0$$

Hence, from (2), we get

$$0_{BA} = -\frac{1}{2} 0_{AB}$$

Substituting this in (1) and noting that $M_{AB} = \mu$, we get

$$\mu = \frac{2EI}{L} \left[ 2\theta_{AB} - \frac{0_{AB}}{2} \right] = \frac{3EI}{L} \theta_{AB} \quad [10.1 \text{ (a)}]$$

If $\theta_{AB} = \text{unity}$, we have

$$\mu = k = \frac{3EI}{L} \quad (10.1)$$

Eq. 10.1 gives the following important proposition:

**Proposition 1**

The moment $k$ required to rotate the near end of a prismatic beam through a unit angle, without translation, the far end being freely supported, is given by

$$k = \frac{3EI}{L}$$

This moment $k$ is known as absolute stiffness or simply, stiffness. The stiffness of a member is the moment required to rotate the end under consideration through unit angle.

2. **Beam hinged at one end and fixed at the other end**

Fig. 10.4 (a) shows a prismatic beam hinged at $A$ and fixed at $B$, and subjected to a moment $\mu$ at the hinged end (called the near end). Due to moment $\mu$, the end $A$ rotates through angle $\theta_{AB}$ while the rotation of end $B$ is zero since it is fixed. Let the induced moment at the end $B$ be $\mu'$. Since there is no external loading on the beam except the moment $\mu$, the fixed end moments $M_{FAB}$ and $M_{FBA}$ are zero. Hence the slope deflection equation are:

$$M_{AB} = \mu = \frac{2EI}{L} [2\theta_{AB} + 0] \quad (1)$$

and

$$M_{B} = \frac{2EI}{L} [0_{AB} + 29_{BA}] \quad (2)$$

From (1), we get

$$\mu = \frac{4EI}{L} \theta_{AB} \quad (3) \quad [10.2]$$

or

$$\theta_{AB} = \frac{\mu L}{4EI} \quad (4)$$

Substituting in (2) the value of $\theta_{AB}$, we get

$$\mu' = \frac{\mu}{2} \quad (10.3)$$

Eqs. 10.2 and 10.3 give the following two important propositions:

**Proposition 2**

A moment $k$ required to rotate the near end of a prismatic beam through unit angle, without translation, the far end being fixed, is given by

$$k = \frac{4EI}{L} \quad [10.2(a)]$$
Proposition 3

A moment which rotates the near end of prismatic beam without translation, the far end being fixed, induces at the far end a moment of one half its magnitude and in the same direction.

3. Several members meeting at a joint

Fig. 10-5 (a) shows members AO, BO, CO and DO meeting at a rigid joint O. Let $L_1$, $L_2$, $L_3$ and $L_4$ be the lengths and $I_1$, $I_2$, $I_3$ and $I_4$ be the moments of inertia of the respective members AO, BO, CO and DO. When an external moment $\mu$ is applied at the rigid joint $O$, the joint will rotate through an angle $\theta$. Due to the rigidity of the joint, all the members will be rotated through the same angle $\theta$.

The applied moment $\mu$ will be resisted collectively by all the four members. Let $\mu_1$, $\mu_2$, $\mu_3$ and $\mu_4$ be the shares of the applied moment $\mu$ resisted by the members $OA$, $OB$, $OC$ and $OD$ respectively, as shown in Fig. 10-5 (b), so that $\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4$. The magnitudes of these moments depend upon the stiffness of the members, and are given by Eqs. 10-1 (a) and 10-2.

$$\mu_1 = \frac{3EI_1}{L_1} \theta = k_1 \theta$$  \hspace{1cm} (1) [Eq. 10-1 (a)]

$$\mu_2 = \frac{4EI_2}{L_2} \theta = k_2 \theta$$  \hspace{1cm} (2) [Eq. 10-2]

$$\mu_3 = \frac{3EI_3}{L_3} \theta = k_3 \theta$$  \hspace{1cm} (3) [Eq. 10-1 (a)]

The quantities $k_1$, $k_2$, $k_3$ and $k_4$ are called distribution factors for the members $OA$, $OB$, $OC$ and $OD$ respectively. The moments $\mu_1$, $\mu_2$, $\mu_3$ and $\mu_4$ are called the distributed moments and are in a direction opposite to that of $\mu$ since they finally restore the equilibrium at the joint.

Relative stiffness ($K$): When several members meeting at a joint have different conditions of support at their other ends, it is always convenient to express the stiffness of the members in terms of relative stiffness. We have seen that the absolute stiffness ($k$) of a prismatic member with far end fixed is $\frac{4EI}{L}$. If all the prismatic members meeting at the joint are of the same material and are fixed at the far end, the stiffness of each member relative to the others can be expressed as:

$$\mu_1 = \frac{4EI}{L_1} k_1$$  \hspace{1cm} (4) (Eq. 10-2)

where $k_1$, $k_2$, $k_3$ and $k_4$ are the stiffness of the members $OA$, $OB$, $OC$ and $OD$ respectively. From (1) to (4), we get

$$\mu_1 : \mu_2 : \mu_3 : \mu_4 = k_1 : k_2 : k_3 : k_4$$  \hspace{1cm} (10-4)

Also, from statics

$$\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4$$  \hspace{1cm} (II)

From (1) and (II), we get the following expressions for the moments $\mu_1$, $\mu_2$, $\mu_3$ and $\mu_4$:

$$\mu_1 = \frac{k_1}{k_1 + k_2 + k_3 + k_4} \mu$$  \hspace{1cm} (a)

$$\mu_2 = \frac{k_2}{k_1 + k_2 + k_3 + k_4} \mu$$  \hspace{1cm} (b)

$$\mu_3 = \frac{k_3}{k_1 + k_2 + k_3 + k_4} \mu$$  \hspace{1cm} (c)

$$\mu_4 = \frac{k_4}{k_1 + k_2 + k_3 + k_4} \mu$$  \hspace{1cm} (d)

Eq. 10-4 gives the following proposition:

Proposition 4

A moment which tends to rotate a joint without translation, will be divided amongst the connecting members at the joint in proportion to their "stiffness".

The quantities $\frac{k_1}{\Sigma k}$, $\frac{k_2}{\Sigma k}$, $\frac{k_3}{\Sigma k}$ and $\frac{k_4}{\Sigma k}$ are called distribution factors for the members $OA$, $OB$, $OC$ and $OD$ respectively. The moments $\mu_1$, $\mu_2$, $\mu_3$ and $\mu_4$ are called the distributed moments and are in a direction opposite to that of $\mu$ since they finally restore the equilibrium at the joint.

Fig. 10-5 (a) shows members meeting at a joint. Several members meeting at a joint.
may be represented by \( \frac{I}{L} \). However, if some of the prismatic members meeting at the joints are freely supported at the other end, their relative stiffness may be taken equal to \( \frac{3}{4} \cdot \frac{I}{L} \), since the absolute stiffness of such members is \( \frac{3EI}{L} \) and is \( \frac{3}{4} \) of that of the members fixed at the far end.

**Example 10-1.** Calculate the distributed moments for the members OA, OB, OC and OD (Fig. 10-5), if their lengths are 150, 200, 100 and 200 cm and moments of inertia are 300, 400, 300 and 200 cm\(^4\) units respectively. The applied moment at joint O is 8100 N-cm.

**Solution**

The calculations are arranged in Table 10-1.

<table>
<thead>
<tr>
<th>Member</th>
<th>Length (cm)</th>
<th>I ( \text{cm}^4 )</th>
<th>Absolute stiffness ( \frac{kN\cdot m}{\text{cm}} )</th>
<th>Distributed moment ( \text{N-cm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>OA</td>
<td>150</td>
<td>300</td>
<td>( \frac{3\times300}{150} = 6E )</td>
<td>6 ( \frac{27}{2} )</td>
</tr>
<tr>
<td>OB</td>
<td>200</td>
<td>400</td>
<td>( \frac{4\times400}{200} = 8E )</td>
<td>8 ( \frac{27}{2} )</td>
</tr>
<tr>
<td>OC</td>
<td>100</td>
<td>300</td>
<td>( \frac{3\times300}{100} = 9E )</td>
<td>9 ( \frac{27}{2} )</td>
</tr>
<tr>
<td>OD</td>
<td>200</td>
<td>200</td>
<td>( \frac{4\times200}{200} = 4E )</td>
<td>4 ( \frac{27}{2} )</td>
</tr>
</tbody>
</table>

Sum \( 3k = 27E \)

\( \sum \text{Distribution factor} \)

10.3. THE MOMENT DISTRIBUTION METHOD

The basic mechanism of moment distribution can be best understood with reference to Fig. 10-6 which shows a two span continuous beam ABC with ends A and C fixed. The method essentially consists of first locking all the joints (which are not fixed) so that each span of the structure behaves like a fixed beam. For the present case of Fig. 10-6 (a), joint B is locked against rotation. The fixed end moments of the two beams AB and BC are:

\[
M_{FAB} = \frac{-W_{ab}^2}{L^2} = -\frac{5 \times 3 \times 2^2}{5^2} = -2.40 \text{ kN-m}
\]

\[
M_{FBA} = \frac{+W_{ab}^2}{L^2} = +\frac{5 \times 3^2 \times 2}{5^2} = +3.60 \text{ kN-m}
\]

\[
M_{FBC} = \frac{-WL}{8} = -\frac{8 \times 5}{8} = -5.00 \text{ kN-m}
\]

\[
M_{FCB} = \frac{+WL}{8} = +\frac{8 \times 5}{8} = +5.00 \text{ kN-m}
\]
The fixed end moment $M_{FAB}$ and $M_{FBC}$ are negative since they act in the anti-clockwise direction, while $M_{FBA}$ and $M_{FCB}$ act in the clockwise direction and hence positive according to our new sign convention. Fig. 10.6(b) shows the fixed end moments marked in the appropriate directions. At the joint $B$, there is an unbalanced moment of $1\,4 \text{kN-m}$ ($i.e.$ $5\,000 - 3\,600 = 1\,400$) acting in the anti-clockwise direction. It is this unbalanced moment which keeps the joint $B$ locked against rotation. Actually, joint $B$ is free to rotate.

In order to permit it to rotate or to unlock it, a balancing moment of $+1\,4 \text{kN-m}$ ($i.e.$ clockwise) is applied at $B$ as shown in Fig. 10.6(d). Now according to proposition 4, end moment applied at a joint is to be resisted by the members meeting at the joint in proportion to their stiffnesses. Here, both the beams have equal span of 5m, and their $EI$ is also the same. Hence their stiffnesses are equal. Due to this the distribution factors at $B$ will be 0.5 for each of the beams $BA$ and $BC$ as marked in Fig. 10.6(e) showing the moment distribution in the tabular form. Thus, the balancing moment of $+1\,4 \text{kN-m}$ is distributed equally to both the beams $BA$ and $BC$, each one getting a balancing moment of $+0\,70 \text{kN-m}$, as indicated in step 2 of Fig. 10.6(e). Since the far ends of the beams $BA$ and $BC$ are fixed, the carry over moments at the far ends $A$ and $C$ are each $+0\,35 \text{kN-m}$ as indicated in step 3. Thus, all the three steps (step 1: calculation of F.E.Ms; step 2: balancing the joint, and step 3: carry over to the far ends) constitute one cycle of moment distribution.

At the third step, Fig. 10.6(e), when the moments are carried over from the near ends to the far ends there is no carry over moments from joint $A$ to $B$, or from joint $C$ to $B$, since joints $A$ and $C$ are originally fixed. It should be remembered that a fixed joint does not need any balancing and it absorbs all the moments carried over to it from the other end. Thus, in the third step, joint $B$ does not have any unbalanced moment. The second cycle consists of the balancing of the joint $B$, which, in the present case, is not required. Hence there are no balancing moments against step 4 and the moment distribution is over. The last step consists of finding the final moments at each joint by taking the algebraic sum of the moments in each of the vertical columns for $A$, $B$ and $C$. The final moments are:

At $A$, $M_{AB} = -1\,05 \text{kN-m}$

At $B$, $M_{BA} = +4\,30 \text{kN-m}$, $M_{BC} = -4\,30 \text{kN-m}$ joint perfectly balanced

At $C$, $M_{CB} = +5\,35 \text{kN-m}$

Fig. 10.6(f) shows the final bending moment diagram for the beam.

In general, the complete moment distribution process consists of a number of cycles. The carried over moments (step 3) may constitute the unbalanced moments for the next cycle. The following examples explain the procedure for other cases when the far ends of the beam may or may not be fixed. For example, if joints $A$ and $C$ are not initially fixed, but are hinged, they are locked first, and then balanced in step 2. Then, in step 3 joint $B$ receives carry over moments from both joints $A$ as well as $C$. These moments become the unbalanced moments for the second cycle, as indicated in Example 10.2.

Example 10.2. Solve problem of Fig. 10.6(a) if ends $A$ and $C$ are simply supported (or hinged).

[Diagram of the beam with loads and moments as shown in Fig. 10.7(a)]

Solution

Step 1. Lock the joints $A$, $B$ and $C$, and calculate the fixed end moments by treating $AB$ and $BC$ as two fixed beams. The moments are:

$M_{FAB} = -2\,40 \text{kN-m} ;
M_{FBA} = +3\,60 \text{kN-m} ;
M_{FBC} = -5\,00 \text{kN-m} ;
M_{FCB} = +5\,00 \text{kN-m}$
Since both the beams have the same values of \( E, I \) and \( L \), and have the same end conditions, their relative stiffness will be equal. Thus the distribution factors will be 0.5 each at joint \( B \) for members \( BA \) and \( BC \).

**Step 2.** Unlock the joints \( A, B \) and \( C \) in succession, and balance them by applying balancing moments in a direction opposite to the unbalanced moment at that joint. Thus, at joint \( A \), the unbalanced moment is \(-2.40\, \text{kN-m}\), and hence the balancing moment will be \(+2.40\, \text{kN-m}\). At joint \( C \), the balancing moment will be \(-5.00\, \text{kN-m}\). At joint \( B \), the unbalanced moment is \(-1.40\, \text{kN-m}\) and hence the balancing moment will be \(+1.40\, \text{kN-m}\), which will be applied equally (i.e., \(+0.70\, \text{kN-m}\)) to each of the spans \( BA \) and \( BC \) at joint \( B \). This is indicated in step 2 of Table 10.2.

**Table 10.2**

<table>
<thead>
<tr>
<th>Cycle</th>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.5</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-2.40</td>
<td>+3.60</td>
<td>-5.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>+2.40</td>
<td>+0.70</td>
<td>+0.70</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>+0.35</td>
<td>+1.20</td>
<td>-2.50</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-0.35</td>
<td>+0.65</td>
<td>+0.65</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>+0.32</td>
<td>-0.17</td>
<td>-0.17</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-0.32</td>
<td>+0.17</td>
<td>+0.17</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>+0.09</td>
<td>-0.16</td>
<td>-0.16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-0.09</td>
<td>+0.16</td>
<td>+0.16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>+0.08</td>
<td>-0.04</td>
<td>-0.04</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-0.08</td>
<td>+0.04</td>
<td>+0.04</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>+6.15</td>
<td>-6.15</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Step 3.** Half of the balancing moment, with the same sign (see proposition 3) is carried over to the opposit joint. These carried over moments constitute unbalanced moment for the second cycle. Hence a line is drawn after step 2. The complete moment distribution is shown in Table 10.2. The moment distribution procedure can be stopped at the end of any cycle when the needed accuracy is achieved. In the present case, the carried over moments at joint \( B \) from joints \( A \) and \( C \) are equal in the third cycle. Hence the procedure can be terminated at the end of the third cycle.

Fig. 10.8 shows the final bending moment diagram and the deflected shape of the beam.

**Example 10.3.** A continuous beam \( ABCD \) consists of three span, and is loaded as shown in Fig. 10.8(a). Ends \( A \) and \( D \) are fixed. Determine the bending moments at the supports and plot the bending moment diagram.

**Solution**

(a) Fixed end moments (kN-m units)

\[
M_{FA} = -\frac{2 \times 6^3}{12} = -6 \quad \text{and} \quad M_{AF} = +\frac{2 \times 6^2}{12} = +6
\]

\[
M_{FB} = -\frac{5 \times 3 \times 2^3}{5^2} = -2.4 \quad \text{and} \quad M_{FB} = +\frac{5 \times 2 \times 3^3}{5^2} = +3.6
\]

\[
M_{FC} = -\frac{8 \times 5}{8} = -5 \quad \text{and} \quad M_{FC} = +\frac{8 \times 5}{8} = +5
\]
### Table 10.3

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>Distribution factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>BA</td>
<td>1/6</td>
<td>17/6</td>
<td>5/17</td>
</tr>
<tr>
<td></td>
<td>BC</td>
<td>21/5</td>
<td>30/5</td>
<td>12/17</td>
</tr>
<tr>
<td>C</td>
<td>CB</td>
<td>21/5</td>
<td>3/5</td>
<td>2/3</td>
</tr>
<tr>
<td></td>
<td>CD</td>
<td>1/5</td>
<td></td>
<td>3/3</td>
</tr>
</tbody>
</table>

(c) Moment distribution

The moment distribution is carried out in Table 10.4. The final bending moment diagram and the deflected shape of the beam are shown in Fig. 10.8 (b) and (c) respectively.

### Table 10.4

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image.png" alt="Image" /></td>
<td><img src="image.png" alt="Image" /></td>
<td><img src="image.png" alt="Image" /></td>
<td><img src="image.png" alt="Image" /></td>
</tr>
</tbody>
</table>

Note. It should be noted that the last cycle ends with the balancing of hinged or continuous joints and with carry over to the fixed joints.

### Example 10.4

Solve example 10.3 if the ends A and D are simply supported (or hinged).

Solution:

(a) Fixed end moments

- Lock all the joints against rotation. The fixed end moments will be the same as found in the previous example.

(b) Distribution factors

- Considering joints A and D locked when the other joints are balanced, the same distribution factors, as calculated in the previous example will be applicable.

(c) Moment distribution (Table 10.5)

### Table 10.5

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image.png" alt="Image" /></td>
<td><img src="image.png" alt="Image" /></td>
<td><img src="image.png" alt="Image" /></td>
<td><img src="image.png" alt="Image" /></td>
</tr>
</tbody>
</table>

Final moments

Note. It should be noted that the last cycle ends with the balancing of hinged or continuous joints and with carry over to the fixed joints.
Alternative Solution

An alternative method of solving the problem is to keep ends A and D free throughout the operation, except in the beginning when the fixed end moments are calculated. In that case, the relative stiffness of BA and CD will be that of what have been taken in the previous example. The revised distribution factors are calculated in Table 10.6.

![Diagram of an engineering problem with moments and forces]

Table 10.6

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative Stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>BA</td>
<td>$\frac{3}{4}$ : $\frac{1}{6}$</td>
<td>$63I$</td>
<td>$\frac{15}{63} = \frac{5}{21}$</td>
</tr>
<tr>
<td></td>
<td>BC</td>
<td>$\frac{21}{5}$</td>
<td>$120$</td>
<td>$\frac{48}{63} = \frac{16}{21}$</td>
</tr>
<tr>
<td>C</td>
<td>CB</td>
<td>$\frac{21}{5}$</td>
<td>$11I$</td>
<td>$\frac{11}{11}$</td>
</tr>
<tr>
<td></td>
<td>CD</td>
<td>$\frac{3}{4}$ : $\frac{1}{5}$</td>
<td>$2I$</td>
<td>$\frac{3}{11}$</td>
</tr>
</tbody>
</table>

In the first cycle of the moment distribution (Table 10.7), only joint A and D are released and balanced, and half of the balanced moments are carried over to B and C respectively. A line is then drawn and the initial moments are found by taking the sum of the first two lines of the first cycle. In process of the balancing and carry over that follows in the subsequent cycles, ends B and D are kept permanently free so that they neither need balancing nor is any moment carried over to them. The moment distribution is shown in Table 10.7.

Table 10.7

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>16</td>
<td>8</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative Stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>BA</td>
<td>$\frac{3}{4}$ : $\frac{1}{6}$</td>
<td>$63I$</td>
<td>$\frac{15}{63} = \frac{5}{21}$</td>
</tr>
<tr>
<td></td>
<td>BC</td>
<td>$\frac{21}{5}$</td>
<td>$120$</td>
<td>$\frac{48}{63} = \frac{16}{21}$</td>
</tr>
<tr>
<td>C</td>
<td>CB</td>
<td>$\frac{21}{5}$</td>
<td>$11I$</td>
<td>$\frac{11}{11}$</td>
</tr>
<tr>
<td></td>
<td>CD</td>
<td>$\frac{3}{4}$ : $\frac{1}{5}$</td>
<td>$2I$</td>
<td>$\frac{3}{11}$</td>
</tr>
</tbody>
</table>

Example 10.5. Solve example 10.3 if there is no support at the end D.
Solution.

(a) Fixed end moments

Ends B and C are clamped and AB and BC are considered as fixed beams. The overhanging portion CD becomes a cantilever fixed at C. Hence

\[ M_{FAB} = -6.00 \, \text{kN-m} ; \quad M_{FBA} = +6.00 \, \text{kN-m} \]
\[ M_{FBC} = -2.40 \, \text{kN-m} ; \quad M_{FCB} = +3.60 \, \text{kN-m} \]
\[ M_{FCD} = -8 \times 2.5 = -20 \, \text{kN-m} \]

(b) Distribution factors

The stiffness of the cantilever CD is zero since it has no resistance to rotation if an external moment is applied at the freely supported end C. The distribution factors at B are calculated on the premise that end C will be kept free throughout the process of moment distribution.

Table 10:3

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative Stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>BA</td>
<td>( \frac{1}{6} )</td>
<td>71</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>BC</td>
<td>( \frac{3}{4} )</td>
<td>15</td>
<td>9</td>
</tr>
<tr>
<td>C</td>
<td>CB</td>
<td>( \frac{21}{5} )</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>CD</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

The process of balancing and carry over, end C is kept permanently free so that it neither needs balancing nor is any moment carried over to it.

Table 10:9

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>14</td>
<td>14</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

-6.00  +6.00 -2.40 +3.60 -20.00 0.00  F.E.M.
-       -       +8.20 +16.40 -
-6.00  +6.00 +5.80 +20.00 -20.00 - Initial
-2.11  -4.21 -7.59 -
-8.11  +1.79 -1.79 +20.00 -20.00 - Final moments

10.4. SINKING OF SUPPORTS

(a) Beam fixed at both the ends

In the previous treatment, we have considered continuous beams resting on rigid supports which do not yield or sink under loads. However, if one of the ends of the beam sinks, additional moments will be induced at both the ends of the beam. If the sinking of the support is such as to rotate the beam as a whole in the clock-
wise direction, the moments at both the ends will be in the anti-clockwise direction and of equal magnitude. Similarly, if the sinking rotates the beam in anti-clockwise direction, the induced moments at both the ends will be in clockwise direction.

Fig. 10'11(a) (I) shows a fixed beam of span $L$, with right hand support sunk by an amount $\delta$. Fig. 10'11 [a(ii)] shows the component bending moment diagram due to the moments $M$ induced at each end, while Fig. 10'12 [a(iii)] shows the net bending moment diagram. The beam has evidently, a point of contraflexure at its middle point.

\[ M = \frac{6EI}{L^2} \]

Fig. 10'11
Sinking of Supports

From the conjugate beam method, we get

\[ \delta = \frac{1}{EI} \left( ML \times \frac{2}{3} L - \frac{ML}{2} \times \frac{1}{3} L \right) = \frac{ML^2}{6EI} \]

Hence

\[ M = \frac{6EI\delta}{L^2} \quad (10'6) \]

This is thus, the expression for the moment induced at both the ends. According to our new sign convention since this moment acts in anti-clockwise direction at each end, it is negative.

Similarly, if the left hand of the beam sinks by an amount $\delta$ [Fig. 10'11 b (i)], the moment induced at each end is given by Eq. 10'6 and will be positive at each end.

(b) Beam fixed at one end and freely supported at the other end [Fig. 10'11 (c)].

If the beam is freely supported at the other end, the moment induced at the fixed end will be in the anti-clockwise direction if the rotation of the beam as a whole is clockwise and vice versa. The bending moment diagram will be a triangle having an ordinate $M$ at the fixed end. From conjugate beam method, we get

\[ M = \frac{3EI\delta}{L^2} \]

or

\[ \delta = \frac{1}{EI} \left( \frac{ML}{2} \times \frac{2}{3} L \right) = \frac{ML^2}{3EI} \]

\[ M = \frac{6EI\delta}{L^2} \quad (10'7) \]

10'5. CONTINUOUS BEAM ON ELASTIC PROPS

In the case of a continuous beam supported on elastic props, there are usually more than two members meeting at a joint. In such a case, proposition 4 is used. The unbalanced moment at any joint is distributed amongst all the members meeting at the joint in the ratio of their relative stiffness. See examples 10'8 to 10'11 for illustration.

10'6. PORTAL FRAMES WITH NO SIDE SWAY

A simple portal frame consists of a beam resting on two columns. The junction of the beam with the columns consist of rigid joints. The analysis of such a frame, when the loading conditions and the geometry of the frame is such that there is no joint translation or sway, is similar to that of continuous beam on elastic props. A portal frame may, in general, have more than one span and more than one storey. Examples 10'12, 10'13 and 10'14 illustrate the procedure of analysis of such frames when they do not have any joint translation.

Example 10'6. A continuous beam ABC is shown in Fig. 10'12
(a). Calculate the moments induced at the ends if support B settle by 30 mm. Draw the bending moment diagram and the deflected shape of the beam. Take $E=2 \times 10^6$ N/mm² and $I=3 \times 10^6$ mm⁴ constant for the whole beam.

Solution
(a) Fixed end moments
Clamp the supports B and C against rotation so that each span behaves as the fixed beam. Due to downward settlement of support
B, beam AB rotates clockwise as a whole. Hence fixed end moments at A and B will be anti-clockwise (i.e. negative):

\[ M_{FA} = M_{FBA} = \frac{-6EI}{L^2} = \frac{-6 \times 2 \times 10^6 \times 3 \times 10^6 \times 30}{(3000)^2} \text{ N-mm} \]
\[ = -12 \times 10^6 \text{ N-mm} = -12 \text{ kN-m}. \]

Similarly, for the fixed beam BC,

\[ M_{FBC} = M_{FCB} = \frac{6EI}{L^2} = \frac{6 \times 2 \times 10^6 \times 3 \times 10^6 \times 30}{(2000)^2} \text{ N-mm} \]
\[ = +27 \times 10^6 \text{ N-mm} = +27 \text{ kN-m}. \]

For the span CD, there will be no fixed end moments because neither of the ends sinks.

The bending moment diagram and the deflected shape of the beam are shown in Fig. 10.12 (b) and (c) respectively.

Example 10.7. A horizontal beam ABCD is carried on hinged supports and is continuous over three equal spans each of 3 m. All the supports are initially at the same level. The beam is loaded as shown in Fig. 10.13 (a). Plot the bending moment diagram and sketch the deflected shape of the beam if the support A settles by 10 mm, B settles by 30 mm and C settles by 20 mm. The moment of inertia of the whole beam is \(2.4 \times 10^6 \text{ mm}^4\) units. Take \(E = 2 \times 10^5 \text{ N/mm}^2\).
Solution.

(a) Fixed end moments

Clamp all the joints against rotation so that each span behaves as a separate fixed beam. The fixed end moment at each end, will be the algebraic sum of the fixed end moments caused by the external loading and the settlement of supports.

For the span AB, end B sinks by $30 - 10 = 20$ mm $\downarrow$ relative to end A, and hence the F.E.M. due to this settlement will be of negative sign.

\[ M_{FAB} = \frac{WL}{8} - \frac{6EI\theta}{L^3} = \frac{8 \times 3}{8} \times 6 \times 2 \times 10^5 \times 2.4 \times 10^4 \times 20 \]
\[ = -3 - 6.4 = -9.4 \text{ kN-m} \]

\[ M_{FBA} = +\frac{WL}{8} + \frac{6EI\theta}{L^3} = +\frac{8 \times 3}{8} \times 6 \times 2 \times 10^5 \times 2.4 \times 10^4 \times 20 \]
\[ = +3 - 6.4 = -3.4 \text{ kN-m}. \]

For the span BC, end B sinks $30 - 20 = 10$ mm $\downarrow$ relative to end C. The F.E.M. due to this settlement will be clockwise (i.e. positive) since the sinking of the supports rotate the beam as a whole in the anticlockwise direction.

\[ M_{FBC} = -\frac{3L^3}{12} + \frac{6EI\theta}{L^3} = -\frac{2 \times 3^2}{12} + \frac{6 \times 2 \times 10^5 \times 2.4 \times 10^4 \times 10}{(3000)^2 \times 10^4} \]
\[ = -1.5 + 3.2 = -1.7 \text{ kN-m} \]

For the span CD, end C moves $20$ mm $\downarrow$ relative to D, and hence the F.E.M. will be positive.

\[ M_{FCD} = +\frac{WB\alpha}{L^3} + \frac{6EI\theta}{L^3} = +\frac{9 \times 1 \times 2^2}{3^2} + \frac{6 \times 2 \times 10^5 \times 2.4 \times 10^4 \times 20}{(3000)^2 \times 10^4} \]
\[ = 4 + 6.4 = 10.4 \text{ kN-m} \]

\[ M_{FDC} = +\frac{WB\alpha}{L^3} + \frac{6EI\theta}{L^3} = +\frac{9 \times 2 \times 1^2}{3^2} + \frac{6 \times 2 \times 10^5 \times 2.4 \times 10^4 \times 20}{(3000)^2 \times 10^4} \]
\[ = +2 + 6.4 = 8.4 \text{ kN-m} \]

Distribution factors (Table 10.12)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>BA</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{1}{L}$</td>
<td>$\frac{71}{4}$</td>
</tr>
<tr>
<td>B</td>
<td>BC</td>
<td>$\frac{1}{L}$</td>
<td></td>
<td>$\frac{4}{7}$</td>
</tr>
<tr>
<td>C</td>
<td>CB</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{L}$</td>
<td>$\frac{71}{4L}$</td>
</tr>
<tr>
<td>D</td>
<td>CD</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{L}$</td>
<td></td>
</tr>
</tbody>
</table>
(c) Moment distribution (Table 10.13)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>BA</td>
<td>1</td>
<td>6</td>
<td>1</td>
<td>-0.406</td>
</tr>
<tr>
<td>BC</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0.254</td>
</tr>
<tr>
<td>BD</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>0.340</td>
</tr>
</tbody>
</table>

The B.M.D. and the deflected shape are shown in Fig. 10.13 (b) and (c) respectively. The beam has four points of contraflexure.

Example 10.8. A continuous beam ABC is supported on an elastic column BD, and is loaded as shown in Fig. 10.14. Treating joint B as rigid, analyse the frame and plot the B.M.D. and sketch the deflected shape of the structure.

Solution

(a) Fixed end moments

\[
M_{FAB} = \frac{W_{ab} L^3}{3} = \frac{-10 \times 2 \times 3^2}{5} = -12 \text{ kN-m}
\]

\[
M_{FBA} = \frac{W_{ba} L^3}{3} = \frac{10 \times 3 \times 2^2}{5} = 48 \text{ kN-m}
\]

\[
M_{FBC} = \frac{W_{bc} L^3}{12} = \frac{-2 \times 3^2}{12} = -1.5 \text{ kN-m}
\]

\[
M_{FCB} = +1.5 \text{ kN-m}; M_{FBD} = M_{FDB} = 0
\]

(b) Distribution factors (Table 10.14)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>BA</td>
<td>1</td>
<td>6</td>
<td>1</td>
<td>-0.406</td>
</tr>
<tr>
<td>BC</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0.254</td>
</tr>
<tr>
<td>BD</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>0.340</td>
</tr>
</tbody>
</table>

(c) Moment distribution (Table 10.15)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>BA</td>
<td>0.106</td>
<td>0.21</td>
<td>0.20</td>
<td>-0.75</td>
</tr>
<tr>
<td>BD</td>
<td>0.340</td>
<td>0.34</td>
<td>0.30</td>
<td>-0.75</td>
</tr>
<tr>
<td>BC</td>
<td>0.254</td>
<td>0.25</td>
<td>0.25</td>
<td>-0.75</td>
</tr>
</tbody>
</table>

The B.M.D. and the deflected shape are shown in Fig. 10.13 (b) and (c) respectively. The beam has four points of contraflexure.
From proposition 3,

\[ M_{DB} = \frac{1}{2} M_{BD} = \frac{1}{2} (-0.87) = -0.44 \text{ kN-m} \]

The B.M.D. is shown in Fig. 10.15. The dotted lines show the deflected shape of the beam.

**Example 10.9.** The continuous beam shown in Fig. 10.16 has rigidly fixed ends at C and D, is pinned at E and has rigid joints at A and B. The members are of uniform section and material throughout. Sketch the bending moment diagram for the frame, showing all important values. Also, find the values of the horizontal and vertical reactions at D and E.

**Solution**

(a) Fixed end moments:

\[ M_{FAB} = \frac{-12 \times 1 \times 2^2}{3^2} - \frac{12 \times 2 \times 1^2}{3^2} = -16 \times \frac{8}{3} \]

\[ = -8 \text{ kN-m} \]

\[ M_{FBA} = \frac{-12 \times 2 \times 1^2}{3^2} + \frac{12 \times 1 \times 2^2}{3^2} = 8 \times \frac{16}{3} + \frac{16}{3} \]

\[ = +8 \text{ kN-m} \]

(b) Distribution factors (Table 10.16)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>AD</td>
<td>1/3</td>
<td>21/5</td>
<td>1/2  = 0.5</td>
</tr>
<tr>
<td>A</td>
<td>AB</td>
<td>1/3</td>
<td></td>
<td>1/2  = 0.5</td>
</tr>
<tr>
<td>B</td>
<td>BA</td>
<td>1/3</td>
<td></td>
<td>0.4</td>
</tr>
<tr>
<td>B</td>
<td>BE</td>
<td>3/4, 1/3, 1/4</td>
<td>10/12</td>
<td>0.3</td>
</tr>
<tr>
<td>B</td>
<td>BC</td>
<td>1/4</td>
<td></td>
<td>0.3</td>
</tr>
</tbody>
</table>

**Fig. 10.16**
(d) Calculation of reactions

Considering the equilibrium of AD and taking moments about A,

\[ H_D = \frac{M_D + M_A}{3} = \frac{2.24 + 4.47}{3} = 2.24 \text{ kN} \rightarrow. \]

Similarly, taking moments about B, of all forces below B, we get,

\[ -1.47 + 3H_E = 0 \]

or

\[ H_E = \frac{1.47}{3} = 0.49 \text{ kN} \leftarrow. \]

Taking moments about B, of all forces to the right to B,

\[ -6.80 + 4.60 - 4C + 4 \times 4 \times 2 = 0 \]

or

\[ V_C = 7.45 \text{ kN} \rightarrow. \]

Taking moments about B, of all forces to the left of B,

\[ 8.27 + 2.24 + 3V_D - (3 \times 2.24) - (12 \times 1) - (12 \times 2) = 0 \]

\[ V_D = 10.74 \text{ kN} \leftarrow. \]

Considering the vertical equilibrium of the whole frame

\[ V_E + 7.45 + 10.74 - 12 = 12 \times 4 \times 4 = 0 \]

\[ V_E = 21.81 \text{ kN} \leftarrow. \]

Example 10.10. Analyse the rigid frame shown in Fig. 10.18.

---

(c) Moment distribution (Table 10.17)

<table>
<thead>
<tr>
<th>D</th>
<th>A</th>
<th>AB</th>
<th>B</th>
<th>BE</th>
<th>BC</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>-8.00</td>
<td>8.00</td>
<td>0.00</td>
<td>-5.33</td>
<td>+5.33</td>
</tr>
<tr>
<td></td>
<td>+4.00</td>
<td>+4.00</td>
<td>-1.07</td>
<td>-0.80</td>
<td>-0.80</td>
<td>-</td>
</tr>
<tr>
<td>+2.00</td>
<td>-</td>
<td>-0.53</td>
<td>+2.00</td>
<td>-</td>
<td>-</td>
<td>-0.40</td>
</tr>
<tr>
<td></td>
<td>+0.26</td>
<td>+0.27</td>
<td>-0.86</td>
<td>-0.60</td>
<td>-0.60</td>
<td>-</td>
</tr>
<tr>
<td>+0.13</td>
<td>-</td>
<td>-0.40</td>
<td>+0.13</td>
<td>-</td>
<td>-</td>
<td>-0.30</td>
</tr>
<tr>
<td></td>
<td>+0.20</td>
<td>+0.20</td>
<td>-0.05</td>
<td>-0.04</td>
<td>-0.04</td>
<td>-</td>
</tr>
<tr>
<td>+0.10</td>
<td>-</td>
<td>-0.02</td>
<td>+0.10</td>
<td>-</td>
<td>-</td>
<td>-0.02</td>
</tr>
<tr>
<td>+0.01</td>
<td>+0.01</td>
<td>+0.01</td>
<td>-0.04</td>
<td>-0.03</td>
<td>-0.03</td>
<td>-0.01</td>
</tr>
<tr>
<td>+2.24</td>
<td>+4.47</td>
<td>-4.47</td>
<td>+8.27</td>
<td>-1.47</td>
<td>-6.80</td>
<td>+4.60</td>
</tr>
</tbody>
</table>

The bending moment diagram is shown in Fig. 10.17.
Solution

(a) Fixed end moments
Clamp all joints.

\[ M_{FAB} = -\frac{2 \times 4^2}{12} = -2.67 \text{ kN-m} \]
\[ M_{FBA} = +2.67 \text{ kN-m} \]
\[ M_{FBC} = -2 \times 2 = -4 \text{ kN-m} \]
\[ M_{FDD} = -\frac{4 \times 4}{8} = -2 \text{ kN-m} \]
\[ M_{FDB} = +2 \text{ kN-m} \]

(b) Distribution factors (Table 10.18)

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Joint & Member & Relative stiffness & Sum & D.F. \\
\hline
 & BA & \frac{2}{4} & 2 & \frac{2}{3} \\
B & BC & 0 & \frac{3}{4} & 0 \\
 & BD & \frac{1}{4} & 1 & \frac{1}{3} \\
\hline
\end{tabular}
\end{table}

(c) Moment distribution (Table 10.19)

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
A & B & D & Joint \\
\hline
\text{Distribution factor} & BA & BC & BD & DB & Member \\
\hline
2/3 & 0 & 1/3 & & & \\
-2.67 & +2.67 & -4.00 & -2.00 & +2.00 & F.E.M. \\
1 & +2.22 & 0.00 & +1.11 & & Balance \\
+1.11 & & & & +0.56 & Carry over \\
& & & & Balance & \\
-1.56 & +4.89 & -4.00 & -0.89 & +2.56 & Final moments \\
\hline
\end{tabular}
\end{table}

Fig. 10.19 shows the bending moment diagram and the deflected shape of the structure.

Example 10.11. Draw the bending moment diagram and sketch the deflected shape of the frame shown in Fig. 10.20.

Solution

(a) Fixed end moments,

\[ M_{FBC} = -\frac{4 \times 3^2}{12} = -3 \text{ kN-m} \]
\[ M_{FCD} = +\frac{4 \times 3^2}{12} = +3 \text{ kN-m} \]
\[ M_{FCE} = +\frac{4 \times 4}{8} = +2 \text{ kN-m} \]
\[ M_{FDC} = -\frac{4 \times 4}{8} = -2 \text{ kN-m} \]

(b) Distribution factors (Table 10.20)

(c) Moment distribution (Table 10.21)

The bending moment diagram and the deflected shape of the frame are shown in Fig. 10.21.
**Strength of Materials and Mechanics of Structures**

**Table 10-20**

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>BA</td>
<td>( \frac{1}{2} = \frac{3l}{6} )</td>
<td>( \frac{3}{7} = 0.43 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>BC</td>
<td>( \frac{2l}{3} = \frac{4l}{6} )</td>
<td>( \frac{7l}{6} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CB</td>
<td>( \frac{2l}{3} = \frac{4l}{6} )</td>
<td>( \frac{4}{7} = 0.57 )</td>
<td></td>
</tr>
<tr>
<td>AB</td>
<td>CE</td>
<td>( \frac{2l}{4} = \frac{3l}{6} )</td>
<td>( \frac{10l}{6} )</td>
<td>0.44</td>
</tr>
<tr>
<td></td>
<td>CD</td>
<td>( \frac{3}{4} \times \frac{2l}{4} = \frac{3l}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 10.12. Analyse the portal frame shown in Fig. 10.22 by moment distribution method. The frame is fixed at A and D and has rigid joints at B and C. Draw the bending moment diagram and sketch the deflected shape of the structure.

Solution

\[ M_{F_{BC}} = -\frac{6 \times 4^3}{22} = -8 \text{ kN-m} ; \quad M_{F_{CB}} = +\frac{6 \times 4^3}{12} = +8 \text{ kN-m} \]

(b) Distribution factors (Table 10.22)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>t/D.F</th>
</tr>
</thead>
<tbody>
<tr>
<td>BA (or CD)</td>
<td>( \frac{I}{2} = \frac{2I}{4} )</td>
<td>( \frac{3I}{4} )</td>
<td>2</td>
<td>( \frac{3}{2} )</td>
</tr>
<tr>
<td>BC (or CB)</td>
<td>( \frac{I}{4} )</td>
<td>( \frac{3I}{4} )</td>
<td>1</td>
<td>( \frac{3}{3} )</td>
</tr>
</tbody>
</table>

(b) Moment distribution (Table 10.23)

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 [ \frac{I}{3} ]</td>
<td>1 [ \frac{I}{2} ]</td>
<td>2 [ \frac{I}{3} ]</td>
<td>3 [ \frac{I}{3} ]</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>-80</td>
<td>+80</td>
</tr>
<tr>
<td>-</td>
<td>+5.33</td>
<td>+2.67</td>
<td>-2.67</td>
</tr>
<tr>
<td>+2.67</td>
<td>-</td>
<td>-1.34</td>
<td>+1.34</td>
</tr>
<tr>
<td>-</td>
<td>+0.89</td>
<td>+0.45</td>
<td>-0.45</td>
</tr>
<tr>
<td>+0.44</td>
<td>-</td>
<td>-0.22</td>
<td>+0.22</td>
</tr>
<tr>
<td>-</td>
<td>+0.15</td>
<td>+0.07</td>
<td>-0.07</td>
</tr>
<tr>
<td>+0.07</td>
<td>-</td>
<td>-0.03</td>
<td>+0.03</td>
</tr>
<tr>
<td>+0.01</td>
<td>-</td>
<td>+0.02</td>
<td>+0.01</td>
</tr>
</tbody>
</table>

The bending moment diagram and the deflected shape of the structure have been shown in Fig. 10.23(a) and (b) respectively.

For finding out the horizontal reaction at B, consider the equilibrium of AB. Taking moments about B, we get

\[ H_B = + \frac{3.19 + 6.39}{2} = +4.79 \text{ kN} \rightarrow \]

Similarly,

\[ H_D = -\frac{3.19 - 6.39}{2} = -4.79 \text{ kN} \leftarrow \]

\[ V_A = V_B = 12 \text{ kN} \uparrow \]
Example 10-13. Analyse the portal frame shown in Fig. 10-24. Draw the bending moment diagram and the deflected shape of the structure.

Solution

\[
\begin{align*}
\text{B} & \quad 20 \text{kN} \\
\text{C} & \quad 20 \text{kN} \\
\text{D} & \\
\text{E} & \\
\text{A} & \quad 21 \\
\text{B} & \quad 1.5 \text{m} \\
\text{C} & \quad 2 \text{m} \\
\text{D} & \quad 2 \text{m} \\
\text{E} & \quad 3 \text{m}
\end{align*}
\]

Fig. 10-24.

(a) Fixed end moments

\[
\begin{align*}
M_{FBC} &= M_{FCD} = \frac{20 \times 4}{8} = -10 \text{ kN-m} \\
M_{FBC} &= M_{FDC} = \frac{20 \times 4}{8} = +10 \text{ kN-m}.
\end{align*}
\]

(b) Distribution factors (Table 10-24)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>B (or D)</td>
<td>BA (or DE)</td>
<td>(\frac{21}{3} = \frac{81}{12})</td>
<td>12 51</td>
<td>0.64</td>
</tr>
<tr>
<td>B (or D)</td>
<td>BC (or DC)</td>
<td>(\frac{1.51}{4} = \frac{4.51}{12})</td>
<td>12 51</td>
<td>0.36</td>
</tr>
<tr>
<td>C</td>
<td>CB</td>
<td>(\frac{1.51}{4} = \frac{31}{8})</td>
<td>3 2</td>
<td>0.333</td>
</tr>
<tr>
<td>C</td>
<td>CF</td>
<td>(\frac{3}{4} \times \frac{1}{2} = \frac{31}{8})</td>
<td>9 1</td>
<td>0.333</td>
</tr>
<tr>
<td>C</td>
<td>CD</td>
<td>(\frac{1.51}{4} = \frac{31}{8})</td>
<td>3 2</td>
<td>0.333</td>
</tr>
</tbody>
</table>

(c) Moment distribution (Table 10-25)

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>BA</td>
<td>BC</td>
<td>CB</td>
<td>CF</td>
<td>CD</td>
</tr>
<tr>
<td>0.64</td>
<td>0.36</td>
<td>0.333</td>
<td>0.333</td>
<td>0.333</td>
</tr>
</tbody>
</table>

Moment at column base = 0

The bending moment diagram and the deflected shape of the structure have been shown in Fig. 10-25.

Example 10-14. Analyse the single span double storey portal frame shown in Fig. 10-26. The ends A and F are fixed.

Solution

(a) Fixed end moments

\[
\begin{align*}
M_{FBE} &= \frac{3 \times 92}{12} = -20 \text{ kN-m} \\
M_{FBA} &= +20 \text{ kN-m}
\end{align*}
\]
**Table 10.26**

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>( k )</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>BA</td>
<td>( \frac{21}{9} )</td>
<td>( \frac{1}{3} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>BE</td>
<td>( \frac{21}{9} )</td>
<td>( \frac{1}{3} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>BC</td>
<td>( \frac{21}{9} )</td>
<td>( \frac{1}{3} )</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>CB</td>
<td>( \frac{21}{9} )</td>
<td>( \frac{2}{3} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CD</td>
<td>( \frac{21}{9} )</td>
<td>( \frac{1}{3} )</td>
<td></td>
</tr>
</tbody>
</table>

(b) Distribution factors (Table 10.26)

(c) Moment distribution (Table 10.27) on page 285.

The bending moment diagram and the deflected shape of the frame have been shown in Fig. 10.27.
Example 10.15. Analyse the box culvert shown in Fig. 10.28. All the joints A, B, C and D are rigid. Plot the bending moment diagram and the deflected shape of the frame.

Solution

(a) Fixed end moments

\[ M_{FAD} = - \frac{wL^3}{20} = - \frac{2 \times 4^3}{20} = -1.6 \text{ kN-m} \]

\[ M_{FBA} = + \frac{wL^3}{30} = + \frac{2 \times 4^3}{30} = +1.07 \text{ kN-m} \]

\[ M_{FBC} = - \frac{12 \times 6}{8} = -9 \text{ kN-m} \]

\[ M_{FCD} = -1.07 \text{ kN-m} \text{ and } M_{FDC} = +1.6 \text{ kN-m} \]

\[ M_{FAD} = + \frac{2 \times 6^3}{12} = +6 \text{ kN-m} \text{ and } M_{FDA} = -6 \text{ kN-m} \]

(b) Distribution factors (Table 10.29)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>AD</td>
<td>( \frac{1}{6} = \frac{21}{12} )</td>
<td>51</td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td>AB</td>
<td>( \frac{1}{4} = \frac{31}{12} )</td>
<td>51</td>
<td>0.6</td>
</tr>
<tr>
<td>B</td>
<td>BA</td>
<td>( \frac{1}{4} = \frac{31}{12} )</td>
<td>51</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>BC</td>
<td>( \frac{1}{6} = \frac{21}{12} )</td>
<td>51</td>
<td>0.4</td>
</tr>
</tbody>
</table>

(c) Moment distribution

The frame may be cut in AD and opened out. The moment distribution will be carried out as usual as shown in Table 10.29.

<table>
<thead>
<tr>
<th>(D)</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>(A)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.4</td>
<td>0.6</td>
<td>0.6</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>0.4</td>
<td>0.6</td>
<td>0.6</td>
<td>0.4</td>
</tr>
<tr>
<td>+6.00</td>
<td>-1.60</td>
<td>+1.07</td>
<td>-9.00</td>
<td>+9.00</td>
<td>-1.07</td>
</tr>
<tr>
<td>-1.76</td>
<td>-2.64</td>
<td>+4.76</td>
<td>+3.17</td>
<td>-3.17</td>
<td>-4.76</td>
</tr>
<tr>
<td>+0.88</td>
<td>+2.38</td>
<td>-1.32</td>
<td>-1.59</td>
<td>+1.59</td>
<td>+1.32</td>
</tr>
<tr>
<td>-1.30</td>
<td>-1.96</td>
<td>+1.75</td>
<td>+1.16</td>
<td>-1.16</td>
<td>-1.75</td>
</tr>
<tr>
<td>+0.65</td>
<td>+0.87</td>
<td>-0.98</td>
<td>-0.58</td>
<td>+0.58</td>
<td>+0.98</td>
</tr>
<tr>
<td>-0.61</td>
<td>-0.91</td>
<td>+0.94</td>
<td>+0.62</td>
<td>-0.62</td>
<td>-0.94</td>
</tr>
<tr>
<td>+0.30</td>
<td>+0.47</td>
<td>-0.45</td>
<td>-0.31</td>
<td>+0.31</td>
<td>+0.45</td>
</tr>
<tr>
<td>-0.31</td>
<td>-0.46</td>
<td>+0.45</td>
<td>+0.30</td>
<td>-0.30</td>
<td>-0.46</td>
</tr>
</tbody>
</table>
The B.M.D. and the deflected shape have been shown in Fig. 10.29.

![Diagram showing B.M.D. and deflected shape](image)

10.7. PORTAL FRAMES WITH SIDE SWAY

In the case of continuous beams, etc., the effect of yielding or settlement of supports was taken into account by introducing initial fixed end moments. In the case of portal frames, however, the amount of 'sway' or joint movement is not known and the analysis is done by assuming some arbitrary fixed moments. These assumed fixed moments due to side sway are then distributed and the reactions (horizontal as well as vertical) are found. The algebraic sum of the horizontal reactions due to the assumed sway moments must be equal to the sway force. If not, the assumed sway moments are reduced proportionately as discussed below.

**Causes of Side Sway**

The portal frames sway due to one of the following reasons:

1. Eccentric or unsymmetrical loading on the portal frame.
2. Unsymmetrical out-line of portal frame.
3. Different end conditions of the columns of the portal frame.
4. Non-uniform section of the members of the frame.
5. Horizontal loading on the columns of the frame.
6. Settlement of the supports of the frame.
7. A combination of the above.

**Method of Analysis**

The analysis of portal frames with side sway is done in the following steps:

**Step 1.** (a) Hold the joints against side sway by applying a force $P$ (Fig. 10.35). Calculate the fixed end moments due to external loads and distribute the moments.

(b) Calculate the horizontal and vertical reactions. The algebraic sum of the two horizontal reactions at the column bases will give the value of the restraining or holding force $P$. The sway force $S$ will then be in the opposite direction and of the magnitude of $P$.

**Step 2.** (a) Remove the holding force $P$ and permit the joints to sway. This will cause a set of fixed end moments. To start with, assume suitable sway moments at the four joints $A$, $B$, $C$, and $D$ of the frame, in proportion given by Eq. 10.8 or 10.9 or 10.10 (see below), as the case may be. Distribute these arbitrary sway moments.

(b) Calculate the horizontal and vertical reactions due to the assumed sway moments. The algebraic sum of the horizontal reactions of the two column bases must be equal to the sway force $S$ calculated in step (1 b). If not, reduce the assumed sway moments proportionately as discussed below. The sway moments must be of such magnitude that the algebraic sum of the horizontal reactions due to sway is equal to the sway force $S$.

Let $H_1$ and $H_2$ be the horizontal reactions.

Let $c(H_1 + H_2) = S$.

Then, Actual Sway Moments = $c \times$ Assumed Sway Moments.

Thus the actual sway moments are known.

**Step 3.** (a) The final moments at each joint will be equal to the algebraic sum of the moments due to initial moments (as obtained in step 1 (a)) and the moments due to actual sway (as obtained in step 2).

(b) The final reactions will be equal to the algebraic sum of those found in 1 (b) and 2 (b).

**Ratio of Sway Moments at Column Heads**

When the joints sway, a set of moments are introduced at the two column heads (and bases) of a portal frame. The ratio of the sway moments at the two column heads (i.e. $M_{BA} : M_{CD}$) will depend upon the end conditions. Let us now take different end conditions to derive the standard expressions for the ratio of the sway moments.
Case I. Both ends hinged (Fig. 10.30)

Consider a portal frame with dimensions as shown in Fig. 10.30. Let a force \( P \) cause the frame to sway, so that the joint \( B \) moves to \( B' \) through a horizontal distance \( \delta \). Considering no change in the length of \( BC \), joint \( C \) will move to \( C' \) through distance \( \delta \).

From 10.4, for a beam hinged at one end and fixed at the other, the fixed end moment due to movement or settlement of the support is given by Eq. 10.7:

\[
M_{BA} = \frac{3EI\delta}{L_1^2}
\]

Similarly, the fixed end moment induced at \( C \) due to movement, is:

\[
M_{CD} = \frac{3EI\delta}{L_2^2}
\]

Dividing (i) by (ii), we get

\[
\frac{M_{BA}}{M_{CD}} = \frac{L_2^3}{L_1^3}
\]  \hspace{1cm} (10.8)

Since both the columns rotate in the same direction, the moments \( M_{BA} \) and \( M_{CD} \) will be of the same sign (either positive or negative, as the case may be). For the case of Fig. 10.30, both the moments will be of the negative sign since the columns tend to rotate in the clockwise direction.

Case II. Both ends fixed

Fig. 10.31 shows a portal frame fixed at both the ends. The movement \( BB' = CC' = \delta \).

For the beam \( AB \), fixed at ends \( A \) and \( B \), the fixing moments due to the movement of the joint \( B \) by \( \delta \) is given by

\[
M_{BA} = M_{AB} = \frac{6EI\delta}{L_1^2}
\]  \hspace{1cm} (i)

For the beam \( CD \), fixed at ends \( C \) and \( D \), the fixing moment due to the movement of the joint \( C \) by \( \delta \) is given by

\[
M_{CD} = M_{DC} = \frac{6EI\delta}{L_2^2}
\]  \hspace{1cm} (ii)

Example 10.16. Analyse the portal frame shown in Fig. 10.33. The end \( A \) is fixed and \( D \) is hinged. The joints \( B \) and \( C \) are rigid. Draw the bending moment diagram and sketch the deflected shape of the frame.

Solution

As the load is acting on the joint, there will be no fixed end moments. However, due to side sway, moments will be induced at joint \( A, B \) and \( C \).
**Fig. 10-33**

**Distribution factors** (Table 10'30).

**Table 10'30**

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BA</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{21}{4}$</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>BC</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{21}{4}$</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>CB</td>
<td>$\frac{1}{4} = \frac{4f}{16}$</td>
<td>$\frac{71}{16}$</td>
<td>0.57</td>
</tr>
<tr>
<td></td>
<td>CD</td>
<td>$\frac{3}{4} \times \frac{1}{4} = \frac{3f}{15}$</td>
<td>$\frac{71}{16}$</td>
<td>0.43</td>
</tr>
</tbody>
</table>

**Side Sway**

Under the action of the 10 kN load, there will be side sway to the right and the columns $AB$ and $CD$ will rotate in a clockwise direction. Thus negative moments will be induced at $A$, $B$ and $C$ in these columns. As the end $A$ is fixed and $D$ is hinged, the ratio of moments will be:

$$\frac{M_{BA}}{M_{CD}} = \frac{2L_1/L_2^2}{I_2/L_2^2} = \frac{2}{1}$$

Also,

$$M_{BA} = M_{AB}$$

Let us, first of all, assume arbitrary values of these moments and find out the corresponding sway force.

Let

$$M_{CD} = -5 \text{ kN-m}$$

$$\therefore M_{BA} = M_{AB} = -10 \text{ kN-m}$$

**Moment distribution method** (Table 10'31)

**Table 10'31**

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.5</td>
<td>-0.5</td>
<td>0.57</td>
<td>-0.57</td>
</tr>
<tr>
<td>$10.0$</td>
<td>0.0</td>
<td>10.0</td>
<td>-5.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>+5.0</td>
<td>+5.0</td>
<td>+2.86</td>
<td>+2.14</td>
</tr>
<tr>
<td>$0.36$</td>
<td>0.0</td>
<td>0.72</td>
<td>0.71</td>
<td>1.43</td>
</tr>
<tr>
<td></td>
<td>1.07</td>
<td>0.0</td>
<td>-1.07</td>
<td>-0.07</td>
</tr>
<tr>
<td>$0.36$</td>
<td>0.0</td>
<td>0.72</td>
<td>0.71</td>
<td>1.43</td>
</tr>
<tr>
<td></td>
<td>1.07</td>
<td>0.0</td>
<td>-1.07</td>
<td>-0.07</td>
</tr>
<tr>
<td>$0.18$</td>
<td>0.0</td>
<td>0.10</td>
<td>0.18</td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td>-0.10</td>
<td>0.0</td>
<td>-0.10</td>
<td>-0.10</td>
</tr>
<tr>
<td>$0.03$</td>
<td>0.0</td>
<td>0.05</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>-0.08</td>
<td>0.0</td>
<td>-0.08</td>
<td>-0.08</td>
</tr>
<tr>
<td>$0.01$</td>
<td>0.0</td>
<td>0.03</td>
<td>0.02</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>0.15</td>
<td>0.0</td>
<td>-0.15</td>
<td>-0.15</td>
</tr>
<tr>
<td>$0.08$</td>
<td>0.0</td>
<td>0.36</td>
<td>0.36</td>
<td>0.36</td>
</tr>
<tr>
<td></td>
<td>0.21</td>
<td>0.0</td>
<td>-0.21</td>
<td>-0.21</td>
</tr>
<tr>
<td>$0.03$</td>
<td>0.0</td>
<td>0.38</td>
<td>0.38</td>
<td>0.38</td>
</tr>
<tr>
<td>$0.01$</td>
<td>0.0</td>
<td>0.35</td>
<td>0.35</td>
<td>0.35</td>
</tr>
<tr>
<td></td>
<td>0.15</td>
<td>0.0</td>
<td>-0.15</td>
<td>-0.15</td>
</tr>
<tr>
<td>$0.02$</td>
<td>0.0</td>
<td>0.35</td>
<td>0.35</td>
<td>0.35</td>
</tr>
<tr>
<td>$0.10$</td>
<td>0.0</td>
<td>0.38</td>
<td>0.38</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>0.15</td>
<td>0.0</td>
<td>-0.15</td>
<td>-0.15</td>
</tr>
<tr>
<td>$0.03$</td>
<td>0.0</td>
<td>0.38</td>
<td>0.38</td>
<td>0.38</td>
</tr>
</tbody>
</table>

**Final moment**

$$\text{Horizontal reaction at } A = -7.70 - 5.38 = 13.08 \text{ kN}$$

$$\text{Horizontal reaction at } D = -3.85 = 0.963 \text{ kN}$$

The sway force causing the assumed moments $= 3.27 + 0.963 = 4.233 \text{ kN}$

But actual sway force is 10 kN; hence the moments will be increased proportionately in the ratio of $\frac{10}{4.203}$, as shown in Table 10'32.
The horizontal reaction at \( A = \frac{3.27}{4.233} \times 10 = 7.72 \) kN.

The horizontal reaction at \( D = \frac{0.963}{4.233} \times 10 = 2.28 \) kN.

The bending moment diagram and the deflected shape have been shown in Fig. 10.34.

**Example 10.17.** Draw the bending moment diagram and sketch the deflected shape of the frame shown in Fig. 10.35. The ends \( A \) and \( D \) are fixed and \( B \) is loaded with U.D.L. of 6 kN/m.

**Table 10.32**

<table>
<thead>
<tr>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sway=4.233 kN</td>
<td>-7.70</td>
<td>-5.48</td>
<td>+5.38</td>
</tr>
<tr>
<td>Sway=10 kN</td>
<td>-18.18</td>
<td>-12.73</td>
<td>+12.73</td>
</tr>
</tbody>
</table>

**Solution**

(a) Fixed end moments

\[
M_{FBC} = -\frac{6 \times 2^2}{12} = -2 \text{ kN-m} \quad M_{FCB} = +2 \text{ kN-m}
\]

Let a horizontal force \( P \) be applied at \( B \) to hold it against translation. The moment distribution is then done as usual.

(b) Distribution factors (Table 10.33).

**Table 10.33**

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( BA )</td>
<td>( \frac{21}{3} ) = ( \frac{41}{6} )</td>
<td>( \frac{71}{6} )</td>
<td>0.57</td>
<td></td>
</tr>
<tr>
<td>( BC )</td>
<td>( \frac{I}{2} = \frac{3J}{6} )</td>
<td>( \frac{I}{2} )</td>
<td>0.43</td>
<td></td>
</tr>
<tr>
<td>( CB )</td>
<td>( \frac{I}{2} )</td>
<td>( I )</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td>( CD )</td>
<td>( \frac{I}{2} )</td>
<td>( I )</td>
<td>0.50</td>
<td></td>
</tr>
</tbody>
</table>

(c) Moment distribution (Table 10.34)

**Table 10.34**

<table>
<thead>
<tr>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.57</td>
<td>0.43</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-2.0</td>
<td>+2.0</td>
</tr>
<tr>
<td>0</td>
<td>+1.14</td>
<td>+0.86</td>
<td>-1.0</td>
</tr>
<tr>
<td>+0.57</td>
<td>-</td>
<td>-0.50</td>
<td>+0.43</td>
</tr>
<tr>
<td>+0.29</td>
<td>+0.21</td>
<td>-0.21</td>
<td>-0.22</td>
</tr>
<tr>
<td>+0.15</td>
<td>-</td>
<td>-0.11</td>
<td>+0.10</td>
</tr>
<tr>
<td>+0.06</td>
<td>+0.05</td>
<td>-0.05</td>
<td>-0.05</td>
</tr>
<tr>
<td>+0.03</td>
<td>-</td>
<td>-0.03</td>
<td>+0.03</td>
</tr>
<tr>
<td>+0.01</td>
<td>+0.02</td>
<td>+0.01</td>
<td>-0.02</td>
</tr>
<tr>
<td>+0.76</td>
<td>+1.51</td>
<td>-1.51</td>
<td>+1.28</td>
</tr>
</tbody>
</table>
Horizontal reaction at \( A \), \( h = \frac{0.76 + 1.51}{3} = \frac{2.27}{3} = 0.76 \text{ kN} \),

and horizontal reaction at \( D \), \( h = \frac{1.28 + 0.64}{2} = \frac{1.92}{2} = 0.96 \text{ kN} \).

The value of \( P \) preventing side sway is: \( 0.96 - 0.76 = 0.20 \text{ kN} \).

\[ (d) \text{ Side Sway} \]

Now let a sway force \( S = 0.20 \text{ kN} \) be applied at \( C \). This will cause the columns \( AB \) and \( DC \) to rotate in an anticlockwise direction and thus clockwise moments will be induced at column heads such that

\[ M_{BA} = M_{AB} = 8 \text{ kN-m} \] and \( M_{CD} = M_{DC} = 9 \text{ kN-m} \).

We shall assume arbitrary values of sway moments in the above proportion.

Let \( M_{BA} = M_{AB} = +8 \text{ kN-m} \) and \( M_{CD} = M_{DC} = +9 \text{ kN-m} \).

\[ (e) \text{ Distribution of correcting moments} (\text{Table 10.35}) \]

<table>
<thead>
<tr>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.57</td>
<td>0.42</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>8.0</td>
<td>8.0</td>
<td>-</td>
<td>-9.0</td>
</tr>
<tr>
<td>-</td>
<td>-4.57</td>
<td>-3.43</td>
<td>-4.5</td>
</tr>
<tr>
<td>-2.29</td>
<td>-2.25</td>
<td>-1.72</td>
<td>-</td>
</tr>
<tr>
<td>-</td>
<td>+1.29</td>
<td>+0.96</td>
<td>+0.86</td>
</tr>
<tr>
<td>+0.64</td>
<td>-</td>
<td>+0.43</td>
<td>+0.48</td>
</tr>
<tr>
<td>-</td>
<td>-0.25</td>
<td>-0.18</td>
<td>-0.24</td>
</tr>
<tr>
<td>-0.13</td>
<td>-0.12</td>
<td>-0.09</td>
<td>-</td>
</tr>
<tr>
<td>-</td>
<td>+0.07</td>
<td>+0.05</td>
<td>+0.05</td>
</tr>
<tr>
<td>+0.04</td>
<td>-</td>
<td>+0.02</td>
<td>+0.03</td>
</tr>
<tr>
<td>-</td>
<td>-0.01</td>
<td>-0.02</td>
<td>-0.01</td>
</tr>
<tr>
<td>+6.26</td>
<td>+4.53</td>
<td>-5.15</td>
<td>+5.15</td>
</tr>
</tbody>
</table>

The final reactions are as follows:

Horizontal reaction at \( A = \frac{6.26 + 4.53}{3} = \frac{10.79}{3} = 3.60 \text{ kN} \)

Horizontal reaction at \( D = \frac{5.15 + 7.08}{2} = \frac{12.23}{2} = 6.12 \text{ kN} \)

So the sway force, which induces the assumed moments is: \( 3.6 + 6.12 = 9.72 \text{ kN} \)

But the actual sway force is: \( 0.20 \text{ kN} \)

The correction in the moments may now be carried out as shown in Table 10.36.

<table>
<thead>
<tr>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Sway=9.72 kN</td>
<td>+6.26</td>
<td>+4.53</td>
<td>-4.53</td>
</tr>
<tr>
<td>2. Sway=0.20 kN</td>
<td>+0.12</td>
<td>+0.09</td>
<td>-0.09</td>
</tr>
<tr>
<td>3. Non-sway</td>
<td>+0.76</td>
<td>+1.51</td>
<td>-1.51</td>
</tr>
<tr>
<td>4. Final moments</td>
<td>+0.88</td>
<td>+1.60</td>
<td>-1.60</td>
</tr>
</tbody>
</table>

The bending moment diagram and the deflected shape have been shown in Fig. 10.36.
Example 10.18. The frame shown in Fig. 10.37 is hinged at A. The end D is fixed and the joints B and C are rigid. The column CD is subjected to a horizontal loading of 2 kN/m. A concentrated load of 6 kN acts on BC at 1 m from B. Analyse the frame completely and sketch its deflected shape.

Solution

![Frame Diagram]

**Table 10.37**

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>BA</td>
<td>$\frac{3}{4} \times \frac{1.51}{3} = \frac{3}{8}$</td>
<td></td>
<td>$\frac{3}{7}$</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>$\frac{71}{8}$</td>
<td></td>
<td>$\frac{4}{7}$</td>
</tr>
<tr>
<td>BC</td>
<td>$\frac{1}{2} = \frac{41}{8}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CB</td>
<td>$\frac{1}{2} = \frac{21}{4}$</td>
<td></td>
<td>$\frac{2}{3}$</td>
<td></td>
</tr>
<tr>
<td>CD</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{31}{4}$</td>
<td></td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

**Table 10.38**

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{3}{7}$</td>
<td>$\frac{4}{7}$</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-1.50</td>
<td>1.50</td>
</tr>
<tr>
<td>-</td>
<td>+0.64</td>
<td>+0.86</td>
<td>+0.78</td>
</tr>
<tr>
<td>-</td>
<td>+0.39</td>
<td>+0.43</td>
<td></td>
</tr>
<tr>
<td>-</td>
<td>-0.17</td>
<td>-0.22</td>
<td>-0.29</td>
</tr>
<tr>
<td>-</td>
<td>+0.06</td>
<td>+0.09</td>
<td>+0.07</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>+0.04</td>
<td>+0.05</td>
</tr>
<tr>
<td>-</td>
<td>-0.02</td>
<td>-0.02</td>
<td>-0.03</td>
</tr>
<tr>
<td>-</td>
<td>+0.51</td>
<td>-0.51</td>
<td>+2.40</td>
</tr>
</tbody>
</table>

Horizontal reaction at $A = \frac{0.51}{3} = 0.17$ kN →

For horizontal reaction at $D$ i.e., $h_D$, taking moments about $C$ for equilibrium of $CD$,

$h_D \times 4 = 4 \times 2 \times 2 + 2.81 - 2.40 = 16.41$

or $h_D = 410$ kN →

So $P = 8 - 0.17 - 410 = 373$ kN →
(d) Side Sway

As actually there is no force like \( P \) acting at joint \( B \), so apply an equal opposite force \( S = 3.73 \) kN at joint \( C \) to neutralise the effect of \( P \).

This sway force \( S \) will rotate the column in anticlockwise direction and thus inducing clockwise moments at the end of the columns. The ratio of moments is as under:

\[
\frac{M_{CD}}{M_{BA}} = \frac{27/16}{3/2 \times 9} = \frac{6}{8}
\]

So let \( M_{CD} = 6'0 \) kN and \( M_{BA} = +8'0 \) kN.

(e) Distribution of correcting moments (Table 10.39)

<table>
<thead>
<tr>
<th></th>
<th>F.E.M.</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{3}{7} )</td>
<td>+8'0</td>
<td>-3'43</td>
<td>-2'0</td>
</tr>
<tr>
<td>( \frac{4}{7} )</td>
<td>-2'0</td>
<td>-2'28</td>
<td>-1'0</td>
</tr>
<tr>
<td>( \frac{2}{3} )</td>
<td>+0'86</td>
<td>+1'14</td>
<td>+0'76</td>
</tr>
<tr>
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<td>-0'33</td>
<td>-0'43</td>
<td>-0'19</td>
</tr>
<tr>
<td>( \frac{2}{3} )</td>
<td>+0'08</td>
<td>+0'11</td>
<td>+0'07</td>
</tr>
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<td>+0'07</td>
<td>-0'04</td>
</tr>
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<td>+0'01</td>
<td>+0'01</td>
</tr>
<tr>
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<td>+0'01</td>
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<td>+0'01</td>
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<tr>
<td>( \frac{2}{3} )</td>
<td>-0'33</td>
<td>+0'11</td>
<td>-0'04</td>
</tr>
<tr>
<td>( \frac{1}{3} )</td>
<td>+0'07</td>
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<td>-0'04</td>
</tr>
<tr>
<td>( \frac{2}{3} )</td>
<td>+0'01</td>
<td>-0'02</td>
<td>-0'02</td>
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<td>-0'01</td>
<td>+0'01</td>
<td>+0'01</td>
</tr>
<tr>
<td>( \frac{2}{3} )</td>
<td>-0'19</td>
<td>+0'14</td>
<td>+0'07</td>
</tr>
<tr>
<td>( \frac{1}{3} )</td>
<td>-0'21</td>
<td>+0'06</td>
<td>+0'01</td>
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<td>( \frac{2}{3} )</td>
<td>+0'11</td>
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<td>-0'02</td>
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<tr>
<td>( \frac{1}{3} )</td>
<td>+0'07</td>
<td>-0'06</td>
<td>-0'04</td>
</tr>
</tbody>
</table>

Table 10.39

Horizontal reaction at \( A = \frac{5'16}{3} = 1.72 \) kN →

Horizontal reaction at \( D = \frac{4'63 + 5'31}{4} = \frac{9'94}{4} = 2'485 \) kN

Corresponding sway force \( = 1'72 + 2'485 = 4'205 \) kN

But the actual sway force is \( 3.73 \) kN only. Hence the moments will be reduced proportionately and added algebraically to the non-sway moments as shown in Table 10.40.

<table>
<thead>
<tr>
<th></th>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Sway ( = 4'205 ) kN</td>
<td>0</td>
<td>+5'16</td>
<td>-5'16</td>
<td>-4'63</td>
</tr>
<tr>
<td>2. Sway ( = 3'73 ) kN</td>
<td>0</td>
<td>+4'58</td>
<td>-4'58</td>
<td>-4'11</td>
</tr>
<tr>
<td>3. Non-sway moments</td>
<td>0</td>
<td>+0'51</td>
<td>-0'51</td>
<td>+2'40</td>
</tr>
<tr>
<td>4. Final moments</td>
<td>0</td>
<td>+5'09</td>
<td>-5'09</td>
<td>-1'72</td>
</tr>
</tbody>
</table>

Fig. 10.38
Horizontal reaction at \( A = \frac{5.09}{3} - 1.70 \text{ kN} \rightarrow \)

Horizontal reaction at \( D = \frac{8 \times 2 + 1.71 + 7.52}{4} = 6.3 \text{ kN} \rightarrow \)

For vertical reaction at \( A \), taking moments about \( D \),
\[ V_a \times 2 = 6 \times 1 + 8 \times 2 - 1 \times 1.70 \times 1 - 7.52 = 12.78 \]
\[ \text{or } V_a = 6.39 \text{ kN} \uparrow \]
\[ V_a = 0.39 \text{ kN} \downarrow \]

The bending moment diagram and the deflected shape of the frame have been shown in Fig. 10.38.

**Example 10.19.** Use the method of moment distribution to analyse the portal frame shown in Fig. 10.39 if the hinged support \( D \) sinks by an amount \( \Delta \). The members have the same uniform cross-section.

**Solution**

**Fig. 10.39**

(a) **Fixed end moments**

When the hinged end \( D \) sinks, the end \( C \) of the beam will also settle by the same amount. Due to settlement, there will be side sway in the right side. Let a force \( P \) be applied at \( C \) to prevent this side sway.

The settlement of end \( C \) will induce moments in \( BC \) in anticlockwise direction.

\[ M_{FC} = M_{BC} = - \frac{6EI\Delta}{L^3} = -6c \text{ (Say)} \]

where

\[ c = \frac{EI\Delta}{L^3} \]

(b) **Distribution factors** (Table 10.41).

<table>
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<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B )</td>
<td>( BA )</td>
<td>( \frac{1 \times 2}{3 \times \frac{3}{2}} = \frac{2}{3} )</td>
<td>( \frac{5}{3} )</td>
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</tr>
<tr>
<td>( B )</td>
<td>( BC )</td>
<td>( \frac{1}{3L} )</td>
<td>( \frac{3}{3L} )</td>
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</tr>
<tr>
<td>( C )</td>
<td>( CB )</td>
<td>( \frac{1}{3 \times \frac{3}{2}} = \frac{1}{3L} )</td>
<td>( \frac{11}{3L} )</td>
<td>0.73</td>
</tr>
<tr>
<td>( C )</td>
<td>( CD )</td>
<td>( \frac{3}{4} \times \frac{1}{2L} = \frac{3}{8L} )</td>
<td>( \frac{3}{8L} )</td>
<td>0.27</td>
</tr>
</tbody>
</table>

(c) **Moment distribution**

<table>
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<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
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<td>0.6</td>
<td>0.73</td>
<td>0.27</td>
</tr>
<tr>
<td>( B )</td>
<td>( - )</td>
<td>( - )</td>
<td>-6.0c</td>
<td>-6.0c</td>
</tr>
<tr>
<td>( C )</td>
<td>( +2.4c )</td>
<td>( +3.6c )</td>
<td>( +4.36c )</td>
<td>( +1.64c )</td>
</tr>
<tr>
<td>( D )</td>
<td>( +1.2c )</td>
<td>( +2.18c )</td>
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<td>( -1.31c )</td>
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</tr>
<tr>
<td>( b )</td>
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<td>( +0.48c )</td>
<td>( +0.18c )</td>
</tr>
<tr>
<td>( c )</td>
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<td>( -0.14c )</td>
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</tr>
<tr>
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</tr>
<tr>
<td>( e )</td>
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<td>( +0.04c )</td>
<td>( +0.05c )</td>
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</tr>
<tr>
<td>( f )</td>
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<td>( +0.03c )</td>
<td>( +0.02c )</td>
<td>( - )</td>
</tr>
<tr>
<td>( g )</td>
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<td>( +0.01c )</td>
<td>( +0.02c )</td>
<td>( -0.01c )</td>
</tr>
<tr>
<td>( h )</td>
<td>( +0.05c )</td>
<td>( +0.04c )</td>
<td>( +0.05c )</td>
<td>( +0.02c )</td>
</tr>
<tr>
<td>( i )</td>
<td>( +0.02c )</td>
<td>( +0.03c )</td>
<td>( +0.02c )</td>
<td>( - )</td>
</tr>
<tr>
<td>( j )</td>
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<td>( +0.01c )</td>
<td>( +0.02c )</td>
<td>( -0.01c )</td>
</tr>
<tr>
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<td>( +0.04c )</td>
<td>( +0.05c )</td>
<td>( +0.02c )</td>
</tr>
<tr>
<td>( l )</td>
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<td>( +0.02c )</td>
<td>( - )</td>
</tr>
<tr>
<td>( m )</td>
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<td>( +0.01c )</td>
<td>( +0.02c )</td>
<td>( -0.01c )</td>
</tr>
<tr>
<td>( n )</td>
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<td>( +0.04c )</td>
<td>( +0.05c )</td>
<td>( +0.02c )</td>
</tr>
<tr>
<td>( o )</td>
<td>( +0.02c )</td>
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<td>( +0.02c )</td>
<td>( - )</td>
</tr>
<tr>
<td>( p )</td>
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<td>( +0.01c )</td>
<td>( +0.02c )</td>
<td>( -0.01c )</td>
</tr>
<tr>
<td>( q )</td>
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<td>( +0.04c )</td>
<td>( +0.05c )</td>
<td>( +0.02c )</td>
</tr>
<tr>
<td>( r )</td>
<td>( +0.02c )</td>
<td>( +0.03c )</td>
<td>( +0.02c )</td>
<td>( - )</td>
</tr>
<tr>
<td>( s )</td>
<td>( -0.01c )</td>
<td>( +0.01c )</td>
<td>( +0.02c )</td>
<td>( -0.01c )</td>
</tr>
<tr>
<td>( t )</td>
<td>( +0.05c )</td>
<td>( +0.04c )</td>
<td>( +0.05c )</td>
<td>( +0.02c )</td>
</tr>
<tr>
<td>( u )</td>
<td>( +0.02c )</td>
<td>( +0.03c )</td>
<td>( +0.02c )</td>
<td>( - )</td>
</tr>
<tr>
<td>( v )</td>
<td>( -0.01c )</td>
<td>( +0.01c )</td>
<td>( +0.02c )</td>
<td>( -0.01c )</td>
</tr>
<tr>
<td>( w )</td>
<td>( +0.05c )</td>
<td>( +0.04c )</td>
<td>( +0.05c )</td>
<td>( +0.02c )</td>
</tr>
<tr>
<td>( x )</td>
<td>( +0.02c )</td>
<td>( +0.03c )</td>
<td>( +0.02c )</td>
<td>( - )</td>
</tr>
<tr>
<td>( y )</td>
<td>( -0.01c )</td>
<td>( +0.01c )</td>
<td>( +0.02c )</td>
<td>( -0.01c )</td>
</tr>
<tr>
<td>( z )</td>
<td>( +0.05c )</td>
<td>( +0.04c )</td>
<td>( +0.05c )</td>
<td>( +0.02c )</td>
</tr>
</tbody>
</table>

F.E.M.

Balance

Carry over

Balance

Carry over

Balance

Carry over

Balance

Carry over

Balance

Carry over

Balance

Carry over

Balance

Carry over

Balance

Carry over

Balance

Carry over

Balance

Carry over

Balance

Carry over

Balance

Carry over

Balance

Carry over

Balance

Carry over

Balance

Carry over

Balance
The moment distribution, assuming absence of sway, will be as shown in Table 10-42.

Horizontal reaction at \( A = \frac{0.85c + 1.71c}{3/2L} = \frac{1.713c}{L} \)  
Horizontal reaction at \( D = \frac{1.28c}{2L} = \frac{0.64EI\Delta}{L^3} \)  
\( P = (1.713 + 0.64) \frac{EI\Delta}{L^3} = \frac{2.353EI\Delta}{L^3} \)

(d) Side Sway

Now apply a force \( S = \frac{2.353EI\Delta}{L^3} \) at \( B \) in opposite direction to that of \( P \). Side sway will induce anticlockwise moment \( A \), \( B \) and \( C \), such that

\[
M_{BA} \text{ or } M_{AS} = \frac{2L_1L_2}{I_2} = \frac{8}{9} = 32
\]

Arbitrary values of sway moments in above proportion are assumed as given below:

\[
M_{AS} = M_{AB} = -8.0 \quad ; \quad M_{CD} = -2.25
\]

(e) Distribution of correcting moments:

<table>
<thead>
<tr>
<th>Table 10-43</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
</tr>
<tr>
<td>-------------</td>
</tr>
<tr>
<td>0.4 0.6</td>
</tr>
<tr>
<td>-8.0 -8.0</td>
</tr>
<tr>
<td>- - +0.320</td>
</tr>
<tr>
<td>+1.60</td>
</tr>
<tr>
<td>- - +0.33</td>
</tr>
<tr>
<td>-0.16</td>
</tr>
<tr>
<td>- - +0.35</td>
</tr>
<tr>
<td>+0.18</td>
</tr>
<tr>
<td>- - -0.04</td>
</tr>
<tr>
<td>-0.02</td>
</tr>
<tr>
<td>-0.02 +0.04</td>
</tr>
<tr>
<td>-6.38</td>
</tr>
</tbody>
</table>

The bending moment diagram and the deflected shape have been shown in Fig. 10-40. The values marked in the figure are to be multiplied by the factor \( \frac{EI\Delta}{L^3} \).
Example 10.20. Analyse the double span single storey portal frame shown in Fig. 10.41. The beam BC is loaded with a U.D.L. of 3 kN/m run. The ends A, F and E are hinged and joints B, C and D are rigid. The moment of inertia of beams is double the same for the columns.

Solution

(a) Fixed end moments

\[ M_{FBC} = \frac{3 \times 4^1}{12} = -4 \text{ kN} \cdot \text{m}; \quad M_{FCD} = +4.0 \text{ kN} \cdot \text{m} \]

(b) Distribution factors (Table 10.45)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative Stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>BA</td>
<td>( \frac{3}{4} \times \frac{1}{3} = \frac{1}{4} )</td>
<td>( \frac{3}{4} )</td>
<td>0.33</td>
</tr>
<tr>
<td></td>
<td>BC</td>
<td>( \frac{2}{4} )</td>
<td>( \frac{1}{4} )</td>
<td>0.67</td>
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<tr>
<td>C</td>
<td>CB</td>
<td>( \frac{2}{4} )</td>
<td>( \frac{1}{4} )</td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td>CF</td>
<td>( \frac{3}{4} \times \frac{1}{3} = \frac{1}{4} )</td>
<td>( \frac{5}{4} )</td>
<td>0.20</td>
</tr>
<tr>
<td></td>
<td>CD</td>
<td>( \frac{2}{4} )</td>
<td>( \frac{1}{4} )</td>
<td>0.40</td>
</tr>
</tbody>
</table>

(a) Moment distribution

Let a horizontal force \( P \) act at joint D to prevent the side sway. The moment distribution is carried out as shown in Table 10.46.

MOMENT DISTRIBUTION METHOD

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.33</td>
<td>0.67</td>
<td>0.4</td>
<td>0.2</td>
<td>0.67</td>
</tr>
<tr>
<td>0.41</td>
<td>0.40</td>
<td>0.40</td>
<td>0.33</td>
<td>0.33</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>+1.33</td>
<td>+2.67</td>
<td>-1.60</td>
<td>-0.80</td>
<td>-1.60</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>+0.33</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>-0.80</td>
<td>0.33</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
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<td>-</td>
<td>-</td>
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<td>-0.53</td>
<td>+0.53</td>
</tr>
<tr>
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<td>-</td>
<td>-0.27</td>
<td>+0.27</td>
<td>-</td>
</tr>
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<td>-0.10</td>
<td>+0.18</td>
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<td>+0.04</td>
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<td>-0.07</td>
<td>-0.04</td>
<td>-0.07</td>
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<td>-0.03</td>
<td>+0.04</td>
<td>-</td>
</tr>
<tr>
<td>-0.03</td>
<td>+0.04</td>
<td>-0.03</td>
<td>-0.03</td>
<td>+0.02</td>
</tr>
<tr>
<td>-</td>
<td>+1.74</td>
<td>-1.74</td>
<td>+3.28</td>
<td>-1.22</td>
</tr>
</tbody>
</table>

Horizontal reaction at \( A = \frac{1.74}{3} = 0.58 \text{ kN} \)

Horizontal reaction at \( E = \frac{0.41}{3} = 0.137 \text{ kN} \)

Horizontal reaction at \( F = \frac{1.22}{3} = 0.407 \text{ kN} \)

So \( P = 0.58 + 0.137 - 0.407 = 0.31 \text{ kN} \)

(d) Side sway

Let a force \( S = 0.31 \text{ kN} \) be made to act at joint B to neutralise the effect of \( P \). The joints B, C and D each will move on right side by equal amount inducing anticlockwise moments in columns at B, C and D. As the end condition of all the three columns and \( EI \) is same, \( M_{BA} = M_{CF} = M_{DE} \).

Let us assume an arbitrary value of 10 kN·m for these moments.
### Distribution of sway moments (Table 10'47)

#### Table 10'47

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F.E.M.</th>
<th>C.O.</th>
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</thead>
<tbody>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
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<td>+4'0</td>
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<td>+4'0</td>
<td>+6'67</td>
</tr>
<tr>
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<td>+6'67</td>
<td>+4'0</td>
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<td>+4'0</td>
<td>+6'67</td>
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<td>+6'67</td>
<td>+4'0</td>
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</tr>
<tr>
<td>-100</td>
<td>+3'33</td>
<td>+6'67</td>
<td>+4'0</td>
<td>+2'0</td>
<td>+4'0</td>
<td>+6'67</td>
</tr>
<tr>
<td>-100</td>
<td>+3'33</td>
<td>+6'67</td>
<td>+4'0</td>
<td>+2'0</td>
<td>+4'0</td>
<td>+6'67</td>
</tr>
<tr>
<td>-100</td>
<td>+3'33</td>
<td>+6'67</td>
<td>+4'0</td>
<td>+2'0</td>
<td>+4'0</td>
<td>+6'67</td>
</tr>
<tr>
<td>-100</td>
<td>+3'33</td>
<td>+6'67</td>
<td>+4'0</td>
<td>+2'0</td>
<td>+4'0</td>
<td>+6'67</td>
</tr>
<tr>
<td>-100</td>
<td>+3'33</td>
<td>+6'67</td>
<td>+4'0</td>
<td>+2'0</td>
<td>+4'0</td>
<td>+6'67</td>
</tr>
</tbody>
</table>

Horizontal reaction at $A = \frac{1'46}{3} = 0'485$ kN

Horizontal reaction at $F = \frac{1'60}{3} = 0'53$ kN

Horizontal reaction at $E = \frac{6'13}{3} = 0'045$ kN

For vertical reactions taking moment about $C$ for equilibrium of left side portion,

$$0'485 \times 3 + 3 \times 4 \times 2 - 3'47 = V_x \times 4$$

or

$$4V_x = 21'99$$

or

$$V_x = 5'5 \text{ kN}$$

Taking moments about $C$ for the equilibrium of frame on right hand side,

$$V_x \times 4 = -0'045 \times 3 - 1'87 = -2'005$$

---

**Moments Distribution Method**

But the actual sway force is 0'31 kN. Thus the sway moments will have to be reduced proportionately. These may then be added to non-sway moments to obtain the final moments as shown in Table 10'48.

#### Table 10'48

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-6'93</td>
<td>+6'93</td>
<td>+6'61</td>
<td>-9'22</td>
</tr>
<tr>
<td>0</td>
<td>-6'93</td>
<td>+6'93</td>
<td>+6'61</td>
<td>-9'22</td>
</tr>
<tr>
<td>0</td>
<td>-6'93</td>
<td>+6'93</td>
<td>+6'61</td>
<td>-9'22</td>
</tr>
<tr>
<td>0</td>
<td>-6'93</td>
<td>+6'93</td>
<td>+6'61</td>
<td>-9'22</td>
</tr>
<tr>
<td>0</td>
<td>-6'93</td>
<td>+6'93</td>
<td>+6'61</td>
<td>-9'22</td>
</tr>
</tbody>
</table>

---

Fig. 10 42.
or
\[ V_f = 0.50 \]
\[ V_f = 12 + 0.5 - 5.5 = 7.0 \text{ kN} \]
The bending moment diagram and the deflected shape have been shown in Fig. 10.42.

10.8. PORTAL FRAMES WITH INCLINED MEMBERS

In the previous article, we have considered the sway moments due to the moment of the column heads in a direction parallel to the centre line of the columns. Due to such movement, the beam BC has a motion of translation only. If, however, the columns are inclined to the vertical, the column head will move due to side sway in a direction perpendicular to the centre line of the column thereby giving a motion of rotation to the beam BC in addition to the motion of translation. Due to motion of rotation, the beam BC will also have sway moments at joints B and C. In order to find the ratio of sway moments we shall consider the three cases with different end conditions.

Case I: Both Ends Hinged

Let us consider a portal frame ABCD, hinged at A and D. Let the column AB be of length \( L_1 \), moment of inertia \( I_1 \) and its inclination to the horizontal be \( \theta_1 \). Similarly, let the column CD be of length \( L_4 \), moment of inertia \( I_4 \) and its inclination with horizontal be \( \theta_4 \). Let the beam BC be of length \( L \) and its moment of inertia be \( I \).

Let the frame be distorted as shown in Fig. 10.43 due to sway. Since the displacements are small, joint B will move to \( B' \) in direction \( BB' \) perpendicular to \( AB \). Similarly, joint C will move to \( C' \) in direction \( CC' \) perpendicular to \( CD \). Consequently, the beam BC will be distorted to \( B'C' \).

Let \( BB' = \delta_1 \) and \( CC' = \delta_4 \). Draw lines \( B_1B' \) and \( C_1C' \) perpendicular to \( BC \) from \( B_1 \) and \( C \), respectively.

Let \( \delta = \text{Vertical displacement of } C \text{ with regard to } B \) after distortion,

or
\[ \delta = B_1B' + C'C_1 \]

Considering no change in the length of the beam BC, we have
\[ BB' = CC' \]

or
\[ \delta_1 \sin \theta_1 = \delta_4 \sin \theta_4 \] (1)

\[ \delta_1 \sin \theta_1 = \delta_4 \sin \theta_4 \] (2)

MOMENT DISTRIBUTION METHOD

Equations (1) and (2) give the relationship between the displacement of various joints. From the above equations, \( \delta_1 \) and \( \delta_4 \) can be calculated in terms of \( \delta \) for given values of \( \theta_1 \) and \( \theta_4 \).

Now considering column AB as a beam fixed at B and hinged at A with the joint displacement \( \delta_1 \), we have
\[ M_{BA} = \frac{3EI\delta_1}{L_1^2} \] (3)

Similarly, for the column CD, we have,
\[ M_{CD} = \frac{3EI\delta_4}{L_4^2} \] (4)

Considering the beam BC as fixed at B and C and the joint \( C \) having displacement \( \delta \) with respect to \( B \), we have,
\[ M_{BC} = M_{BA} : M_{CD} : = \frac{2I_4}{I_1} : \frac{I_1}{L_1} : \frac{I_4}{L_4} \] (10.11)

From equations (3), (4) and (5), we get
\[ M_{BC} = M_{BA} : M_{CD} : \frac{2I_4}{I_1} : \frac{I_1}{L_1} : \frac{I_4}{L_4} \] (10.11)

It should be noted here that the moments \( M_{BA} \) and \( M_{CD} \) will be of the same sign and \( M_{BC} \) (or \( M_{CD} \)) will be of different signs. For the case illustrated in Fig. 10.43 \( M_{BA} \) and \( M_{CD} \) are of negative sign as the columns rotate in clockwise direction while \( M_{BC} \) (or \( M_{BA} \)) of positive sign since the beam BC rotates in the anticlockwise direction.
Case II: Both Ends Fixed (Fig. 10.44)

With the same notations as that of Fig. 10.43, we have

\[ \delta_1 \sin \theta_1 = \delta_2 \sin \theta_2 \]  
(1)

and

\[ \lambda = \delta_1 \cos \theta_1 + \delta_2 \cos \theta_2 \]  
(2)

Considering column \( AB \) to be fixed at \( A \) and \( B \), the movement of the joint \( B \) by \( \delta_1 \) will cause moments in the column \( AB \). Thus

\[ M_{BA} = M_{AB} = \frac{6EI\delta_1}{L_1^2} \]  
(3)

Similarly, for the column \( CD \), we have,

\[ M_{CD} = M_{DC} = \frac{6EI\delta_2}{L_2^2} \]  
(4)

and for the beam \( BC \), we have

\[ M_{BC} = M_{CB} = \frac{6EI\lambda}{L^2} \]  
(5)

Hence from (3), (4) and (5), we get

\[ \frac{M_{BC}}{M_{BA}} : \frac{M_{BA}}{M_{CD}} : \frac{M_{CD}}{M_{CB}} = \frac{I_1}{L_1^2} : \frac{I_1^2}{L_1} : \frac{I_2}{L_2^2} : \frac{L_2}{L_1} \]  
(10.12)

For the case of sway illustrated in Fig. 10.44, \( M_{BC} \) will be of positive sign while \( M_{BA} \) and \( M_{CD} \) will be of negative sign.

Case II: One End Fixed, Other End Hinged (Fig. 10.45)

In this case also, we have

\[ \delta_1 \sin \theta_1 = \delta_2 \sin \theta_2 \]  
(1)

and

\[ \lambda = \delta_1 \cos \theta_1 + \delta_2 \cos \theta_2 \]  
(2)

For the column \( AB \), fixed at \( A \) and \( B \), we have

\[ M_{BA} : M_{AB} = \frac{6EI\delta_1}{L_1^2} \]  
(3)

For the column \( CD \), fixed at \( C \) and hinged at \( D \), we have

\[ M_{CD} = M_{DC} = \frac{3EI\delta_2}{L_2^2} \]  
(4)

For the beam \( BC \), fixed at \( B \) and \( C \), we have

\[ M_{BC} : M_{CB} = \frac{2I\delta_1}{L_1^2} : \frac{2I\delta_2}{L_2^2} : \frac{I_1}{L_1} : \frac{I_2}{L_2} \]  
(10.13)

Example 10.21. Analyse the portal frame shown in Fig. 10.46. The end \( A \) is fixed and \( D \) is hinged. The beam \( BC \) is loaded with a U.D.L. of 5 kN/m.

Fig. 10.21.
Solution

(a) Fixed end moments
\[ M_{FBC} = \frac{5 \times 3^3}{12} = -3.75 \text{ kN-m} \]
\[ M_{FCB} = +3.75 \text{ kN-m} \]
(b) Distribution factors (Table 10.49)

<table>
<thead>
<tr>
<th>Joints</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>BA</td>
<td>( \frac{1}{5} = \frac{3l}{15} )</td>
<td>( 8l )</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>BC</td>
<td>( \frac{1}{3} = \frac{5l}{15} )</td>
<td>( 15 )</td>
<td>0.62</td>
</tr>
<tr>
<td>C</td>
<td>CB</td>
<td>( \frac{1}{3} = \frac{0.33l}{3} )</td>
<td>0.51</td>
<td>0.65</td>
</tr>
</tbody>
</table>
   \[ \frac{3}{2} \times \frac{1}{3} = \frac{0.18l}{3} \]

(c) Moment distribution

\[ V_x \times 3 + 1.04 + 2.08 = 4 \times h_a \]
\[ 4h_a = 3V_x + 3.12 \]

Taking moments about C for equilibrium of CD
\[ V_x \times 3 + 1.91 = h_a \times 3 \]
\[ 3V_x + 1.91 = 3h_a \]

Also,
\[ P = h_a - h_a \]

Taking moments about A for the equilibrium of the whole frame we get,
\[ V_x \times 9 - P \times 4 = 5 \times 3 \times 4.5 - h_a \times 1 + 1.04 \]
\[ 9V_x + h_a - 4P = 66.46 \]

Substituting the value of \( P \) in (5), we get
\[ 9V_x + h_a + 4h_a - 4h_a = 66.46 \]
\[ 9V_x + 5h_a = 66.46 + 3V_x + 3.12 \]
\[ 12V_x + 5h_a = 114.58 \]

Substituting the value of \( h_a \) from equation (2),
\[ 12V_x + 5(15 - V_x) = 114.58 \]
\[ V_x = 5.93 \text{ kN} \]

Hence
\[ V_x = 15 - 6.93 = 8.07 \text{ kN} \]

Substituting in equation (2), we get,
\[ 3h_a = 3V_x + 1.91 = 3 \times 6.93 + 1.91 = 22.7 \]
\[ h_a = 7.57 \text{ kN} \]

Similarly from (1) we get \( h_a = 6.83 \text{ kN} \)

From equation (3), we get,
\[ P = 7.57 - 6.83 = 0.74 \text{ kN} \]
(d) Side Sway

Now apply a force $S=0.74$-at $C$ in opposite direction to that of $P$ to neutralise its effect. The frame will now experience side sway as shown in Fig. 10.46(e).

Now, let $BB_1=\delta_1$; $CC_2=\delta_2$

Difference in level of $B_1$ and $C_1=\delta=\delta_1 \cos \theta_1 + \delta_2 \cos \theta_2$

or $\delta_1 = \frac{3}{5} + \frac{\delta_2}{\sqrt{2}}$

Also $\delta_1 \sin \theta_1 = \delta_2 \sin \theta_2$

or $\delta_1 = \frac{4\sqrt{2}}{5} \delta_2$

or $\delta_1 = \frac{5}{4\sqrt{2}} \delta_2$

So $\delta = \frac{3}{4\sqrt{2}} \delta_2 + \frac{1}{\sqrt{2}} \frac{7}{4\sqrt{2}} \delta_2 = \frac{7}{4\sqrt{2}} \delta_2$

$\delta_1 = \frac{7}{5} \delta_1$

Hence $\delta_1 = \frac{5}{7} \delta = 0.7148$

and $\delta_2 = \frac{4\sqrt{2}}{7} \delta = 0.808 \delta$

Now $M_{AB} = \frac{6EI\delta_1}{l_1^2} = \frac{6EI\times0.7148}{25} = 0.171EI\delta = M_{BA}$

$M_{BC} = M_{CB} = - \frac{6EI\delta}{9} = -0.667 EI\delta$

and $M_{CD} = \frac{3EI\delta_2}{l_2^2} = \frac{3EI\times0.808 \delta}{18} = -0.135 EI\delta$

Keeping these relations between the fixed end moments we can put arbitrary value of each bending moment as follows:

$M_{AB} = M_{BA} = +5.00$ kN-m

$M_{BC} = M_{CB} = -19.50$ kN-m

$M_{CD} = +3.95$ kN-m

(e) Distribution of sway moments

The moments may now be distributed as shown in Table 10.51.

### Table 10.51

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.38</td>
<td>0.62</td>
<td>0.65</td>
<td>0.35</td>
</tr>
<tr>
<td>5.00</td>
<td>5.00</td>
<td>-19.50</td>
<td>-19.50</td>
</tr>
<tr>
<td>5.52</td>
<td>8.98</td>
<td>+10.10</td>
<td>+5.45</td>
</tr>
<tr>
<td>-2.76</td>
<td>-</td>
<td>+5.05</td>
<td>+4.49</td>
</tr>
<tr>
<td>-1.92</td>
<td>-3.13</td>
<td>-2.92</td>
<td>-1.57</td>
</tr>
<tr>
<td>-0.96</td>
<td>-1.46</td>
<td>-1.57</td>
<td>-</td>
</tr>
<tr>
<td>-0.55</td>
<td>+0.91</td>
<td>+1.02</td>
<td>+0.55</td>
</tr>
<tr>
<td>+0.28</td>
<td>-</td>
<td>+0.51</td>
<td>+0.46</td>
</tr>
<tr>
<td>-0.19</td>
<td>-0.32</td>
<td>-0.30</td>
<td>-0.16</td>
</tr>
<tr>
<td>-0.10</td>
<td>-0.15</td>
<td>-0.16</td>
<td>-</td>
</tr>
<tr>
<td>+0.03</td>
<td>+0.06</td>
<td>+0.09</td>
<td>+0.10</td>
</tr>
<tr>
<td>+7.01</td>
<td>+9.02</td>
<td>-9.02</td>
<td>-8.28</td>
</tr>
</tbody>
</table>

Taking moments about $B$ for equilibrium of $AB$,

$3V_s + 7.01 + 9.02 = h_s \times 4$

or $4h_s = 3V_s + 16.03$ (1)

Taking moments about $C$ for equilibrium of $DC$,

$h_s \times 3 = V_s \times 3 + 8.28$ (2)

Also $S = h_s + h_s'$ (3)

and $V_s = V_s'$ (4)

Taking moment about $A$ for the equilibrium of the whole frame,

$9V_s + 7.01 + h_s' \times 1 = 4h_s + 4h_s'$ (5)

Subtracting (1) from (5), we get

$9V_s + 7.01 + h_s' - 3V_s - 16.03 = 4h_s$ or $6V_s - 9.02 = 3h_s'$
Substituting the value of $3h'_e$ from (2), we get

$$6V'_d - 9.02 = 3V'_d + 8.28$$

or

$$3V'_d = 17.3$$

or

$$V'_d = 5.77 \text{kN}$$

and

$$V'_c = 5.77 \text{kN}$$

Also

$$3h'_e = 3 \times V'_d + 8.28 = 17.30 + 8.28 = 25.28$$

or

$$h'_e = 8.33 \text{kN}$$

From equation (1)

$$4h'_e = 3 \times 5.77 + 16.03 = 33.33$$

or

$$h'_e = 8.33 \text{kN}$$

The actual sway force is 0.74 kN only and hence the moments will have to be reduced accordingly. The final moments will be found out as shown in Table 10.52.

**Table 10.52**

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Sway=16.86 kN</td>
<td>$+7.01$</td>
<td>$-9.02$</td>
<td>$-8.28$</td>
</tr>
<tr>
<td>2. Sway=0.74 kN</td>
<td>$+0.31$</td>
<td>$-0.40$</td>
<td>$-0.36$</td>
</tr>
<tr>
<td>3. Non-sway moment</td>
<td>$+1.04$</td>
<td>$+2.08$</td>
<td>$-2.08$</td>
</tr>
<tr>
<td>4. Final moments</td>
<td>$+1.35$</td>
<td>$+2.48$</td>
<td>$-2.48$</td>
</tr>
</tbody>
</table>

Vertical reaction at $A=8.07 + \frac{5.77}{16.86} \times 0.74 = 8.32 \text{kN}$

Vertical reaction at $D=6.93 - \frac{5.77}{16.86} \times 0.74 = 6.86 \text{kN}$

Horizontal reaction at $A=6.83 + \frac{8.33}{16.86} \times 0.74 = 7.20 \text{kN}$

Horizontal reaction at $D=7.57 - \frac{8.33}{16.86} \times 0.74 = 7.20 \text{kN}$

**Example 10.22.** Analyse the inclined portal frame shown in Fig. 10.48 completely and draw bending moment diagram and deflected shape of the frame.

**Solution**

The bending moment diagram and the deflected shape have been shown in Fig. 10.47.

**Table 10.53**

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>D.F.</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>BA</td>
<td>$I/4A = 0.236I$</td>
<td>0.236I</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>BC</td>
<td>$I/25 = 0.4I$</td>
<td>0.4I</td>
<td>0.63</td>
</tr>
<tr>
<td>C</td>
<td>CB</td>
<td>$I/25 = 0.4I$</td>
<td>0.4I</td>
<td>0.44</td>
</tr>
<tr>
<td></td>
<td>CD</td>
<td>$I/25 = 0.5I$</td>
<td>0.5I</td>
<td>0.56</td>
</tr>
</tbody>
</table>
(c) Moment Distribution

A force $P$ is applied at joint $C$, as shown in Fig. 10.48(a) and moment distribution is carried out as indicated in Table 10.54.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>F.E.M.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.37</td>
<td>0.63</td>
<td>0.44</td>
<td>0.56</td>
<td></td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>-3.13</td>
<td>+3.13</td>
<td>-2.0</td>
</tr>
<tr>
<td>+1.16</td>
<td>+1.97</td>
<td>-0.50</td>
<td>-0.63</td>
<td>Balance</td>
</tr>
</tbody>
</table>

For equilibrium of $AB$, taking moments about $B$,  

$$V_a \times 3 + 0.67 + 1.34 = 3h_a$$

$$3h_a = 3V_a + 2.01$$  \[1\]

Taking moments about $C$ for equilibrium of $DC$,  

$$8 \times 1 + 1.36 - 3.27 = h_d \times 2$$

$$h_d = 3.05 \text{ kN}$$

or

$$P = h_a + h_d - 8 = h_a + 3.05 - 8 = h_a = 4.95$$  \[2\]

Now, taking moments about $A$ for the equilibrium of the whole frame, we get  

$$3.05 \times 1 - P \times 3 + 6.25 \times 4.25 + 1.36 + 0.67 = 8 \times 2 + V_a \times 5.50$$

Putting value of $P$ from (3) and solving, we get,  

$$6.78 - 3h_a = 5.50V_a$$

Adding (1) and (3), we get  

$$6.78 = 5.50V_a + 3V_a + 2.01$$

Also  

$$V_a = 15 - V_a$$

From equation (2), we get  

$$P = 7.40 - 4.95 = 2.45 \text{ kN}$$

(d) Side Sway

Now apply a force $S = 2.45 \text{ kN}$ at $B$ so as to neutralise the effect of force $P$. The frame $ABCD$ will be deformed to $AB'CD$ as shown in Fig. 10.48(b).  

Now  

$$B_1B' = S; \quad CC_1 = \delta_2 \text{ and } BB_1 = \delta_1$$

So  

$$\delta = \delta_1 \cos 45^\circ = \frac{\delta_1}{\sqrt{2}}$$

Also  

$$\delta_1 \sin 45^\circ = \delta_2$$

or  

$$\frac{\delta_1}{\sqrt{2}} = \delta_2$$

So  

$$\delta = \delta_2 = \frac{\delta_1}{\sqrt{2}}$$

The relations between the sway moments may now be put as under:

$$M_{AB} = M_{AB} = -\frac{6EIh_a}{(3\sqrt{2})^2} = -\frac{6EI\times 8\sqrt{2}}{18} = -0.47\frac{6EI}{8}$$

$$M_{BC} = M_{CB} = +\frac{6EI}{2.53} = +0.96\frac{EI}{8}$$

$$M_{CD} = M_{DC} = -\frac{6EIh_a}{25} = -\frac{6EI}{4} = -1.5\frac{6EI}{8}$$

The sway moments are given arbitrary values as below, satisfying the above relations.

$$M_{AB} = M_{AB} = -10.0 \text{ kN-m}$$

$$M_{BC} = M_{CB} = +20.4 \text{ kN-m}$$

and

$$M_{CD} = M_{DC} = -31.9 \text{ kN-m}$$

(c) Distribution of sway moments

The moment distribution may now be carried out as shown in Table 10.55.
Table 10.55

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10.0</td>
<td>-10.0</td>
<td>+20.4</td>
<td>+20.4</td>
</tr>
<tr>
<td>-3.85</td>
<td>-6.55</td>
<td>+5.06</td>
<td>+6.44</td>
</tr>
<tr>
<td>1.92</td>
<td>+2.53</td>
<td>-3.27</td>
<td>-</td>
</tr>
<tr>
<td>-0.94</td>
<td>-1.59</td>
<td>+1.44</td>
<td>+1.83</td>
</tr>
<tr>
<td>-0.47</td>
<td>+0.72</td>
<td>-0.80</td>
<td>-</td>
</tr>
<tr>
<td>-0.27</td>
<td>-0.45</td>
<td>+0.33</td>
<td>-0.47</td>
</tr>
<tr>
<td>-0.14</td>
<td>+0.16</td>
<td>-0.22</td>
<td>-</td>
</tr>
<tr>
<td>-0.66</td>
<td>-0.10</td>
<td>+0.10</td>
<td>+0.12</td>
</tr>
<tr>
<td>-0.03</td>
<td>-0.05</td>
<td>-0.05</td>
<td>-</td>
</tr>
<tr>
<td>-0.02</td>
<td>-0.03</td>
<td>-0.02</td>
<td>+0.03</td>
</tr>
<tr>
<td>-12.57</td>
<td>-15.14</td>
<td>+23.01</td>
<td>-23.01</td>
</tr>
</tbody>
</table>

Taking moments about B for equilibrium of AB,

\[ 3h_s = V_s' \times 3 + 12\times 57 + 15\times 14 = 3V_s' + 27.71 \]

(1)

Taking moments about C for equilibrium of CD,

\[ 2h_s = 23.01 + 27.45 = 50.46 \]

\[ h_s' = 25.23 \text{ kN} \]

Also

\[ S = h_s + h_s' = h_s + 25.23 \]

(2)

(3)

Taking moments about A, for the equilibrium of the whole frame,

\[ 5 \times 3 = h_s' \times 1 + V_s' \times 5 + 27.45 + 12.57 \]

Substituting the values of \( P \) and \( h_s' \) from equations (2) and (3), we get,

\[ 3h_s = 5.5 V_s' - 10.46 \]

Subtracting (1) from (4), we get,

\[ 2.5V_s' = 38.17 \text{ kN} \]

or

\[ V_s' = 15.27 \text{ kN} = V_s' \]

Fig. 10.549.

Horizontal reaction at \( A = 7.40 - \frac{24.51 \times 2.45}{49.74} = 6.20 \text{ kN} \)

Horizontal reaction at \( D = 3.05 - \frac{25.24 \times 24.45}{49.74} = 1.80 \text{ kN} \)

Vertical reaction at \( A = 6.73 - \frac{15.27 \times 24.45}{49.74} = 5.98 \text{ kN} \)

Vertical reaction at \( D = 8.27 + \frac{15.27 \times 24.45}{49.74} = 9.02 \text{ kN} \)

The bending moments diagram and the deflected shape have been plotted in Fig. 10.49.

PROBLEMS

1. A continuous beam \( ABCD \) is fixed at ends \( A \) and \( D \), and is loaded as shown in Fig. 10.50. Spans \( AB, BC \) and \( CD \) have moments of inertia of \( I, 1.5 I \).
and I respectively and are of the same material. Determine the moments at the supports and plot the bending moment diagram.

\[
\text{EI \ CONSTANT}
\]

Fig. 10-50.

2. Solve problem 1 if both ends A and D are freely supported.

3. Solve problem 1 if there is no support at the end D.

4. A beam ABC, 12 m long, fixed at A and C and continuous over support B, is loaded as shown in Fig. 10-51. Calculate the end moments and plot the bending moment diagram.

\[
\text{EI \ CONSTANT}
\]

Fig. 10-51.

5. A beam ABCD, 16 m long is continuous over three spans and is loaded as shown in Fig. 10-52. Calculate the moments and reactions at the supports.

\[
\text{EI \ CONSTANT}
\]

Fig. 10-52.

6. Solve Problem 5 if the support B sinks by 5 mm downwards. I for the beam is \(93 \times 10^6 \text{ mm}^4\) throughout. Take \(E=2.1 \times 10^7 \text{ N/mm}^2\).

7. A continuous beam ABCD, 20 m long is simply supported at its ends and is propped at the same level at points 5 m and 12 m from left hand A. It carries two concentrated loads of 8 kN and 5 kN at 2 m and 9 m respectively from A and a uniformly distributed load of 1 kN/m over the span CD. Find the B.M. at the props if the support B sinks by 10 mm below A and C. Moment of inertia for the whole beam is \(85 \times 10^6 \text{ mm}^4\) and \(E=2.1 \times 10^7 \text{ N/mm}^2\).

8. Draw the bending moment diagram for the frame shown in Fig. 10-53. The frame has stiff joint at B and is fixed at A, C and D.

9. Analyse the continuous one storey frame shown in Fig. 10-54.

10. Fig. 10-55 shows a two span portal frame with the columns fixed at end A, E and F and carries uniformly distributed load of \(w \text{ kN/m}\) along BD. The stiffness ratios of the members are shown in the diagram and all the members are of equal length.

(a) Determine the bending moments throughout the frame and sketch the bending moment diagram.

(b) If Young's modulus \(E\) is constant and the central column sinks by an amount \(\Delta\), determine the changes in the moments at B, C and D in terms of \(E\), \(\Delta\) and \(L\).

11. A portal frame ABCD fixed at ends A and D carries a point load \(P\) as shown in Fig. 10-56. Draw the bending moments diagram and sketch the deflected shape of the beam.

12. A portal frame ABCD is fixed at A and hinged at D. Draw the bending moment diagram due to a point load of 9 kN as shown in Fig. 10-57. Calculate the reactions and sketch the deflected shape of the frame.
13. Analyse completely the portal frame shown in Fig. 10-58.

14. The portal frame shown in 10-59 has fixed ends. If D sinks by 
\( \Delta \), find the moments induced in the frame. All the members have the same 
uniform cross-section.

15. Determine the bending moments at the joints and draw the bending 
moment diagram for the frame shown in Fig. 10-60. The flexural rigidity of 
the members is as shown.

\begin{align*}
M_{AB} &= -4.6 \text{kN} \cdot \text{m} ; M_{BA} = +2.98 \text{kN} \cdot \text{m} ; \\
M_{BC} &= -2.98 \text{kN} \cdot \text{m} ; M_{CB} = +4.27 \text{kN} \cdot \text{m} ; \\
M_{CD} &= +5.7 \text{kN} \cdot \text{m} ; M_{DC} = -5.7 \text{kN} \cdot \text{m} ; \\
M_{DB} &= +16 \text{kN} \cdot \text{m}.
\end{align*}

16. The portal frame shown in 10-59 has fixed ends. If D sinks by 
\( \Delta \), find the moments induced in the frame. All the members have the same 
uniform cross-section.

17. Determine the bending moments at the joints and draw the bending 
moment diagram for the frame shown in Fig. 10-60. The flexural rigidity of 
the members is as shown.
The Column Analogy Method

11. THE COLUMN ANALOGY

The column analogy method was suggested by Prof. Hardy Cross in 1930, and is the most useful in the analysis of beams and curved members with two fixed supports and of rigid frames up to third degree of redundancy. He proved the mathematical similarity or analogy between the stresses created on a column section subjected to eccentric load and the moments imposed on a member due to fixidity of its supports.

To understand the analogy, consider a fixed beam AB with loads \(W_1, W_2, \text{etc.}\) (Fig. 11.1). The indeterminate bending moment diagram (or fixed bending moment diagram or \(M_1\) dia.) is shown in Fig. 11.1 (b) and the static bending moment diagram (or \(Ms\) dia.) is shown in Fig. 11.1 (c). Fig. 11.1 (d) shows a short height of a column having the length \(L\) equal to the span of the beam, and width equal to \(\frac{1}{EI}\).

The ends A and B of the beam are fixed, and hence \(\theta_A\) and \(\theta_B\) are zero. It follows, therefore, that the area of the total bending moment diagram from A to B is zero.

Thus, we have

\[
\int_{A}^{B} EI \theta_0 = - \int_{A}^{B} \text{Area of B.M. diagram} = 0
\]

or

\[
\int_{0}^{L} \frac{M dx}{EI} + \int_{0}^{L} \frac{M_1 dx}{EI} = 0
\]

or

\[
\int_{0}^{L} \frac{M dx}{EI} = - \int_{0}^{L} \frac{M_1 dx}{EI}
\]  

(1)

Let us now load the column by loading diagram equal to \(\frac{Ms}{EI}\) diagram [Fig. 11.1 (e)]. The pressure distribution diagram on the column will be as shown in Fig. 11.1 (f). Studying Figs. 11.1 (b) and 11.1 (f), it is evident that both the diagrams are similar and this similarity can be utilised to find the fixed end moments \(M_A\) and \(M_B\).

![Diagram](image-url)
(ii) the moment of the total pressure, at the base of the column, about \( A \) is equal to the moment of the total pressure at the top of the column, about \( A \).

Let us consider a small section \( dx \) of the column, situated at a distance \( x \) from the end \( A \). Let \( f \) be the intensity of pressure at the base of that section.

Thus we have

\[
P = \int_0^L f \cdot dA
\]

where

- \( P \) = total load on the column
- \( dA \) = area of \( \frac{M_S}{EI} \) diagram

and

\[
\int_0^L \frac{M_S dx}{EI} = \int_0^L f \cdot dA
\]

(Thus minus sign has been used because \( M_S \) diagram is negative.)

Thus from (2) and (1),

\[
\int_0^L \frac{M_S dx}{EI} = \int_0^L f \cdot dA
\]

(11.1)

From equation 11.1, it is clear that the fixed end bending moment diagram is analogous to the pressure diagram of an eccentrically loaded column. Since the \( \frac{M_t}{EI} \) diagram and the pressure diagram are trapeziums of equal base \( L \), it follows that

\[
\begin{align*}
M_A &= fA \\
M_B &= fB
\end{align*}
\]

(11.2)

Thus, if the pressure diagram is known, the fixing moment can easily be determined. The column shown in Fig. 11.1 (d) is known as the analogous column having width of such a magnitude that the total pressure at its base is equal to the total load on the column.

- Total load on the column = \( \int_0^L \frac{M_S dx}{EI} = \int_0^L \frac{M_S dx}{EI} \)
- Total pressure of the base = \( \int_0^L f \cdot dA \)

Comparing the two, it is evident that in order that \( M_t \) is equal to \( f \), the area \( dA \) of the section must be equal to \( \frac{dx}{EI} \). But \( dA = \frac{Length \times width}{x \times width} \). Hence the width of the analogous column is equal to \( \frac{1}{EI} \).

### 11.2. APPLICATION OF THE ANALOGY FOR FIXED BEAMS

Let us start with the analysis of a fixed beam which is statically indeterminate to second degree for vertical loading. The stress diagram of the analogous column will also give the \( M_t \) diagram of the fixed beam if the analogous column is loaded with the \( \frac{M_S}{EI} \) diagram. The \( M_S \) diagram may be drawn by releasing statically indeterminate moments and forces until the beam becomes statically determinate. Any basic statically determinate structure may be chosen. The statically indeterminate moment at any point of the beam will be equal to the stress \( f \) of the column at the same point. Once the \( M_t \) diagram is known, the final bending moment at any point of the beam may be found from the relation

\[
M_t = M_S + M_t
\]

**Sign Convention**

The following sign convention will be adopted.

1. A moment bending the beam convex upwards will be taken as positive and that bending the beam concave upwards as negative.
2. A downward load on the column will be taken as positive and upward load as negative. Thus the positive downward load on the column will be the \( \frac{M_S}{EI} \) diagram.
3. An upward pressure at the base of the column will be taken as positive and downward pressure as negative.
11.3. PROPERTIES OF A SYMMETRICAL ANALOGOUS COLUMN

Fig. 11-2 (a) shows an analogous column having width \( \frac{1}{E} \) and length \( L \) equal to the length or span of the statically indeterminate structure. The column is symmetrical, having the axis \( x-x \) along the span or length and the axis \( y-y \) perpendicular to it. Fig. 11-2 (b) shows the plan view. Let \( P \) be the point of application of the load on the column, having the co-ordinates \( (x_1, y_1) \).

For the analogous column,

\[
A = L \cdot \frac{1}{E} = \frac{L}{E} \\
I_{yy} = \frac{1}{12} \cdot \frac{L}{E} = \frac{L^3}{12EI} \\
I_{xx} = \frac{1}{12} \cdot \frac{L}{(EI)^2} \approx \frac{L}{12(EI)^2} = \text{negligible}
\]

The total stress \( f \) at any point having co-ordinates \( (x, y) \) referred to the principal axis, will be given by

\[
f = f_0 \pm \frac{M_{xx} \cdot y}{I_{xx}} \pm \frac{M_{yy} \cdot x}{I_{yy}}
\]

\[\text{(11.3)}\]

Example 11.1. A beam \( AB \) of span \( L \) is fixed at both the ends and carries a uniformly distributed load \( w \) per unit length. Using the column analogy method, compute the fixed end moments.

Solution (A)

Fig. 11-3 (a) shows the loaded beam. If the simply-supported beam is chosen as the basic determinate structure, the \( Ms \) diagram will be a parabola having a central ordinate \( \frac{wL^2}{8} \).

Fig. 11-3 (d) shows the corresponding analogous column. Due to symmetry, \( f_A \) and \( f_B \) will be equal.

Total load \( P = \int_0^L \frac{M_{s} dx}{EI} = \frac{1}{EI} \left[ -\frac{2}{3} \cdot L \cdot \frac{wL^2}{8} \right] = \frac{wL^3}{12EI} \)

Area of the column \( A = \frac{L}{E} \)
Since the loading is symmetrical about \( x-x \) and \( y-y \) axis, \( M_{xx} \) and \( M_{yy} \) are each zero, and the stress \( f \) at any point is given by

\[ f = f_0 = \frac{P}{A} \]

Hence

\[ f_A = M_A = f_0 = \frac{P}{A} = \frac{wL^3}{12EI} \cdot \frac{L}{E} = \frac{wL^4}{12} \]

and

\[ f_B = M_B = f_0 = \frac{wL^3}{12}. \]

**Alternative Solution**

The problem can also be solved by taking the cantilever as the basic determinate structure. For that, the support \( B \) is completely removed, and the end \( A \) is kept fixed. The \( M_s \) diagram will be a parabola having maximum ordinate of \( \frac{wL^3}{2} \) at \( A \), as shown in Fig. 11'3 (c). The corresponding analogous column is shown in Fig. 11'3 (e).

Resultant load \( P = -\int_0^L \frac{M_{s}dx}{EI} = -\frac{1}{EI} \left[ \frac{1}{3} \cdot L \times \frac{wL^2}{2} \right] \)

\[ x \text{ from } A = \frac{L}{4} \]

\[ \therefore \text{ Eccentricity } e = \frac{L}{2} - \frac{x}{2} = \frac{L}{4} = \frac{L}{4} \]

Moment \( M_{yy} = P \cdot e = -\frac{wL^3}{6EI} \times \frac{L}{4} = -\frac{wL^4}{24EI} \)

Moment \( M_{xx} = 0 \)

\[ I_{yy} = \frac{L^3}{12EI} \]

Area \( A = \frac{L}{EI} \)

The stress \( (f) \) at any point \((x, y)\) is given by

\[ f = \frac{P}{A} \pm \frac{M_{yy} \cdot x}{I_{yy}} \]

\[ = \left( -\frac{wL^3}{6EI} \times \frac{L}{E} \right) \pm \left( -\frac{wL^4}{24EI} \times \frac{12EI}{L^2} \cdot x \right) \]

\[ = \left( -\frac{wL^3}{6} \right) \pm \left( -\frac{wL}{2} \cdot x \right) \]  

(1)
STRENGTH OF MATERIALS AND THEORY OF STRUCTURES

For $A, f = f_A$ and $x = \frac{L}{2}$
\[ f_A' = \frac{W_a b^2}{L^3} \cdot \frac{L}{2} = \frac{W_a b^2}{L^3} (L + b - a) = \frac{W_a b^2}{L^3} \]

For $B, f = f_b$ and $x = \frac{L}{2}$
\[ f_b' = \frac{W_a b^2}{L^3} \cdot \frac{L}{2} = \frac{W_a b^2}{L^3} (L - b + a) = \frac{W_a b^2}{L^3} \]

Hence
\[ M_A = f_A' = \frac{W_a b^2}{L^3} \]
\[ M_B = f_B' = \frac{W_a b^2}{L^3} \]

and

Distance of C.G. from $A = \bar{x} = \frac{a}{3}$

Eccentricity $c = \frac{L}{2} - \frac{a}{3} = \frac{3L - 2a}{6} = \frac{a + 3b}{6}$

Area $A = \frac{L}{E I}$; $I_y = \frac{L^3}{12 E I}$

\[ M_{yy} = P \cdot e = -\frac{W_a^2}{2 E I} \cdot \frac{a + 3b}{6} \]

The stress at any point in the column is given by
\[ f = \frac{P}{A} \pm \frac{M_{yy}}{I_y} \cdot x \]
\[ = \left( -\frac{W_a^2}{2 E I} \cdot \frac{E I}{L} \right) \pm \left( -\frac{W_a^2}{2 E I} \cdot \frac{a + 3b}{6} \cdot \frac{12 E I}{L^3} \cdot x \right) \]
\[ = \left( -\frac{W_a^2}{2 L} \right) \pm \left\{ -\frac{W_a^2}{L^3} \left( \frac{a + 3b}{6} x \right) \right\} \]

For $A, x = L/2$ and $f = f_A$
\[ f_A = \left( -\frac{W_a^2}{2 L} \right) \pm \left\{ -\frac{W_a^2}{L^3} \left( a + 3b \right) \cdot \frac{L}{2} \right\} \]
\[ W_a^2 \cdot \frac{L}{2L} = \frac{W_a^2}{L^2} (a + 3b) = -\frac{W_a^2}{L^2} (a + 3b) \]

For $B, x = L/2$ and $f = f_b$
\[ f_b = \left( -\frac{W_a^2}{2 L} \right) - \left\{ -\frac{W_a^2}{L^3} \left( a + 3b \right) \cdot \frac{L}{2} \right\} \]
\[ = -\frac{W_a^2}{2 L} + \frac{W_a^2}{2 L^3} (a + 3b) = -\frac{W_a^2}{2 L^3} (a + 3b) - L = W_a^2/L^3 \]

Hence $M_A = M_A'$, $M_S = f_A + M_S = -\frac{W_a^2}{L^2} (a + 2b) + W_a$
\[ = \frac{W_a}{L^3} (L^2 - a^2 - 2ab) \]

and $M_B = M_B' + M_S = \frac{W_a b}{L^2} + 0 = \frac{W_a b}{L^2}$

Example 11.3. A beam $AB$ of span $L$ is fixed at both the ends and carries a point load $W$ at its centre. The moment of inertia of first half portion of the beam is $I$ and that of the next half is $I$. Compute the fixed end moments.

Solution

Let us first solve the problem by taking the simply-supported beam as the basic determinate structure. The corresponding B.M.
diagram is shown in Fig. 11.5 and the loaded analogous beam is shown in Fig. 11.5 (d).

![Diagram of strength of materials and theory of structures](image)

The total load is given by:

$$P = \frac{W}{2} + \frac{W}{2} = \frac{WL^2}{32EI} \frac{L}{6} \frac{L}{2} \frac{L}{12}$$

The C.G. $\bar{x}$ of the load, measured from A, is given by:

$$\bar{x} = \frac{3WL^2}{32EI} \left( \frac{2}{3} \cdot \frac{L}{2} \right) + \frac{WL^2}{16EI} \left( \frac{L}{2} + \frac{1}{3} \cdot \frac{L}{2} \right)$$

From which $\bar{x} = \frac{5L}{9}$.

Properties of the analogous column:

Area $A$ of the analogous column

$$A = \left( \frac{1}{2EI} \cdot \frac{L}{2} \right) + \left( \frac{1}{2EI} \cdot \frac{L}{2} \right) = \frac{3L}{4EI}$$

The distance $g$ of the centroid of the column section is given by:

$$g = \left( \frac{1}{2EI} \cdot \frac{L}{2} \right) \left( \frac{L}{4} \right) + \left( \frac{1}{2EI} \cdot \frac{L}{2} \right) \left( \frac{L}{2} + \frac{L}{4} \right)$$

From which $g = \frac{7L}{12}$.

Eccentricity $e$ of the load is:

$$e = \left( \frac{7L}{12} \right) - \left( \frac{5L}{9} \right) = \frac{L}{36}$$

$IA = \frac{L}{3} \cdot \frac{1}{EI} \cdot \frac{L}{3} \left( \frac{1}{2EI} \right) \left( \frac{L}{2} \right) = \frac{5L^3}{16EI}$

$IV = \frac{L}{4} - Ag = \frac{5L}{16EI} - \frac{3L}{4EI} \left( \frac{11L}{12} \right)^2 = \frac{11L^3}{192EI}$

The stress $f$ at any point is given by:

$$f = \frac{P}{A} \pm \frac{M_{yy} \cdot x}{I_{yy}}$$

where $x$ is measured from $y-y$ axis

$$f = \frac{WL}{8} \pm \frac{W}{22} \frac{x}{L}$$

For $A$, $x = g = \frac{7L}{12}$

Hence $f_A = \frac{WL}{8} + \frac{W}{22} \cdot \frac{7L}{12} = \frac{-5}{33} WL$

For $B$, $x = L - g = \frac{7L}{12} - \frac{5}{12} L$

Hence $f_B = \frac{WL}{8} - \frac{W}{22} \cdot \frac{5}{12} L = \frac{7}{66} WL$

and $M_A = f_A + M_S = f_A + M_S = \frac{5}{33} WL + 0 = \frac{5}{33} WL$

and $M_B = f_B + M_S = f_B + M_S = \frac{7}{66} WL + 0 = \frac{7}{66} WL$

Alternative Solution

Let us now solve the problem by taking the cantilever as the basic determinate structure. The $Ms$ diagram is shown in Fig. 11.5 (c) and the loaded column in Fig. 11.5 (e).

The total load is given by:

$$P = \frac{WL}{2} + \frac{WL}{2} = \frac{WL^2}{16EI}$$

$$= \left[ \frac{1}{2EI} \cdot \frac{L}{2} \left( \frac{L}{2} + \frac{WL}{2} \right) \right] = \frac{WL^2}{16EI}$$
The C.G. \( \bar{x} \) of the load, measured from \( A \), is given by
\[
\bar{x} = \frac{1}{3} \left( \frac{L}{2} \right) = \frac{L}{6}
\]
For the previous solution:
\[
A = \frac{3L}{4EI}, \quad g = \frac{7L}{12}, \quad l_{yr} = \frac{11L^2}{192} \frac{L^x}{EI}
\]
The eccentricity \( e = g - \bar{x} = \frac{7L}{12} - \frac{L}{6} = \frac{5L}{12} \)
\[
M_{yr} = P \cdot e = -\frac{WL^3}{16EI} \cdot \frac{5}{12} L = -\frac{5WL^3}{192EI}
\]
The stress at any point is given by
\[
f = \frac{P}{A} \pm \frac{M_{yr} \cdot x}{l_{yr}}, \text{ where } x \text{ is measured from } y-y \text{ axis.}
\]
\[
\begin{align*}
&= \left( -\frac{WL^3}{16EI} \cdot \frac{4E}{3L} \right) \pm \left( -\frac{5WL^3}{192EI} \cdot \frac{11x}{12} \right) \\
&= \left( -\frac{WL}{12} \right) \pm \left( -\frac{5wx}{11} \right)
\end{align*}
\]
For \( A \), \( x = g = \frac{7L}{13} \)
\[
f_A = -\left( \frac{WL}{12} \right) + \left( -\frac{5W}{12} \cdot \frac{7L}{12} \right) = -\frac{WL}{12} - \frac{35WL}{12 \times 11} = \frac{23}{66} \frac{WL}{12}
\]
For \( B \), \( x = L - g = L - \frac{7L}{12} = \frac{5L}{12} \)
\[
f_B = -\left( \frac{WL}{12} \right) - \left( -\frac{5W}{11} \cdot \frac{5L}{12} \right) = -\frac{WL}{12} + \frac{25WL}{12 \times 11} = \frac{7WL}{66}
\]
Hence \( M_A = M_1 + M_S = -\frac{23}{66} \frac{WL}{12} + \frac{25WL}{12} = \frac{5}{33} \frac{WL}{12} \)
\[
M_B = M_1 + M_S = +\frac{7WL}{66} + 0 = +\frac{7WL}{66}
\]
\textbf{Example 11.4.} An encastre beam of span \( L \) carries a uniformly distributed load \( w \). The second moment of area of the central half of the beam is \( I_1 \) and that of the end portion is \( I_2 \). Neglecting the weight of the beam itself, find the ratio of \( I_1 \) to \( I_2 \) so that the magnitude of the bending moment at the centre is one-third of that of the fixed moments at the ends.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig11-6}
\caption{Fig. 11-6}
\end{figure}

Taking simply-supported beam as the basic determinate structure, the \( M_S \) diagram is shown in Fig. 11-6(c). The ordinates of \( M_S \) diagram at \( C \) and \( D \) are
\[
M_S \text{ at } C \text{ and } D = -\frac{wL}{4} \left( \frac{L}{4} \right) + \frac{w}{6} \left( \frac{L}{4} \right)^3 = -\frac{3wL^3}{32}
\]
\[
M_S \text{ at the centre of the beam} = -\frac{WL^3}{8}
\]
Fig. 11-6(c) shows the \( \frac{M_S}{EI} \) diagram which also represents the load on the analogous column.

Let area \( acc_0 = bdd_0 \) = load \( P_1 \)
and area \( ccd_1d_1 = \text{Load } P_2 \)
\[
\therefore \quad P_1 = \int_0^{L/4} \frac{M_S dx}{EI_1} = \int_0^{L/4} \frac{1}{EI_1} \left[ \frac{wL}{4} \left( \frac{L}{4} \right) + \frac{w}{6} \left( \frac{L}{4} \right)^3 \right] dx = \frac{384WL^3}{384EI_1}
\]
To get \( \bar{x} \) of \( P_1 \) measured from \( a \), we have
\[
P_1 \bar{x} = \frac{1}{EI_1} \left[ \frac{wL}{4} \left( \frac{L}{4} \right) + \frac{w}{6} \left( \frac{L}{4} \right)^3 \right] x dx
\]
\[
5 \frac{wL^3}{384 EI_1} \bar{x} = \frac{1}{EI_1} \left[ \frac{wL}{6} \left( \frac{L}{4} \right)^3 - \frac{w}{8} \left( \frac{L}{4} \right)^4 \right] = \frac{13}{16 \times 384} \cdot \frac{wL^4}{EI_1}
\]
From which \( x_i = \frac{13}{80} L \).

Similarly, Load \( P_i = -\frac{1}{E_i} (\text{area } a_i d_i) \)

\[ = -\frac{1}{E_i} (\text{area } a_i d_i) \]

And \( x_2 \), measured from \( a \), is equal to \( \frac{L}{2} \).

The net loadings \( P_1, P_2 \) and \( P_3 \) are shown on the analogous column in Fig. 11.5 (d).

Resultant load \( P = 2P_1 + P_2 = \frac{5wL^3}{192E_i} + \frac{11}{192} \frac{wL^3}{E_i} \)

and \( x = \frac{L}{2} \), due to symmetry.

Area

\[ A = \frac{2}{E_i} \left( \frac{1}{4} \right) \]

Due to symmetry, \( M_{yy} = 0 \).

\[ f_a = f_b = \frac{\left( \frac{5wL^3}{192E_i} + \frac{11}{192} \frac{wL^3}{E_i} \right)}{\left( \frac{I}{I_1} \right)} \]

\[ = \frac{wL^3}{96} \left( \frac{5}{I_1} + \frac{11}{I_2} \right) \cdot \frac{1}{96} \left( \frac{I_1 + 11I_3}{I_1 + I_2} \right) \]

Now \( Ms \) at the centre = \( \frac{wL^3}{8} \)

\[ M_1 \text{ at the centre} = f \]

\[ M (\text{net at the centre}) = M_s - M_t = \frac{wL^3}{8} - f \] (numerically)

\[ M (\text{at ends}) = M + M_s = M_1 + 0 = f \] (numerically)

As per given condition

\[ \frac{wL^3}{8} - f = \frac{f}{3} \]

\[ \frac{wL^3}{8} = f + \frac{f}{3} = \frac{4}{3} f = \frac{4}{3} \cdot \frac{wL^3}{96} \left( \frac{5I_1 + 11I_3}{I_1 + I_2} \right) \]

or \( 9I_3 + 9I_1 = 11I_3 + 5I_1 \) (c)

From which \( \frac{I_1}{I_2} = \frac{1}{2} \)

Fig. 11.7.
is downward if \( M_S \) causes compression on the outside of the frame. The pressure on the bottom of the analogous column at any point will then give the indeterminate moment \( M_I \) at the point. The final moment is then given by \( M = M_S + M_I \).

Fig. 11.7(a) shows a portal frame fixed at ends \( A \) and \( D \) and loaded with uniformly distributed load on \( BC \). Fig. 11.7(b) shows a basic determinate structure derived by removing the end \( D \) completely, the \( M_S \) diagram being drawn on the frame itself. Fig. 11.7(d) shows the corresponding load on the analogous column. Another basic determinate structure can be obtained by making end \( A \) hinged and supporting end \( D \) on rollers as shown in Fig. 11.7(c) along with the \( M_S \) diagram. Fig. 11.7(e) shows the corresponding loading on the analogous column. Fig. 11.7(c) and (e) also correspond to the basic determinate structure derived by treating the beam \( BC \) as hinged (or simply supported) at \( B \) and \( C \). In a similar manner, basic determinate structure can be obtained for other end conditions of portal frames.

### 11.5. THE GENERALISED COLUMN FLEXURE FORMULA

Upto this stage, we have analysed a column which is symmetrical about both the principal centroidal axes, as in \$11.3. While analysing the portal frames, we come across such analogous columns which are not symmetrical about the principal centroidal axes. In a generalised case, the stress \( f \) at any point \( x, y \) in a column section can be expressed as

\[
f = a + bx + cy
\]

where \( a, b, c \) are constants to be determined.

Hence load \( P = \int f \cdot dA = a \int dA + b \int x \cdot dA + c \int y \cdot dA \).

Since the principal axes are through the centroid, and \( x \) and \( y \) are measured with reference to these, \( \int x \cdot dA \) and \( \int y \cdot dA \) are each zero.

Hence \( P = a \int dA \),

or \( a = \frac{P}{\int dA} = \frac{P}{A} \) (2)

Also, \( M_{xx} = \int (f \cdot dA)x = \int (a + bx + cy)x \cdot dA \)

\[
= a \int x \cdot dA + b \int x^2 dA + c \int xy dA
\]

\[
= 0 + bx^2 + cy^2 \]

\( \text{Similarly, } M_{yy} = \int (f \cdot dA)y = \int (a + bx + cy)y \cdot dA \)

\[
= a \int y \cdot dA + b \int xy dA + c \int y^2 dA
\]

\[
= 0 + bxy + cy^2 \]

\[
\int x^2 dA = \frac{1}{2} I_{xx}, \quad \int xy dA = \frac{1}{3} I_{xy}, \quad \int y^2 dA = \frac{1}{2} I_{yy}
\]

Substituting the values of \( a, b \) and \( c \) in (1), we get

\[
f = \frac{P}{A} + \left[ \frac{M_{yy} I_{xx} - M_{xx} I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right] x + \frac{M_{xx} I_{yy} - M_{yy} I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} y
\]

(11.4)

If, however, the portal frame is such that one of the principal centroidal axes is the axis of symmetry, \( I_{xy} \) becomes zero. Equation 11.4 then reduces to the form

\[
f = \frac{P}{A} + \frac{M_{yy}}{I_{yy}} x + \frac{M_{xx}}{I_{xx}} y
\]

(11.5)

### Sign Convention

While applying equation 11.4 or 11.5 the following sign convention is to be followed rigidly: \( x \) is reckoned positive when measured to the right, and negative when measured to the left of the \( y-y \) axis. Similarly, \( y \) is reckoned positive when measured upwards and negative when measured downwards to the \( x-x \) axis. Both \( x \) and \( y \) will be measured with centroid of the column section as the origin. This is represented in Fig. 11.8.

The length of every member of the analogous column will be equal to the length of the corresponding member of the frame, and
its width will be equal to \( \frac{1}{EI} \) where \( I \) is the moment of inertia of the member. Since the width \( \frac{1}{EI} \) is extremely small in comparison to its length, the column section may be looked upon as a line diagram when calculating the centroid of the section, or calculating \( I_{xx} \) and \( I_{yy} \).

**Neutral Axis of Analogous Column:**

The neutral axis of the analogous column is that axis, at every point of which the stress \( f \) is zero. Hence the points of intersection of N.A. with the legs of the portal will be the points of contraflexure since \( f \) and hence \( M_{N} \) will be zero at these points. The position of the N.A. (Fig. 11'8) can very easily be located by calculating the co-ordinates \( x_{1} \) and \( y_{1} \) in which it intersects the \( x \)-axis and \( y \)-axis respectively. The co-ordinates \( x_{1} \) and \( y_{1} \) can be calculated on the premise that stress \( f \) on each of these points, i.e. \((x, 0)\) and \((0, y)\) is zero.

11'6. PORTAL FRAME WITH HINGED LEG(S)

A hinge represents the possibility of an indefinitely large rotation. Since it offers no resistance to rotation, the flexural rigidity \( EI \) at a hinge is zero. This results in the following:

(1) Since the width of the analogous column is \( \frac{1}{EI} \), its area of cross-section corresponding to the hinge is infinite.

(2) If the portal frame has one end fixed and the other end hinged, both axes will pass through the hinge as shown in Fig. 11'9 (a). The area of the analogous column will be infinite, and is assumed to be concentrated at the hinge.

(3) If the portal frame has both legs hinged at the ends, the centroid of the column section lies midway between the hinges. The area \( A \) and \( I_{yy} \) of the analogous column becomes infinite. The N.A. of the column section will pass through the hinges, as shown in Fig. 11'9 (b).

The fibre stress at any point, for case of Fig. 11'9 (b) is then given by

\[
f = \frac{P}{A} + \frac{M_{yy} \cdot x}{I_{yy}} + \frac{M_{xx} \cdot y}{I_{xx}}
\]

(11'6)

(4) If both the legs are hinged at the base, but the hinges are not at the same level, the \( x \)-\( x \) axis, and hence the N.A., will pass through both the hinges as shown in Fig. 11'9 (c). The area \( A \) and \( I_{xx} \) of the analogous column becomes infinite and hence the stress...
The column analogy method

To find the position of \( x-x \) axis, take moments about \( BC \)

Thus \( \frac{4L}{EI} \cdot y = \left( \frac{L}{EI} \times \frac{L}{2} \right) + \left( \frac{L}{EI} \times \frac{L}{2} \right) = \frac{L^2}{EI} \)

(the contribution of the section \( BC \) being negligible)

\[
\therefore \quad y = \frac{L}{4}
\]

\[ I_{bc} = \frac{1}{3} \cdot \frac{1}{EI} (L)^3 + \frac{1}{3} \cdot \frac{1}{EI} (L)^3 = \frac{2L^3}{3EI} \]

(the contribution of the section \( BC \) being negligible)

\[ I_{xx} = I_{bc} - Ay^2 = \frac{2L^3}{3EI} - \frac{4L}{EI} \left( \frac{L}{4} \right)^3 = \frac{5L^3}{12EI} \]

\[ I_{yy} = \frac{1}{12} \times \frac{1}{EI} (2L)^3 + 2 \times \frac{L}{EI} (L)^3 = \frac{8L^3}{3EI} \]

(the contribution of \( AB \) and \( CD \) about their own axis being negligible).

\[ I_{xx} = 0, \text{ since } y-y \text{ axis is the axis of symmetry.} \]
Hence the fibre stress at any point is given by
\[ f = \frac{P}{A} + \frac{M_{yy} \cdot x}{I_{yy}} + \frac{M_{yy} \cdot y}{I_{xx}} \]
\[ M_{yy} = Py = \frac{2}{3} \frac{wL^3}{EI} \cdot \frac{L}{4} = \frac{wL^4}{6EI} \]
\[ M_{yy} = 0 \]
\[ f = \frac{2}{3} \frac{wL^3}{EI} \cdot \frac{EI}{4L} + \frac{wL^4}{6EI} \cdot \frac{12EI}{5L^2} \cdot \frac{y}{y} \]
\[ = \frac{wL^3}{6} + \frac{2wL}{5} - y \]

At B and C, \( y = \frac{L}{4} \)
\[ f_B = f_C = \frac{wL^3}{6} + \frac{2wL}{20} = \frac{4}{15} wL^3 \]

At A and D, \( y = -\frac{3}{4} L \)
\[ f_A = f_D = \frac{wL^3}{5} - \frac{2wL}{5} \cdot \frac{3}{4} = -\frac{2}{5} wL^3 \]

Since \( M_s \) is zero at each of the points \( A, B, C \) and \( D \), we have
\[ M_A = M_{BA} = f_A = \frac{2}{15} wL^3 \]
\[ M_B = M_{AB} = f_B = \frac{4}{15} wL^3 \]
\[ M_C = M_{BC} = f_C = \frac{4}{15} wL^3 \]
\[ M_D = M_{CD} = f_D = \frac{2}{15} wL^3 \]

The final B.M.D. is shown in Fig. 11.10 (d).

Example 11.6. A portal frame \( ABCD \) is hinged at \( A \) and \( D \), and has rigid joints at \( B \) and \( C \). The frame is loaded as shown in Fig. 11.11 (a). Plot the bending moment diagram for the frame.

Solution

The basic determinate structure is derived by treating \( BC \) hinged at \( B \) and \( C \). The \( M_s \) diagram, also the load diagram, is shown in Fig. 11.11(b). The ordinate of \( M_s \) diagram and the load
\[ M_{xx} = \frac{4 \times 1.5 \times 3}{1.5} = -4 \text{ kN-m} \]

The total load is
\[ M_{xx} = -\int \frac{M_s}{EI} \cdot dx = -\frac{1}{EI} \left[ \frac{1}{2} \times 4.5(-4) \right] = \frac{9}{EI} \]

acting at \( \frac{1}{3} (1.5+4.5) = 2 \text{ m from } A \).

Since ends \( A \) and \( D \) are hinged, the analogous column will have infinite area at the hinges and the remaining area of legs and beam becomes negligible. Hence the \( x-x \) axis will pass through the hinges.

Fig. 11.11.
and the \( y-y \) axis is the axis of symmetry. The centroid \( G \) is mid-way between the hinges, as shown in Fig. 11.11 (c).

Hence \( I_{yy} = \infty \), and the stress is given by equation 11.6
\[ f = \frac{M_{xx}}{I_{yy}} \cdot y \]

\[ M_{xx} = \frac{9}{EI} \times 4.5 = \frac{40.5}{EI} \]

\[ I_{xx} = 2 \left( \frac{1}{3} \times \frac{1}{EI} \times \frac{4.5}{1.5} \right) + \left( \frac{4.5}{EI} \right)^2 \]
\[ = \frac{151.9}{EI} \]
The basic determinate structure is obtained by treating the beam BC hinged at B and C. The \( M_s \) diagram will be a triangle having a maximum ordinate \( y = \frac{10 \times 1 \times 3}{4} = 7.5 \) kN-m under the load.

For \( B \) and \( C \), \( y = 4.5 \) m

\[ f_B = f_C = \frac{4.5 \times 3}{75} = 1.2 \]

Hence \( M_{IB} = M_{IC} = 1.2 \) kN-m

Since \( M_s \) is zero at \( B \) and \( C \), \[ M_B = M_C = 1.2 \] kN-m

The B.M. diagram is shown in Fig. 11.11(d).

**Example 11.7**. A portal frame \( ABCD \) is fixed at \( A \) and \( D \), and has rigid joints at \( B \) and \( D \), and is loaded as shown. Plot the bending moment diagram for the frame.

**Solution.**

![Frame Diagram](image)

To locate the centroid \( G \), or to determine \( \bar{x} \) and \( \bar{y} \), take moment about \( CD \) and \( BC \).

Thus

\[ \bar{x} = \frac{4 \times 2}{8} = 1.5 \text{ m} \]

\[ \bar{y} = \frac{4 \times 2}{8} = 1.5 \text{ m} \]

\[ M_{XX} = P_y = \frac{7.5 \times 1.5}{6} = 11.25 \text{ kN-m} \]

\[ e = 2.5 - \frac{5}{3} = 2.5 \]

\[ M_{YY} = \frac{7.5}{6} \left( -\frac{2.5}{3} \right) = -6.25 \text{ kN-m} \]

(the minus sign with \( e \) being used because the eccentricity is to the left of \( y-y \) axis).
THE COLUMN ANALOGY METHOD

\[
M_{xy} \cdot I_{xx} - M_{yy} \cdot I_{xx} = \frac{11.25}{EI} \left( \frac{74}{3EI} \right) \left( \frac{22.5}{2EI} \right)
\]

\[
= \frac{277.5 - 12.5}{345 - 4} = 0.0777
\]

Substituting these values in equation 11.5, we get

\[
f = 0.938 - 0.19x + 0.777y
\]

(1) For A, \( x = -2.5 \text{ m}; \ y = -2.5 \text{ m} \)

\[
M_y = M_{1A} = 0.938 - (0.19 \times 2.5) = -0.53 \text{ kN-m}
\]

(2) For B, \( x = -2.5 \text{ m}; \ y = -1.5 \text{ m} \)

\[
M_y = M_{1B} = 0.938 - (0.19 \times 2.5) = -2.58 \text{ kN-m}
\]

(3) For C, \( x = +1.5 \text{ m}; \ y = +1.5 \text{ m} \)

\[
M_y = M_{1C} = 0.938 - (0.19 \times 1.5) = +1.82 \text{ kN-m}
\]

(4) For D, \( x = +1.5 \text{ m}; \ y = +2.5 \text{ m} \)

\[
M_y = M_{1D} = 0.938 - (0.19 \times 1.5) = +1.29 \text{ kN-m}
\]

Since \( Ms \) is zero at points A, B, C and D, we have

\[
M_A = M_{1A} = -0.53 \text{ kN-m}
\]

\[
M_B = M_{1B} = -2.58 \text{ kN-m}
\]

\[
M_C = M_{1C} = +1.82 \text{ kN-m}
\]

\[
M_D = M_{1D} = +1.29 \text{ kN-m}
\]

The B.M.D. is drawn in Fig. 11.12 (d).

Example 11.8. A portal frame ABCD has end A fixed and end D hinged, with rigid joints at B and C. Plot the bending moment diagram if the frame is loaded as shown in Fig. 11.13(a).

Solution

The basic determinate structure is derived by making the beam BC simply supported. The \( Ms/El \) diagram and hence the load diagram is shown in Fig. 11.13 (b). The ordinate of \( Ms \) under the load \( 6 \times 1.5 \times 3 \) \( = -6 \text{ kN-m} \)

The total load \( P = \int_{-1/2}^{1/2} \frac{Ms}{EI} \text{ dx } = -6 \text{ kN-m} \)

Acting at \( \frac{1}{3} (1.5 + 4.5) = 2 \text{ m from B} \).

Since the leg CD is hinged at D, the analogous column has infinite area concentrated at D, and the area of the remaining members is negligible. Hence both the centroidal axes will pass through
the hinge D as shown in Fig. 11'13 (c) which illustrates the analogous.

![Diagram](image)

**Fig. 11'13.**

Column fully dimensioned. P is the point of application of the resultant load on the analogous column.

*Properties of the analogous column:*

\[ A = \infty \]

\[ I_{xx} = \frac{1}{12} \cdot \frac{1}{2EI} (6^2) + \frac{1}{3} \cdot \frac{1}{EI} (3)^3 + \frac{4.5}{1.5EI} (3)^2 = \frac{45}{EI} \]

The contribution of BC about its own centroidal axis (being negligible).

\[ I_{yy} = \frac{6}{2EI} (4.5)^2 + \frac{1}{3} \cdot \frac{1}{1.5EI} \]

\[ I_{yy} = \frac{6}{2EI} (4.5)^2 + \frac{1}{3} \cdot \frac{1}{1.5EI} = \frac{81}{EI} \]

\[ I_{xy} = EIa \bar{y} = \frac{6}{2EI} (-4.5) (0) + \frac{4.5}{1.5EI} (-2.25) (3) + \frac{3}{EI} (0) (1.5) \]

\[ = -\frac{20.25}{EI} \]

**Example 11'9.** A portal frame ABCD has legs hinged at A and D, and has stiff joints at B and C. Draw the B.M. for the loading shown in Fig. 11'14 (a).

**Solution**

The basic determinate structure is derived by considering the beam BC simply supported. The \( M_S \) diagram will be a parabola.
The total load \( P = - \int M_y \, dx = - \frac{1}{EI} \left[ \frac{2}{3} (2L) \left( - \frac{wL^2}{2} \right) \right] \)
\[ = \frac{2wL^3}{3EI} \]

Acting at a distance \( L \) from \( B \) and \( C \).

The analogous column is shown in Fig. ii'14 (b). Since ends \( A \) and \( D \) are hinged, it has infinite area concentrated at \( A \) and \( D \), and the axis \( x-x \), therefore, passes through both the hinges. The axis \( y-y \) is perpendicular to \( x-x \), and passes through a point midway between \( AD \).

Properties of analogous column

\( A = \infty \)

\( I_y = \infty \)

\( I_{xy} = 0 \), since the reference axes are the principal axes of inertia.

The stress at any point is given by equation 11'5, i.e.
\[ f = \frac{M_{xx}}{l_{xx} y} \]
\[ = \frac{2wL^4}{\sqrt{5}E} \times \frac{EI}{6'14L^2} y = \frac{2wL}{6'15\sqrt{5}} y \]

At \( A \) and \( D \), \( y = 0 \)
\[ f_a = f_d = 0 \]

At \( B \), \( y = 2L \cos \alpha = \frac{4L}{\sqrt{5}} \)
\[ f_b = \frac{2wL}{6'14\sqrt{5}} \times \frac{4L}{\sqrt{5}} = 0'26 \, wL^3 \]
At C,
\[ y = L \cos \alpha = \frac{2L}{\sqrt{5}} \]
\[ f_c = \frac{2wL}{6\sqrt{3}} \times 2L = 0.13 \text{ wL}^2 \]
Since \( M_s \) is zero at each of the points A, B, C, and D, we have
\[ M_A = M_{1A} = f_A = 0 \]
\[ M_B = M_{1B} = f_B = 0.26 \text{ wL}^2 \]
\[ M_C = M_{1C} = f_C = 0.13 \text{ wL}^2 \]
\[ M_D = M_{1D} = f_D = 0 \]
The B.M.D. is shown in Fig. 11.14 (c).

Example 11.10. A culvert shown in Fig. 11.14 (a) is of constant section throughout and carries a central load of 4 kN on BC. Determine the moments at the corners of the culvert and draw the B.M.D. Assume a uniformly distributed reactive force under the base.

Solution

![Diagram](image)

Fig. 11.15

The basic determinate structure is derived by treating the joints A, B, C, and D, hinged. The \( M_s \) diagram for BC will be a triangle having maximum central ordinate of \( -\frac{4 \times 4}{8} = -3 \) kN-m.

The \( M_s \) diagram for AD will be a parabola having a maximum central ordinate of \( -\frac{4 \times 6}{8} = -3 \) kN-m.

The total load \( P_1 \) on BC = \[-\frac{1}{E} \left( \frac{1}{2} \times 6(-6) \right) \]
\[ = \frac{18}{E} \]

The total load \( P_2 \) on AD = \[-\frac{1}{E} \left( \frac{1}{2} \times 6(-3) \right) \]
\[ = \frac{12}{E} \]

The points of application of \( P_1 \) and \( P_2 \) are shown in Fig. 11.15 (c) along with the position of the centroidal axes of the analogous column.

The resultant load \( P = P_1 + P_2 = \frac{18}{E} + \frac{12}{E} = \frac{30}{E} \)

Acting at \( \left( \frac{12}{E} \times 6 \right) \frac{E}{30} = 2.4 \) m from face BC.

Since \( y-y \) axis is the axis of symmetry, the stress \( f \) is given by

\[ f = \frac{P}{A} + \frac{M_{xx}}{l_{xx}} \]
\[ y = \left( \frac{30}{E} \times \frac{E}{24} \right) + \left( \frac{18}{E} \times \frac{E}{144} \right) y = 1.25 + \frac{y}{8} \]

At A and D, \( y = -3 \)
\[ f_A = f_D = 1.25 - \frac{3}{8} = 0.875 \text{ kN-m} = M_A = M_D \]

At B and D, \( y = +3 \)
\[ f_B = f_C = 1.25 + \frac{3}{8} = 1.625 \text{ kN-m} = M_B = M_D \]

The B.M.D. is shown in Fig. 11.15 (d).

Example 11.11. A portal frame ABCD is fixed at A and hinged at D and carries a horizontal load of 10 kN at B as shown in Fig. 11.16 (a). Compute the moments at A, B, and C.

Solution

The basic determinate structure is derived by removing the end D completely. The \( M_s \) diagram will be a triangle having a maximum ordinate of \( +10 \times 4 = +40 \) kN-m at A as shown in Fig. 11.16 (b).
Fig. 11*16

Resultant load = \[ \frac{M_s}{EI} dx \]
\[ = -\frac{1}{EI} \left\{ \frac{1}{2} \times 4(40) \right\} = -\frac{80}{EI} \]

Acting at \( \frac{4}{3} \) m from \( A \).

Since the frame is hinged at \( D \), the analogous column will have infinite area concentrated at \( D \). Both the axes will pass through \( D \) as shown in Fig. 11*16 (c).

Properties of analogous column,
\( A = \infty \)
\( I_{xx} = \frac{4}{3} \frac{1}{EI} (4)^2 + \frac{1}{EI} (4)^2 + \frac{4}{EI} (4)^2 = \frac{320}{3EI} \)
\( I_{yy} = \frac{4}{EI} (4)^2 + \frac{1}{3} \frac{1}{EI} (4)^2 = \frac{256}{3EI} \)
\( I_{xy} = \frac{4}{EI} (4)^2 (4)^2 + \frac{4}{EI} (-2)(4) = \frac{64}{EI} \)

The final moments are as follows:
\( M_A = M_{1A} + M_{4A} = -21.82 \) kN-m
\( M_B = M_{1B} + M_{4B} = -12.73 \) kN-m
\( M_C = M_{1C} + M_{4C} = +9.09 \) kN-m
\( M_D = M_{1D} + M_{4D} = 0 \)

The final B.M.D. is shown in Fig. 11*16 (d).

PROBLEMS

1. Find the support moments of a built-in beam loaded at third point by two point loads \( W \) each. \( EI \) is constant throughout.

2. A girder of 36 ft. span is fixed horizontally at the end. A downward vertical load of 12 tons acts on the girder at a distance of 12 ft. from the left hand end and an upward vertical force of 8 tons acts at a distance of 18 ft. from the right hand end. Determine the end reactions and fixity couple and draw the bending moment and shearing force diagram for the girder. (U.L.)

3. An encastre beam of span \( L \) carries a load \( wL \) uniformly distributed over the span. The second moment of area of the beam section is not the same throughout; for a length \( L/4 \) at each end the value is \( 2I \) and for the middle length \( L/2 \), it is \( I \).

Determine the bending moment at the end of the beam and sketch the bending moment diagram, showing on it the values at the ends and at midspan. (U.L.)
4. A beam of 20 m span is fixed at both the ends. A couple of 12 kN-m is applied to the beam at a distance 8 m from the left hand support, about a horizontal axis at right angles to the beam. Find the fixing couple at each end and plot the B.M. diagrams.

5. A beam \( AB \) of span 3 m is fixed at both the ends and carries a point load of 10 kN at \( C \), distant 1 m from \( A \). The moment of inertia of the portion \( AC \) of the beam is \( I \) and that of portion \( CB \) is \( J \). Calculate the fixed end moments.

6. A portal frame \( ABCD \) is fixed at \( A \) and \( D \), and has rigid joints at \( B \) and \( C \) and is loaded as shown in Fig. 11-17. Plot the bending moment diagram for the frame.

Fig. 11-17.

7. Analyse the portal frame shown in Fig. 11-18. \( EI \) is constant for the whole frame.

8. Draw the bending moment diagram and the deflected shape of the frame shown in Fig. 11-19. The ends \( A \) and \( D \) are fixed and \( BC \) is loaded with U.D.L. of 10 kN/m.

Fig. 11-18.

Fig. 11-19.

9. A rectangular box frame is 10 ft. wide and 6 ft. deep. The second moments of area of the horizontal members are twice those of the vertical members. The frame carries an inward uniformly distributed load of 20 tons per foot run along the top and bottom horizontal members only. Calculate the bending moments at the corners of the frame. \( (A.M.I.C.E.) \)

10. A portal \( ABCD \) is hinged at \( A \) and \( D \), and has stiff joints at \( B \) and \( C \). Draw the bending moment diagram due to a point load of 10 kN as shown in Fig. 11-20. \( EI=\text{const} \).

11. A portal frame \( ABCD \) has ends \( A \) and \( D \) and carries U.D.L. of 3 kN/m on \( AB \) as shown in Fig. 11-21. Plot the B.M. diagram.

Fig. 11-20.

Fig. 11-21.

12. Fig. 11-22 gives the dimensions of a continuous frame \( ABCD \) in which \( EI \) is constant. The end \( A \) is fixed and the end \( D \) is hinged. Draw the bending moment diagram for this frame, marking on it all important values, when the frame is subjected to a horizontal load of 6 kN applied at \( B \).

Answers

1. \( \frac{2}{3} WL \).

2. \( R_t=4.89 \); \( R_s=0.89 \).

3. \( M_t=4.89 \); \( M_s=0.89 \).

4. \( M (\text{Left})=1.44 \); \( M (\text{Right})=3.84 \) kN-m both in the same directions as the external moment; \( R=0.864 \) kN.

5. \( M_A=5.60 \) kN-m; \( M_B=1.76 \) kN-m.

6. \( M_A=0.0137 \) WL; \( M_B=0.054 \) WL.

7. \( M_A=0.0461 \) WL; \( M_B=0.0325 \) WL.

8. \( M_A=0.028 \) kN-m; \( M_B=2.31 \) kN-m.

9. \( M_A=0.04 \) kN-m; \( M_B=0.07 \) kN-m.

10. \( M_A=+3.55 \) kN-m; \( M_B=+0.56 \) kN-m.

11. \( M_A=1.97 \) kN; \( M_B=+0.56 \) kN-m.

12. \( M_A=+3.00 \) kN-m; \( M_B=+1.98 \) kN-m.

13. \( M_A=+0.56 \) kN-m; \( M_B=+0.81 \) kN-m.

14. \( M_A=+0.56 \) kN-m; \( M_B=+0.81 \) kN-m.

15. \( M_A=+3.50 \) kN-m; \( M_B=+0.81 \) kN-m.

16. \( M_A=+3.00 \) kN-m; \( M_B=+3.00 \) kN-m.
12

Method of Strain Energy

12.1. GENERAL PRINCIPLES

When external force (i.e. axial load or moment) acts on an elastic body, it deforms. If the elastic limit is not exceeded, the work done in straining the material is stored in it in the form of resilience of internal strain energy. By equating the external work done by applied loads as they deform the elastic body to the internal strain energy stored in the body, we obtain a method of determining deflections that is based on the principle of conservation of energy. The energy principles presented in this chapter have the broad scope of their application to the analysis of redundant systems also.

In pin jointed structures, where the members are in tension or compression, the energy stored depends on direct forces only. However, in beams and frames having rigid joints, shear stress and bending stress may also occur at any section, and the total strain energy stored depends on the magnitudes of direct force, shear and moment. While analysing statically indeterminate structures, the work done by direct and shear forces is neglected since it is very small in comparison to that done by bending. Before discussing the various strain energy theorems, let us first derive standard expressions for strain energy stored in linear elastic systems under various loadings.

12.2. STRAIN ENERGY IN LINEAR ELASTIC SYSTEMS

(i) Axial Loading

Let us consider a straight bar of length \( L \), having uniform cross-sectional area \( A \). If an axial load \( P \) is applied gradually, and if the bar undergoes a deformation \( \Delta \), the work done, stored as strain energy \( U \) in the body, will be equal to average force \( \frac{1}{2} P \) multiplied by the deformation \( \Delta \).

Thus \[ U = \frac{1}{2} P \cdot \Delta \]

If, however, the bar has variable area of cross section, consider a small section of length \( dx \) and area of cross-section \( A(x) \). The strain energy \( dU \) stored in this small element of length \( dx \) will be, from Eq. 12.1,

\[ dU = \frac{P}{2A(x)E} \.

The total strain energy \( U \) can be obtained by integrating the above expression over the length of the bar.

Thus \[ U = \int_0^L \frac{P}{2A(x)E} \, dx \] (12.2)

(ii) Flexural Loading (moment or Couple):

Let us now consider a member of length \( L \) subjected to uniform bending moment \( M \). Consider an element of length \( dx \), and let \( d \) be the change in the slope of the element due to applied moment \( M \). If \( M \) is applied gradually, the strain energy stored in the small element will be,

\[ dU = \frac{1}{2} M \cdot d \]

But \[ \frac{d}{dx} \left( \frac{d}{dx} \right) = \frac{d^2 y}{dx^2} = \frac{M}{EI} \]

or \[ d = \frac{M}{EI} \cdot dx \]

Hence \[ dU = \frac{1}{2} M \cdot \frac{M}{2EI} \cdot dx = \frac{M^2}{2EI} dx \]

Integrating this over the entire length, we get the total strain energy stored in the member. Thus,

\[ U = \int_0^L \frac{M^2 dx}{2EI} \] (12.3)

12.3. CASTIGLIANO'S FIRST THEOREM

The concept of elastic strain energy can be very useful in the study of deflections of various points of structure under load. Instead of directly equating the external work to the internal strain energy, considerable simplification is obtained by Castigliano's first theorem which states that the deflection caused by any external force is equal to the partial derivative of the strain energy with respect to that force.
A generalised statement of the theorem is as follows:

"If there is any elastic system in equilibrium under the action of a set of forces \( W_1, W_2, W_3, \ldots, W_n \) and corresponding displacements \( \delta_1, \delta_2, \delta_3, \ldots, \delta_n \), and a set of moments \( M_1, M_2, M_3, \ldots, M_n \), and corresponding rotations \( \phi_1, \phi_2, \phi_3, \ldots, \phi_n \), then the partial derivative of the total strain energy \( U \) with respect to any one of the forces or moments taken individually would yield its corresponding displacement in its direction of action."

Expressed mathematically,

\[
\frac{\partial U}{\partial W_1} = \delta_1 \quad (12.4)
\]

and

\[
\frac{\partial U}{\partial M_1} = \phi_1 \quad (12.5)
\]

Proof:

Consider an elastic body (Fig. 12.1) subjected to loads \( W_1, W_2, W_3, \ldots \) etc. each applied independently. Let the body be supported at \( A, B \) etc. The reactions \( R_A, R_B \) etc. do not do work while the body deforms because the hinge reaction is fixed and cannot move (and therefore the work done is zero) and the roller reaction is perpendicular to the displacements of the roller. Assume that the material follows Hooke’s law, the displacements of the points of loading will be linear functions of the load and the principle of superposition will hold.

Let \( \delta_1, \delta_2, \delta_3, \ldots \) etc. be the deflections of points 1, 2, 3, etc. in the direction of the loads at these points. The total strain energy \( U \) is then given by

\[
U = \frac{1}{2}(W_1 \delta_1 + W_2 \delta_2 + W_3 \delta_3 + \ldots) \quad (1)
\]

Let the load \( W_1 \) be increased by an amount \( dW_1 \), after the loads have been applied. Due to this, there will be small change in the deformation of the body, and the strain energy will be increased slightly by an amount \( dU \). Expressing this small increase as the rate of change of \( U \) with respect to \( W_1 \) times \( dW_1 \), the new strain energy will be

\[
U + \frac{\partial U}{\partial W_1} \cdot dW_1
\]

On the assumption that principle of superposition applies, the final strain energy does not depend upon the order in which the forces are applied. Hence assuming that \( dW_1 \) is acting on the body, prior to the application of \( W_1, W_2, W_3 \) etc. the deflections will be infinitesimally small and the corresponding strain energy of the second order can be neglected. Now when \( W_1, W_2, W_3 \ldots \) etc. are applied (with \( dW_1 \) still acting initially), the points 1, 2, 3 etc. will move through \( \delta_1, \delta_2, \delta_3 \) etc. in the direction of these forces and the strain energy \( U \) will be as given by (1) above. However, in doing so, the small load \( dW_1 \), which is acting prior to the application of \( W_1 \), rides through a distance \( \delta_1 \), and produces the external work increment \( dU = dW_1 \cdot \delta_1 \). Hence the new strain energy, when the loads are applied in this order, is

\[
U = dW_1 \cdot \delta_1 \quad (3)
\]

Since the final strain energy does not depend upon the order in which the forces are applied, we get, by equating (2) and (3)

\[
U + dW_1 \cdot \delta_1 = U + \frac{\partial U}{\partial W_1} \cdot dW_1
\]

or

\[
\delta_1 = \frac{\partial U}{\partial W_1}
\]

which proves the proposition.

Similarly, it can be proved that \( \phi_1 = \frac{\partial U}{\partial M_1} \).

### 12.4 Deflection of Beams etc. by Castigliano’s First Theorem

In this chapter, we shall compute the deflections of beams and other members connected by rigid joints. The case of joint deflection of an articulated structure has been dealt with separately in chapter 13.

Castigliano’s first theorem (Eq. 12.4) can be used for computing the deflection of beams and frames with rigid joints.
If a member carries an axial force, the energy stored is given by

\[ U = \int_0^L \frac{P^2 dx}{2AE} \] (from Eq. 12.2).

In the above expression, \( P \) is the axial force in the member, and \( M \) is the function of external load \( W_1, W_2 \) etc. If it is required to compute the deflection \( \delta_1 \) in the direction of \( W_1 \), we have, from Castigliano's first theorem,

\[ \delta_1 = \frac{\partial U}{\partial W_1} = \int_0^L \frac{P}{AE} \cdot \frac{\partial P}{\partial W_1} \cdot dx \] (12.6)

In the above expression, \( \frac{\partial P}{\partial W_1} \) is best evaluated by differentiating inside the integral sign before integrating. This is permissible because \( W_1 \) is not a function of \( x \).

If, however, the strain energy is due to bending, and not due to axial load,

\[ U = \int_0^L \frac{M^2 dx}{2EI} \] (from equation 12.3)

(where \( M \) is a function of the load \( W_1 \))

and

\[ \delta_1 = \frac{\partial U}{\partial W_1} = \int_0^L M \left( \frac{\partial M}{\partial W_1} \right) \frac{dx}{EI} \] (12.7)

In the above expression also, \( \frac{\partial M}{\partial W_1} \) is evaluated by differentiating inside the integral sign before integrating.

If no load is acting at a point where the deflection is desired, fictitious load \( W \) is applied at the point, in the direction the deflection is required. Then, after differentiating but before integrating, the fictitious load is set to zero. The method is sometimes known as the fictitious load method.

If, however, the rotation \( \phi_1 \) is required in the direction of \( M_1 \) equation 10.7 is modified as follows:

\[ \phi_1 = \frac{\partial U}{\partial M_1} = \int_0^L M \left( \frac{\partial M}{\partial M_1} \right) \frac{dx}{EI} \] (12.8)

where \( M \) is a function of \( M_1 \).

The procedure will now be illustrated with the help of few worked examples.

Example 12.1. Calculate the central deflection, and the slope at ends of a simply supported beam carrying a U.D.L. \( w \) per unit length over the whole span.

**Solution**

(a) Central deflection

Since no point load is acting at the centre where the deflection is required, apply a fictitious load \( W \) there, as shown in Fig. 12.2(a).

The reactions at \( A \) and \( B \) will \( \left( \frac{wL}{2} + \frac{W}{2} \right) \) to each.

Then

\[ \delta_c = \frac{\partial U}{\partial W} = \frac{1}{EI} \int_0^L M_1 \frac{\partial M_1}{\partial W} \cdot dx \] (1)

where \( M_1 \) is the bending moment at any section distant \( x \) from \( A \).

\[ M_1 = - \left( \frac{wL}{2} + \frac{W}{2} \right) x + \frac{wx^2}{2} \]

\[ \frac{\partial M_1}{\partial W} = - \frac{x}{2} \]

Substituting in (1), we get

\[ \delta_c = \frac{2}{EI} \int_0^{L/2} \left( \frac{wL}{2} x - \frac{wx^2}{2} \right) \frac{x}{2} dx \]

Putting \( W = 0 \),

\[ \delta_c = \frac{2}{EI} \int_0^{L/2} \left( \frac{wL}{2} x - \frac{wx^2}{2} \right) \frac{x}{2} dx \]

\[ = \frac{2}{EI} \left[ \frac{wLx^2}{12} - \frac{wx^4}{48} \right] \left[ \frac{L^2}{12} \right] \]

\[ = \frac{5}{384} \frac{wL^4}{EI} \]

(b) Slope at ends

To obtain the slope at the end \( A \), say, apply a fictitious moment \( M \) as shown in Fig. 12.2(b). The reactions at \( A \) and \( B \) will be respectively \( \left( \frac{wL}{2} - \frac{M}{L} \right) \) and \( \left( \frac{wL}{2} + \frac{M}{L} \right) \).
Measuring $x$ from $B$, we have, Eq. 12:8,

$$
\phi_A = \frac{\partial U}{\partial M} = \frac{1}{EI} \int_0^L M_x \cdot \frac{\partial M_x}{\partial M} \cdot dx
$$

(2)

where $M_x$ is the moment at a point distant $x$ from the origin (i.e., $B$) and is a function of $M$.

$$
M_x = -\left( \frac{wL}{2} + \frac{M}{L} \right) x + \frac{wx^2}{2}
$$

$$
\therefore \frac{dM_x}{dM} = -\frac{x}{L}
$$

Substituting the values in (2), we get

$$
\phi_A = \frac{1}{EI} \left[ \left( \frac{wL}{2} + \frac{M}{L} \right) x - \frac{wx^2}{2} \right] dx
$$

Putting $M=0$

$$
\phi_A = \frac{1}{EI} \left[ \left( \frac{wL}{2} x - \frac{wx^2}{2} \right) \right] dx
$$

$$
= \frac{1}{EI} \left[ \frac{wx^2}{6} - \frac{wx^4}{8L} \right]_0^L = +\frac{wL^3}{24EI}
$$

The plus sign signifies that the rotation is in the direction of the moment $M$, i.e., clockwise.

Example 12:2. Using Castigliano’s first theorem, determine the deflection and rotation of the overhanging end $A$ of the beam loaded as shown in Fig. 12:3(a).

Solution

![Fig. 12:3](image)

(a) Rotation of $A$

For the loading of Fig. 12:3(a), the reaction $R_B = \frac{M}{L}$ and $R_C = \frac{M}{L}$.

For the portion $AB$, $x=0$ at $A$ to $x=L/3$ at $B$,

$$
M_x = -M - \frac{wx}{3}
$$

$$
\frac{\partial M_x}{\partial W} = -x
$$

For the portion $CB$, $x=0$ at $C$ to $x=L$ at $B$,

$$
M_x = -\left( M + \frac{1}{3}WL \right) \frac{1}{L} x
$$

$$
\frac{\partial M_x}{\partial W} = -\frac{x}{3}
$$

(b) Deflection of $A$

To find the deflection at $P$, apply a fictitious load $W$ at $A$, in upward direction as shown in Fig. 12:2(b).

The reaction $R_B = \left( M + \frac{4}{3}WL \right) \frac{1}{L}$.

The reaction $R_C = \left( M + \frac{1}{3}WL \right) \frac{1}{L}$.

Then

$$
\delta_A = \frac{\partial U}{\partial W} = \frac{1}{EI} \left[ \int_0^{L/3} M_x \frac{\partial M_x}{\partial W} \cdot dx + \int_{L/3}^L M_x \frac{\partial M_x}{\partial W} \cdot dx \right]
$$

For the portion $AB$, $x=0$ at $A$ to $x=L/3$ at $B$,

$$
M_x = -M - \frac{wx}{3}
$$

$$
\frac{\partial M_x}{\partial W} = -x
$$

For the portion $CB$, $x=0$ at $C$ to $x=L$ at $B$,

$$
M_x = -\left( M + \frac{1}{3}WL \right) \frac{1}{L} x
$$

$$
\frac{\partial M_x}{\partial W} = -\frac{x}{3}
$$
Substituting the values, we get
\[ \delta_a = \frac{1}{EI} \int_0^{L/3} \left( M + \frac{1}{3} Wx \right) x \, dx + \frac{1}{EI} \int_0^L \left( M + \frac{1}{3} WL \right) \frac{x}{L} \cdot \frac{x}{3} \, dx \]

Putting \( W = 0 \),
\[ \delta_a = \frac{1}{EI} \int_0^{L/3} Mx \, dx + \frac{1}{EI} \int_0^L \frac{Mx^2}{3L} \, dx \]
\[ = \frac{M}{EI} \left[ \frac{x^2}{2} \right]_0^{L/3} + \frac{M}{3EI} \left[ \frac{x^3}{3} \right]_0^L \]
\[ = \frac{ML^3}{18EI} + \frac{ML^3}{9EI} = \frac{ML^3}{6EI} \]

**Example 12.3.** A freely supported beam of span \( L \) carries a central load \( W \). The sectional area of the beam is so designed that the moment of inertia of the section increases uniformly from \( I \) at ends to \( 1.5I \) at the middle. Calculate the central deflection.

**Solution**

![Diagram of beam](image)

The deflection is given by
\[ \delta_c = \frac{\partial U}{\partial W} = \frac{1}{E} \int Mx \cdot \frac{\partial Mx}{\partial W} \, dx \]

In the above integral, \( Ix = I + \frac{L}{2} \cdot \frac{2}{L} \cdot x = I \left( 1 + \frac{x}{L} \right) \) and is a function of \( x \).

\( Mx = -\frac{W}{2} x \), from \( x = 0 \) to \( x = L/2 \)

\[ \frac{\partial Mx}{\partial W} = -\frac{x}{2} \]

Substituting the value, we get
\[ \delta_c = \frac{2}{E} \int_0^{L/2} \frac{1}{I \left( 1 + \frac{x}{L} \right)^2} W \cdot \frac{x}{2} \, dx \]
\[ = \frac{W}{2EI} \int_0^{L/2} \frac{x}{\left( 1 + \frac{x}{L} \right)} \, dx \]

Substituting \( x + L = t \) in the above integral, and simplifying, we get
\[ \delta_c = \frac{W}{2EI} \int_0^{3L/2} \left( -2L + \frac{L^2}{t} \right) dt \]
\[ = \frac{W}{2EI} \left[ \frac{t^2}{2} - 2Lt^2 + \frac{L^3}{3} \log t \right]_L^{3L/2} \]
\[ = 0.015 \frac{W^2L^5}{EI} \]

**Example 12.4.** A beam of uniform section and of length \( 2L \) is freely supported by rigid supports at its ends and by an elastic prop at its centre. If the prop deflects by an amount \( \lambda \) times the load it carries and if the beam carries a total distributed load of \( W \), show that the load carried by the prop is \( \frac{5W}{8 \left( 1 + \frac{6EI}{L^3} \right)} \) (U.L.).

**Solution**

![Diagram of beam](image)

Let the prop reaction be \( P \)
Load on beam = \( W \)

\( \cdot \cdot \cdot \) U.D.L. = \( \frac{W}{2L} \) per unit length

Reaction at ends = \( R = \frac{1}{2} \left( W - P \right) \)

The deflection at the prop is given by
\[ \delta_c = \frac{\partial U}{\partial P} = \frac{1}{EI} \int Mx \frac{\partial Mx}{\partial P} \, dx \]

Since the prop deflects by an amount \( \lambda \) times the load it carries, we have \( \delta_c = -P \cdot \lambda \).

(Minus sign has been used since deflection is in a direction opposite to the line of action of \( P \)).

Hence
\[ -P \lambda = \frac{1}{EI} \int Mx \frac{\partial Mx}{\partial P} \, dx \]
or
\[ -P = \frac{1}{\lambda} \cdot \frac{1}{EI} \int Mx \frac{\partial Mx}{\partial P} \, dx \] (1)
For any section distant \( x \) from \( A \),

\[ M_x = -\frac{1}{2} (W-P)x + \frac{W x^2}{2L} \]

\[ \frac{\partial M_x}{\partial P} = + \frac{x}{2} \]

Substituting the value in (1), we get

or

\[ P = \frac{2}{\lambdaEI} \left[ \frac{L}{0} \left\{ -\frac{1}{2} (W-P)x + \frac{W x^2}{2L} \right\} \frac{x}{2} \right] \]

or

\[ P = \frac{\lambda EI}{2} \left[ \frac{W x}{12} + \frac{PL}{12} - \frac{WL^3}{32} \right] \]

or

\[ P \left( \frac{6EI}{L^2} + 1 \right) = \frac{5W}{8} \]

From which

\[ P = \frac{5W}{8} \left( 1 + \frac{6EI}{L^2} \right) \]

**Example 12.5.** A vertical load \( W \) is applied to the rigid cantilever frame shown in Fig. 12.6. Assuming \( EI \) to be constant throughout the frame, determine the horizontal and vertical displacements of the point \( C \). Neglect axial deformations.

**Solution**

**Vertical deflection of \( C \)**

The vertical deflection of \( C \) is given by

\[ \delta_{CV} = \frac{\partial U}{\partial W} = \frac{1}{EI} \left[ M_x \frac{\partial M_x}{\partial W} \right] dx \]

For \( BC \), measuring \( x \) from \( C \),

\[ M_x = + Wx \]

\[ \frac{\partial M_x}{\partial W} = + x \]

For \( BA \), measuring \( x \) from \( B \)

\[ M_x = + \frac{WL}{2} \quad \text{(constant)} \]

\[ \frac{\partial M_x}{\partial W} = + \frac{L}{2} \]

Substituting the values in (1), we get

\[ \delta_{CV} = \frac{1}{EI} \left[ \int_0^{L/2} Wx dx + \int_0^L \frac{WL}{2} \frac{L}{2} dx \right] \]

\[ \delta_{CV} = \frac{1}{EI} \left[ \left\{ \frac{W x^2}{2} \right\}_0^L + \left\{ \frac{WL^3}{24} \right\}_0^L \right] \]

\[ \delta_{CV} = \frac{1}{EI} \left( \frac{WL^3}{24} + \frac{L^3}{4} \right) \quad \text{(Answer)} \]

**(b) Horizontal deflection of \( C \)**

To compute the horizontal deflection, apply fictitious horizontal load \( P \) at \( C \), as shown in Fig. 12.6. Then

\[ \delta_{CH} = \frac{\partial U}{\partial P} \]

For \( BC \), measuring \( x \) from \( C \),

\[ \frac{\partial M_x}{\partial P} = 0 \]

For \( BA \), measuring \( x \) from \( B \)

\[ M_x = + \frac{WL}{2} + Px \]

\[ \frac{\partial M_x}{\partial P} = + x \]

Substituting the values in (2), we get

\[ \delta_{CH} = \frac{1}{EI} \left[ \int_0^{L/2} Wx(0) dx + \int_0^L \left( \frac{WL}{2} + P \cdot x \right) dx \right] \]

\[ \delta_{CH} = \frac{1}{EI} \left( \frac{W L^3}{4} + \frac{P x^3}{3} \right) \]

Applying the limits and putting \( P = 0 \), we get

\[ \delta_{CH} = \frac{1}{EI} \left( \frac{W L^3}{4} \right) \quad \text{(Answer)} \]

**Example 12.6.** Obtain an expression for the vertical displacement of point \( A \) in the bent cantilever shown in Fig. 12.7(a).
Solution

The vertical displacement of $A$ is given by

$$
\delta_A = \frac{U}{E} \left[ M_x \int_0^x \frac{\partial M_x}{\partial W} \, dx \right]
$$

where the integration is carried over the whole frame. A bending moment will be designated positive if it produces convexity to the inside of the frame.

The B.M.D. for the whole frame is shown in Fig. 12.7(b).

(i) For $AB$:
Width $= \text{unity}

I = \frac{1}{12} \times 1 \times r^3 = \frac{r^3}{12}$

$M_x = W_x$ (measuring $x$ from $A$)

$\frac{\partial M_x}{\partial W} = +x$

Limits of $x$ from 0 at $A$ to $a$ at $B$.

(ii) For $BC$:
Width $= \text{unity}

I = \frac{1}{12} \times 1 \times (2a)^3 = \frac{2a^3}{3}$

$M_x = Wa$

$\frac{\partial M_x}{\partial W} = +a$

Limits of $x$ from 0 at $B$ to $2a$ at $C$.

(iii) For $CD$:
Width $= \text{unity}

I = \frac{1}{12} \times 1 \times (a)^3 = \frac{a^3}{12}$

$M_x = W_x (a-x)$, $x$ being measured from $C$.

(Evidently, $M_x = 0$ when $x = a$ where the line of action of $W$ cuts the member $CD$).

$\frac{\partial M_x}{\partial W} = a-x$

Limits of $x$ from 0 at $5a/2$ at $D$

Substituting the values in (1), we get

$$
\delta_A = \frac{1}{E} \left[ \left[ \int_0^a W_x(x)^{1/2} \, dx + \int_0^{2a} Wa(a) \frac{3}{2} x^2 \, dx + \frac{5a}{2} W(a-x)(a-x) \frac{12}{r^3} \, dx \right] \right]
$$

$$
= \frac{1}{E} \left[ \left[ 4Wx^3 \right]_0^a + \left( \frac{3}{2} Wa^2 x \right)_0^{2a} + 12W \left( a^2 x + \frac{x^3}{3} - \frac{2a x^2}{2} \right)_0^{5a/2} \right]
$$

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$$
\delta_A = \frac{1}{E} \left[ 4Wx^3 + 3Wx^3 + 12W \left( \frac{5}{2} a^2 + \frac{125}{24} a^2 - \frac{25}{4} a^2 \right) \right]
$$

$$
= \frac{24Wa^3}{E} \left( \text{Answer} \right)
$$

Example 12.7. A steel tube having outside and inside diameters of 10 cm and 6 cm respectively is bent into the form of a quadrant of 2 m radius. One end is rigidly attached to a horizontal base plate to which a tangent to that end is perpendicular, and the free end supports a load of 1000 N. Determine the vertical and horizontal deflections of the free end under this load. The dimensions of the cross-section may be considered as small relative to the radius of curvature.

$$
E = 2 \times 10^5 \text{ N/mm}^2
$$

Solution.

(a) Vertical deflection of end $A$

$$
\delta_A = \frac{ \partial U }{ E } \left[ \int M \frac{\partial M}{\partial W} \, ds \right]
$$

where the integration is carried over the whole frame along the curved surface. Hence $ds$ has been used in place of $dx$.

Let us consider a small element of curved length $ds$ subtending $d\theta$ at the centre, and being at an angle $\theta$ from the line $OA$, $O$ being the centre of the quadrant of the circle.

$$
M = WR \sin \theta
$$

$$
\frac{\partial M}{\partial W} = R \sin \theta
$$

$$
ds = R d\theta
$$

Limits of $\theta$ from 0 at $A$ to $\pi/2$ at $B$.

Substituting in (1), we get

$$
\delta_A = \frac{1}{E} \left[ \int_0^{\pi/2} W \sin \theta \, R \sin \theta \, d\theta \right]
$$

$$
= \frac{WR^2}{E} \left[ \int_0^{\pi/2} \sin^2 \theta \, d\theta \right]
$$

$$
= \frac{WR^2}{E} \cdot \frac{\pi}{4}
$$

Now

$$
I = \frac{\pi}{64} (10^4 - 6^4) = 427 \text{ cm}^4 = 427 \times 10^4 \text{ mm}^4
$$

$$
E = 2 \times 10^5 \times 427 \times 10^4 = 854 \times 10^9 \text{ N-mm}^2
$$

$$
R = 2m = 2000 \text{ mm}
$$
**Example 12.8.** A steel bar bent to the shape shown in Fig. 12.9 is fixed at A and carries a vertical load W at C. Calculate the vertical deflection of C. EI is constant throughout.

**Solution**

For any radius vector subtending an angle \( \theta \) with OC, we have (see example 12.7):

\[
M = WR \sin \theta
\]

\[
\frac{\partial M}{\partial W} = R \sin \theta
\]

\[
ds = R \sin \theta \, d\theta
\]

\[
\int_B^C M \frac{\partial M}{\partial W} \, ds = \int_0^{\pi/2} WR \sin \theta \, \frac{\partial M}{\partial W} \, d\theta
\]

\[
= \frac{\pi}{4} WR^3
\]

For BA,

Measuring \( x \) from B,

\[
M = WR
\]

\[
\frac{\partial M}{\partial W} = R
\]

\[
\int_B^A M \frac{\partial M}{\partial W} \, dx = \int_0^L WR \, dx = WR^2 L
\]

Substituting the values in (1), we get

\[
\delta_c = \frac{1}{EI} \left[ \int_0^{\pi/4} \left( \frac{\pi}{4} R^3 + WR^2 \right) \, d\theta \right]
\]

\[
= \frac{WR^2}{4EI} \left( \pi R + 4L \right)
\]

Answer.

**Example 12.9.** A circular arch rib of constant flexural rigidity is encastre at A as shown in Fig. 12.10. The end B is tied horizontally with a force \( H \) such that it can only move vertically when a load \( W \) is hung at B. Find the ratio \( H/W \).**

**Solution**

The horizontal deflection of B is given by

\[
\delta_{BH} = \int_B^A M \frac{\partial M}{\partial H} \, ds = 0
\]

As per condition of the problem, \( \delta_{BH} = 0 \)

\[
\int_B^A M \frac{\partial M}{\partial H} \, ds = 0 \quad (1)
\]

Consider an element \( ds \), the angular distance of its radius vector being at \( \theta \) from OA.

\[
M = -H \left( \frac{\sqrt{3}}{2} R - R \sin \theta \right) + W \left( \frac{1}{2} R + R \cos \theta \right)
\]

\[
\frac{\partial M}{\partial W} = -R \left( \frac{\sqrt{3}}{2} - \sin \theta \right) \quad \text{and} \quad ds = Rd\theta
\]

Substituting in (1), we get

\[
\int_0^{2\pi/3} \left[ H R \left( \frac{\sqrt{3}}{2} - \sin \theta \right) - W R \left( \frac{1}{2} + \cos \theta \right) \right] R \left( \frac{\sqrt{3}}{2} - \sin \theta \right) \, d\theta = 0
\]

or

\[
HR^2 \int_0^{2\pi/3} \left( \frac{\sqrt{3}}{2} - \sin \theta \right)^3 \, d\theta - WR^2 \int_0^{2\pi/3} \left( \frac{1}{2} + \cos \theta \right) \left( \frac{\sqrt{3}}{2} - \sin \theta \right) \, d\theta = 0
\]

\[
\frac{H}{W} = \frac{\int_0^{2\pi/3} \left( \frac{1}{2} + \cos \theta \right) \left( \frac{\sqrt{3}}{2} - \sin \theta \right)^3 \, d\theta}{\int_0^{2\pi/3} \left( \frac{\sqrt{3}}{2} - \sin \theta \right)^3 \, d\theta}
\]
The numerator of the right hand side
\[ \frac{1}{2} \left( 2 \frac{3}{4} \sqrt{3} + 2 \sqrt{3} \cos \theta - 2 \sin \theta - 2 \sin 2 \theta \right) d\theta \]
\[ = \frac{3}{4} \left[ 2 \sqrt{3} - \sqrt{3} \left( \frac{1}{2} \right) + \frac{\pi}{3} - \frac{2}{4} \left( - \frac{\sqrt{3}}{2} \right) \right] = 1.28 \]

The denominator of the right hand side
\[ \frac{1}{2} \frac{1}{\sqrt{3} \left( \frac{2}{3} \right)} \sin \theta + \sin 2 \theta \]
\[ = \frac{3}{4} \left( \frac{1}{2} \right) + \frac{\pi}{3} - \frac{2}{4} \left( - \frac{\sqrt{3}}{2} \right) = 1.97 \]

Hence \( \frac{H}{W} = 1.28 = 0.65 \).

12.5. MINIMUM STRAIN ENERGY AND CASTIGLIANO’S SECOND THEOREM

1. Minimum Strain Energy

In the sixth chapter, we have discussed two methods of analysing the statically indeterminate structures: the displacement method (equilibrium method or stiffness coefficient method) and the force method (compatibility method or flexibility coefficient method). For the displacement method of analysis, Castigliano’s first theorem may be used to express the conditions of equilibrium. For the force method of analysis, Castigliano’s second theorem may be used to express the conditions of compatibility. Castigliano stated that, among all the statically possible states of stress in a structure subjected to a variation of stress during which the conditions of equilibrium are maintained, the correct one is that which makes the strain energy of the system a minimum.

Thus, if \( U \) is the strain energy stored in an elastic body, and if \( R_1 \) and \( R_2 \), etc., are the redundant reactions or forces, then if there are no support movements and no change in the temperature, the redundant forces \( R_1, R_2, \) etc., must be such as to make the strain energy a minimum.

Expressed mathematically,
\[ \frac{\partial U}{\partial R_1} = 0 \quad (1) \]
\[ \frac{\partial U}{\partial R_2} = 0 \quad (2) \]

This set of equations is interpreted as follows: of all possible set of values that redundant forces in the system may assume, the correct set of values is that which makes the strain energy a minimum. There will be one equation for each redundant force and a set of equations corresponding to the conditions of compatibility will be obtained. For example, if \( \lambda \) is a small strain or displacement, within the elastic limit, in the direction of the redundant force \( T \), we have
\[ \frac{\partial U}{\partial T} = \lambda \quad (12.9) \]

The self-straining may be caused by the settlement of the support of a redundant reaction by an amount \( \lambda \), or by the initial misfit of a member by an amount \( \lambda \) too short. Redundant support

Actually, equation (12.9) represents a theorem imposing the conditions of compatibility and need not be associated with a minimum of strain energy. In this text, equation (12.9) will be called Castigliano’s theorem, while equation (12.8) will be called Castigliano’s theorem of minimum strain energy which is a particular case of Castigliano’s second theorem when \( \lambda = 0 \) (i.e. when the redundant supports do not yield or when there is no initial lack of fit in the redundant members).

3. Proof

Fig. 12.11.

Consider a redundant frame shown in Fig. 12.11(a), in which \( FC \) is a redundant member of geometrical length \( L \). Let the actual length of the member \( FC \) be \( (L-\lambda) \), \( \lambda \) being the initial lack of fit. The member \( FC \) is shown in Fig. 12.11(b), in which the lack of fit \( \lambda \)
has been own option. At the actual length (L-x) of the member, when it is fitted to the frame, the member will have to be pulled such that F2 and F coincide. In doing so, a tensile force will be induced in the member and this tensile force will pull the joint P towards F. Let F be the final position of the end (and of the join P), such that the end joint moved to F and the member has been extended by an amount F.P by the fixing operation. According to Hooke's Law,

\[ F = T(L - x) \]

where \( T \) is the force (tensile) induced in the member, and \( L \) is the length of the member.

Hence, \( F = \frac{T(L - x)}{AE} \) (approx).

Let the member FC be removed, and consider the tensile force \( T \) applied at the corners F and C as shown in Fig. 12.11(b), so that the basic system is not changed. Now, \( FF \) is the relative deflection of F and C of the beam.

Equating (1) and (2), we get

\[ \partial U \over \partial T = \lambda - \frac{T(L)}{AE} \]

or

\[ \partial U \over \partial T + \frac{T(L)}{AE} = \lambda \]

The strain energy stored in the member FC due to a force \( T \) is

\[ U_{FC} = \frac{1}{2} T \frac{TL}{AE} = \frac{T^2L}{2AE} \]

Substituting the value of \( T \frac{TL}{AE} \) in (3), we get

\[ \partial U \over \partial T = \frac{T(L)}{AE} \]

or

\[ \partial U \over \partial T = \lambda \]

where \( U = U^1 + U_{FC} \).

Equation 12.9 represents Castigliano's theorem. If, however, there is no initial lack of fit, \( \lambda = 0 \), and hence

\[ \partial U \over \partial T = 0 \]

Example 12.10. A continuous beam of two equal spans \( L \) is uniformly loaded over its entire length. Find the magnitude \( R \) of the middle reaction by using the Castigliano's theorem.

Solution

Fig. 12.12.
Let \( R \) be the redundant reaction at \( B \),

\[
\frac{\partial U_{AC}}{\partial R} = \frac{1}{E} \int_A^C M \frac{\partial M}{\partial R} \, dx = 0
\]

The reactions at \( A \) and \( C = \left( wL - \frac{R}{2} \right) \) each.

At any point distant \( x \) from \( A \)

\[
M = -\left( wL - \frac{R}{2} \right) x + \frac{wx^2}{2}
\]

\[
\frac{\partial M}{\partial R} = \frac{x}{2}
\]

Substituting the values in (1), we get

\[
\frac{2}{E} \int_0^L \left\{ -\left( wL - \frac{R}{2} \right) x + \frac{wx^2}{2} \right\} \frac{x}{2} \, dx = 0
\]

or

\[
\left[ -\left( wL - \frac{R}{2} \right) \frac{x^3}{6} + \frac{wx^4}{4} \right]_0^L = 0
\]

or

\[
-\frac{wL^4}{6} + \frac{RL^2}{12} + \frac{wL^4}{16} = 0
\]

From which \( R = \frac{5}{4} wL \).

**Example 12.11.** Two wood beams of identical cross-section are supported at their ends and cross at their midpoints as shown in Fig. 12.13. What interactive force \( R \) will exist between the two beams at \( C \) when a vertical load \( W \) is applied to the upper beam as shown?

**Solution**

![Fig. 12.13](image)

Let us treat the interactive forces \( R \) as a generalised force. The corresponding displacement in the direction of each interactive force \( R \) as shown in Fig. 12.13 (b) is the relative displacement between the midpoints of the two beams. Since both the beams remain in contact, the relative displacement is zero. Hence if \( U \) is the total strain energy stored in both the beams \( AB \) and \( DE \), we have

\[
\frac{\partial U}{\partial R} = 0 = \frac{\partial U_{AB}}{\partial R} + \frac{\partial U_{DE}}{\partial R}
\]

For any simply supported beam of span \( l \) and loaded with a central point load \( P \), we have

\[
M_x = -\frac{P}{2} x \text{, for } x = 0 \text{ to } x = l/2
\]

\[
U = \int_0^l \frac{M^2}{2E} \, dx
\]

\[
= \frac{1}{2} \int_0^{l/2} \frac{P^2}{8EI} \, x^2 \, dx
\]

\[
= \frac{P^4}{4EI} \left[ \frac{x^3}{3} \right]_0^{l/2} = \frac{P^4}{96EI}
\]

Hence, for the beam \( AB \)

Span \( l = 2L \)

The net downward load \( P = (W - R) \)

\[
U_{AB} = \frac{(W - R)PL^3}{96EI}
\]

\[
\frac{\partial U_{AB}}{\partial R} = 8(W - R) \frac{L^3}{96EI}
\]

Similarly, for the span \( DE \)

Span \( l = D \)

The net downward load \( P = R \)

\[
U_{DE} = \frac{RL^3}{96EI}
\]

\[
\frac{\partial U_{DE}}{\partial R} = R \frac{L^3}{96EI}
\]

Substituting the values in (1), we get

\[
\frac{\partial U}{\partial R} = \left( \frac{(W - R)L^3}{6EI} + \frac{RL^3}{48EI} \right) = 0
\]

or

\[
-8(W - R) + R = 0
\]

from which \( R = \frac{8}{9} W \).

**Example 12.12.** A uniform continuous bar \( ABCD \) is built-in at \( A \) and laterally supported at \( B \) as shown in Fig. 12.14 (a). Find the
relative force \( R \) at \( B \) due to the action of a vertical load \( W \) at \( D \) as shown. Neglect the effect of direct compression in the vertical portions of the bar. Joint \( C \) is stiff. Sketch the B.M.D. for the frame.

**Solution**

Let the reactive force at \( B \) be \( R \) as shown in Fig. 12.14(a).

Since the support \( B \) does not yield, we have

\[
\frac{\partial U_{DA}}{\partial R} = \frac{\partial U_{DC}}{\partial R} + \frac{\partial U_{CB}}{\partial R} + \frac{\partial U_{BA}}{\partial R}
\]

\[= 0 \quad (1) \]

\((i)\) For member \( DC \)

\[M_x = wx, \quad x \text{ being measured from } D\]

\[
\frac{\partial M_x}{\partial R} = 0
\]

\[\therefore \frac{\partial U_{DC}}{\partial R} = \frac{1}{EI} \int_0^L M_x \frac{\partial M_x}{\partial R} \, dx = 0 \quad (2)
\]

\((ii)\) For member \( CB \)

Measuring \( x \) from \( B \), towards \( B \),

\[M_x = +wl\]

\[
\frac{\partial M_x}{\partial R} = 0
\]

\[\therefore \frac{\partial U_{CB}}{\partial R} = \frac{1}{EI} \int_0^L M_x \frac{\partial M_x}{\partial R} \, dx = 0 \quad (3)
\]

\[(ii)\) For member \( BA \)

Measuring \( x \) from \( B \) towards \( A \).

\[M_x = +wl\]

\[
\frac{\partial M_x}{\partial R} = 0
\]

\[\therefore \frac{\partial U_{BA}}{\partial R} = \frac{1}{EI} \int_0^L (WL - R_x)(-x) \, dx = \frac{1}{EI} \left[ \frac{Rx^3}{3} - \frac{Wlx^2}{2} \right]_0^L
\]

\[= \frac{1}{EI} \left( \frac{RL^3}{3} - \frac{Wl^2}{2} \right) \quad (4)
\]

Substituting the values in (1), we get

\[
\frac{\partial U_{DA}}{\partial R} = \frac{1}{EI} \left( \frac{RL^3}{3} - \frac{Wl^2}{2} \right) = 0
\]

From which \( R = \frac{3W}{2} \)

The B.M.D. has been shown in Fig. 12.14(b).

**Example 12.13.** A portal frame \( ABCD \) is hinged at \( A \) and \( D \), and has rigid joints \( B \) and \( C \). The frame is loaded as shown in Fig. 12.15. Using the method of minimum strain energy, analyse the frame and plot the B.M. diagram.

**Solution**

The frame is statically indeterminate to first degree.

Let the horizontal reaction at \( A \) be \( H \). Since there is no external horizontal force acting, the horizontal reaction at \( D \) will also be \( H \). Let us treat \( H \) as the redundant reaction. Since the support \( A \) does not move,
Hence \( V_D = 6 - 4 = 2 \) kN.

The work is most conveniently done in the tabular form below:

<table>
<thead>
<tr>
<th>Member</th>
<th>( M )</th>
<th>( \delta M/\delta H )</th>
<th>Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( AB )</td>
<td>(-Hy)</td>
<td>(+y)</td>
<td>0 to 3</td>
</tr>
<tr>
<td>( DC )</td>
<td>(+Hy)</td>
<td>(+y)</td>
<td>0 to 3</td>
</tr>
<tr>
<td>( BE )</td>
<td>(-4x+3H)</td>
<td>(+3)</td>
<td>0 to 1</td>
</tr>
<tr>
<td>( CB )</td>
<td>(-2x+3H)</td>
<td>(+3)</td>
<td>0 to 2</td>
</tr>
</tbody>
</table>

For \( AB \), \( \frac{\partial U_{AB}}{\partial H} = \frac{1}{EI} \int_0^3 (Hy) (y) \, dy - \frac{H}{3EI} (3)^3 = \frac{-9H}{EI} \)

For \( DC \), \( \frac{\partial U_{DC}}{\partial H} = \frac{1}{EI} \int_0^3 (Hy) (y) \, dx = \frac{H}{3EI} (3)^3 = \frac{9H}{EI} \)

For \( BC \), \( \frac{\partial U_{BC}}{\partial H} = \frac{1}{EI} \int_0^1 (-4x+3H)(+3) \, dx 
\quad = \frac{1}{EI} \left[ (-6+9H) - (-12+18H) \right] = \frac{9}{EI} (-2+3H) \)

Substituting the values in (1), we get
\[
\frac{9H}{EI} + \frac{9H}{EI} + \frac{9}{EI} (-2+3H) = 0
\]

From which \( H = 0.4 \) kN

Hence \( M_A = 0 \)
\( M_B = 3H = 0.4 \times 3 = 1.2 \) kN-m
\( M_C = 3H = 1.2 \) kN-m
\( M_D = 0 \).

The B.M.D. is shown in Fig. 12'15(b).

Example 12'14. Using the principle of least work, analyse the portal frame shown in Fig. 12'16(a). Also, plot the B.M. diagram.

**Solution**

The structure is statically indeterminate to first degree. Let us treat the horizontal reaction \( H(\leftarrow) \) at \( A \) as redundant. The horizontal reaction at \( D \) will evidently be \( -(3-H) \leftarrow \).

![Fig. 12'16.](image)

By taking the moments at \( D \), we get
\[
(V_A \times 3) + H(3-2) + (3 \times 1)(2 \times 1')kN = 0.
\]

\( V_A = 3.5 - \frac{H}{3} \)

and hence \( V_D = 6 - V_A = 2.5 + \frac{H}{3} \)

By the theorem of minimum strain energy,
\[
\frac{\partial U}{\partial H} = 0 = \frac{\partial U_{AB}}{\partial H} + \frac{\partial U_{BE}}{\partial H} + \frac{\partial U_{CE}}{\partial H} + \frac{\partial U_{DC}}{\partial H}
\]

(1) For member \( AB \)

Taking \( A \) as the origin,
\[
M = \frac{1}{2} x^2 - H \cdot x
\]

\[
\frac{\partial M}{\partial H} = -x
\]

\[
\frac{\partial U_{AB}}{\partial H} = \frac{1}{EI} \int_0^3 M \frac{\partial M}{\partial H} \, dx = \frac{1}{EI} \int_0^3 \left( \frac{x^2}{2} - Hx \right) (-x) \, dx
\]

\[
= \left[ \frac{1}{EI} \left( \frac{Hx^3}{3} - \frac{x^4}{8} \right) \right]_0^3 = \frac{1}{EI} (9H - 10'12)
\]

(1)
METHOD OF STRAIN ENERGY

or

\[ 30.67 \, H = 41.80 \]

From which \( H = 1.36 \, \text{kN} \).

Hence

\[ V_A = 3.5 - \frac{H}{3} = 3.5 - \frac{1.36}{3} = 3.05 \, \text{kN} \]

\[ V_D = 6 - 3.05 = 2.95 \, \text{kN} \]

\[ M_A = 0 \]

\[ M_B = \frac{1}{2} (1.36 \times 3) = 0.42 \, \text{kN-m} \]

\[ M_C = (3 - H)2 = (3 - 1.36)2 = 3.28 \, \text{kN-m} \]

\[ M_D = 0 \]

The B.M. diagram is shown in Fig. 12.16(b).

**Example 12.15.** A frame ABC is hinged at A and C, and has stiff joints at B, as shown in Fig. 12.17(a). Analyse the frame completely, and draw the B.M. diagram.

**Solution**

The structure is statically indeterminate, to single degree. Horizontal reactions at A and C will be equal, of magnitude \( H \), say. Treating \( H \) as redundant.

\[ M = (3 - H) \times 2 \left( 2.5 + \frac{H}{3} \right) x \]

\[ \frac{dM}{dH} = -2 - \frac{x}{3} \]

\[ \frac{\partial U_{CE}}{\partial H} = \frac{1}{E I} \left[ \int_0^2 M \frac{dM}{dH} dx \right] \]

\[ = \frac{1}{E I} \left[ \int_0^2 \left( 3 - H \right) \times 2 \left( 2.5 + \frac{H}{3} \right) x \right] \left[ -2 - \frac{x}{3} \right] dx \]

\[ = \frac{1}{E I} (10.96H - 15.78) \]

**Example 12.16.** A frame ABC is hinged at A and C, and has stiff joints at B, as shown in Fig. 12.17(b). Analyse the frame completely, and draw the B.M. diagram.

**Solution**

The structure is statically indeterminate, to single degree. Horizontal reactions at A and C will be equal, of magnitude \( H \), say. Treating \( H \) as redundant.
\[
\frac{\partial U}{\partial H} = 0 = \frac{\partial U_{AD}}{\partial H} + \frac{\partial U_{DB}}{\partial H} + \frac{\partial U_{CB}}{\partial H}
\]  

Taking moments about C, \( V_A = 7.5 \); Hence \( V_c = 2.5 \) kN.

It must always be remembered that in the case of curved or inclined members, the integrations must be taken along the members of frame.

If \((x, y)\) are the co-ordinates of any point, and \(s\) is its distance along the member, from the origin, we have

\[ x = y = 0.707 s \text{ (from the geometry of the frame).} \]

(1) For the member \( AD \):

- Length \( AD = 2.83 \) m.
- Taking \( A \) as the origin,
- \( M = -7.5x + Hy \)

Writing \( x \) and \( y \) in terms of \( s \),

\[ M = -5.3s + 0.707 Hs. \]

\[ \frac{\partial M}{\partial H} = 0.707 s \]

\[ \frac{\partial U_{AD}}{\partial H} = \frac{1}{EJ} \left( \frac{2.83}{2} \right) \int (-5.3s - 0.707 Hs)(0.707s) ds \]

\[ = \frac{1}{EI} (3.75H - 28) \]  

(2) For the member \( DB \):

- Taking \( A \) as the origin,
- \( M = -7.5x + Hy + 10(x - 2) \)

\[ = -5.3s + 0.707 Hs + 10(0.707s - 2) \]

\[ = 1.77s + 0.707 Hs - 20 \]

\[ \frac{\partial M}{\partial H} = 0.707 s; \text{ Length } AB = 5.66 \text{ m.} \]

\[ \frac{\partial U_{DB}}{\partial H} = \frac{1}{EJ} \int^{5.66} 2.83 \left( 1.77s + 0.707 Hs - 20 \right)(0.707s) ds \]

\[ = \frac{1}{EI} (26.6H - 103) \]  

(3) For the member \( BC \):

- Length \( BC = 5.66 \) m.
- Taking \( C \) as the origin,
- \( M = -2.5x + Hy \)

\[ = -1.77s + 0.707 Hs \]

\[ \frac{\partial M}{\partial H} = 0.707 s \]

**Example 12.16.** A beam \( AB \) of span 3 m is fixed at both the ends and carries a point load of 9 kN at \( C \) distant 1 m from \( A \). The moment of inertia of the portion \( AC \) of the beam is 2I and that of portion \( CB \) is \( I \). Calculate the fixed end moments and reactions.

**Solution**

This is a problem of second degree indeterminacy. There are four unknowns \( M_A, R_A, M_B \) and \( R_B \). Only two equations of statics are available, i.e., \( \Sigma V = 0 \) and \( \Sigma M = 0 \). Let us choose \( M_A \) and \( R_A \) as redundants, as shown in Fig. 12.18(b).

Since the end \( A \) does not settle, we have

\[ \delta_A = \frac{\partial U_{AB}}{\partial R_A} = 0 = \left( \frac{B}{A} \right) \frac{M_x}{EJ} \frac{\partial M_x}{\partial R_A} dx \]  

(1)

Again, since the end \( A \) does not rotate, we have

\[ \phi_A = \frac{\partial U_{AB}}{\partial M_A} = 0 = \left( \frac{B}{A} \right) \frac{M_x}{EJ} \frac{\partial M_x}{\partial M_A} dx \]  

(2)
Thus, there are two unknowns: \( M_A \) and \( R_A \), and we have two equations from the theorem of minimum strain energy.

(1) For portion \( AC \)

Taking \( A \) as the origin, we have

\[
\begin{align*}
\frac{\partial M_x}{\partial R_A} &= -x \\
\frac{\partial M_x}{\partial M_A} &= +1
\end{align*}
\]

Moment of inertia = 2

Limits of \( x \): 0 to 1 m

Hence

\[
\int_A^C \frac{M_x}{EI} \frac{\partial M_x}{\partial R_A} \, dx = \int_0^1 \left( \frac{M_A - R_A x}{2} \right) \, dx
\]

\[
= \frac{-M_A}{2} + \frac{R_A^3}{3} = \frac{1}{2EI} \left[ R_A - \frac{M_A}{2} \right]
\]

and

\[
\int_A^C \frac{M_x}{EI} \frac{\partial M_x}{\partial M_A} \, dx = \int_0^1 \left( \frac{M_A - R_A x}{2} \right) \, dx
\]

\[
= \frac{1}{2EI} \left[ \frac{M_A (1 - R_A)^2}{2} \right] = \frac{1}{2EI} \left[ \frac{M_A - R_A}{2} \right]
\]

(2) For portion \( CB \)

Taking \( A \) as the origin, we have

\[
\begin{align*}
\frac{\partial M_x}{\partial R_A} &= -x \\
\frac{\partial M_x}{\partial M_A} &= +1
\end{align*}
\]

Moment of inertia = 1

Limits of \( x \): 1 to 3 m

Hence

\[
\int_C^B \frac{M_x}{EI} \frac{\partial M_x}{\partial R_A} \, dx = \int_1^3 \left( \frac{M_A - R_A x + 9(x-1)}{2EI} \right) \, dx
\]

\[
= \frac{1}{EI} \left[ -4M_A + \frac{26}{3} R_A - 42 \right]
\]

and

\[
\int_C^B \frac{M_x}{EI} \frac{\partial M_x}{\partial M_A} \, dx = \int_1^3 \left( \frac{M_A - R_A x + 9(x-1)}{2EI} \right) \, dx
\]

\[
= \frac{1}{EI} \left( 2M_A - 4R_A + 18 \right)
\]

Substituting these values in Eqs. (1) and (3), we get

\[
\frac{\partial U_{AB}}{\partial R_A} = 0 = \frac{1}{2EI} \left[ \frac{R_A}{3} - \frac{M_A}{2} \right] + \frac{1}{EI} \left[ -4M_A + \frac{26}{3} R_A - 42 \right]
\]

or

\[
2.08 R_A - M_A = 9.88 \quad \text{...(3)}
\]

and

\[
\frac{\partial U_{AB}}{\partial M_A} = 0 = \frac{1}{2EI} \left[ M_A - \frac{R_A}{2} \right] + \frac{1}{EI} \left[ 2M_A - 4R_A + 18 \right]
\]

or

\[
M_A - 7.2 R_A = -1.7 \quad \text{...(4)}
\]

Solving (3) and (4), we get

\[
M_A = 4.8 \text{ kN-m (i.e. assumed direction is correct)}
\]

\[
R_A = 7.05 \text{ kN-m (i.e. the assumed direction is correct)}
\]

To find \( M_B \), take moments at \( B \), and apply the condition \( \Sigma M = 0 \) there. Take clockwise moment as positive and anti-clockwise moment as negative. Taking \( M_B \) clockwise, we have

\[
M_B - M_A + R_A (3) - 9 \times 2 = 0
\]

or

\[
M_B = 4.8 + (7.05 \times 3) - 18 = 0
\]

or

\[
M_B = 1.65 \text{ kN-m (i.e. assumed direction is correct)}
\]

To find \( R_B \), apply \( \Sigma V = 0 \) for the whole frame

\[
R_B = 9 - R_A = 9 - 7.05 = 1.95 \text{ kN}.
\]

**Example 12.17.** Using Castigliano's theorem of minimum strain energy, analyse the frame shown in Fig. 12.19 (a). \( EI \) is constant for the whole frame.

**Solution**

Fig. 12.19.

There are five unknowns: \( M, R, \) and \( H \) at \( A, Ho \) and \( Ro \) at \( D \). Let all these reactions act in the directions as shown in Fig. 12.19 (a). Since three equations (i.e. \( \Sigma H = 0, \Sigma V = 0, \Sigma M = 0 \)) are available from statical equilibrium, the frame is statically indeterminate to second degree. Let \( M \) and \( H \) (both at \( A \)) be the redundants.

(i) Since \( \Sigma H = 0 \) is zero for the whole frame:

\[
H_D + H - 10 = 0
\]

or

\[
H_D = 10 - H
\]
(ii) Since \( \Sigma V \) is zero for the whole frame:
\[
R_d = R \text{(numerically)}
\]
where \( R \) is the vertical reaction at \( A \), acting downwards.

(iii) To find the value of \( R \) in terms of \( M \) and \( H \), apply the third equation of statical equilibrium, i.e., \( \Sigma M = 0 \). Taking moments at \( D \), treating clockwise as positive, we get
\[
-M - 4R + 40 = 0.
\]

or
\[
R = 10 - \frac{M}{4} = R_d
\]

Thus the other three reactions (i.e., \( R, R_d \) and \( H_d \)) are known in terms of the redundants \( H \) and \( M \).

From the theorem of minimum strain energy,
\[
\frac{\partial U}{\partial M} = 0 = \int \frac{M_x}{EI} \cdot \frac{\partial M_x}{\partial M} dx
\]
and
\[
\frac{\partial U}{\partial H} = 0 = \int \frac{M_x}{EI} \cdot \frac{\partial M_x}{\partial H} dx
\]

The calculations for various quantities are performed in the tabular form below:

<table>
<thead>
<tr>
<th>Member</th>
<th>( M_x )</th>
<th>( \frac{\partial M_x}{\partial M} )</th>
<th>( \frac{\partial M_x}{\partial H} )</th>
<th>Limits of ( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( AB )</td>
<td>( -M - Hx )</td>
<td>1</td>
<td>(-x)</td>
<td>0 to 4</td>
</tr>
<tr>
<td>( DC )</td>
<td>( +10 - Hx )</td>
<td>0</td>
<td>(-x)</td>
<td>0 to 4</td>
</tr>
<tr>
<td>( CB )</td>
<td>( 10 - \frac{M}{4} )</td>
<td>( x )</td>
<td>(-4)</td>
<td>0 to 4</td>
</tr>
</tbody>
</table>

\[
\frac{\partial U}{\partial M} = 0 = \int_{0}^{4} (M - Hx)(1) dx + \int_{0}^{4} (10 - H)x(0) dx + \int_{0}^{4} \left\{ (10 - H) \left( 10 - \frac{M}{4} \right) x \right\} \left( \frac{1}{4} \right) dx
\]

which gives \( M - 3H + 5 = 0 \)

Similarly,
\[
\frac{\partial U}{\partial H} = 0 = \int_{0}^{4} (M - Hx)(-x) dx + \int_{0}^{4} (10 - H)x(-x) dx + \int_{0}^{4} \left\{ (10 - H) \left( 10 - \frac{M}{4} \right) x \right\} (-4) dx
\]

which gives \( -M + \frac{20}{3} H + \frac{100}{3} = 0 \)

Solving (I) and (II), we get
\[
H = +7.73 \text{ kN}
\]
and
\[
M = +18.19 \text{ kN-m}
\]

Plus sign with the numerical values of \( H \) and \( M \) indicates that the assumed directions are correct.

The other reactions are
\[
R = 10 - \frac{M}{4} = +5.45 \text{ kN (i.e. downwards)}
\]
\[
R_d = +5.45 \text{ kN}
\]
\[
H_d = 10 - H = +2.27 \text{ kN}
\]
\[
M_B = +M - 4H = +18.19 - 30.92 = -12.73 \text{ kN-m}
\]
\[
M_C = +4H_d = +9.08 \text{ kN-m}
\]

The B.M. diagram has been shown in Fig. 12.19(b).

PROBLEMS

1. A simply supported beam with overhang is loaded as shown in Fig. 12.20. Using the theorem of Castigliano find the vertical deflection of point C.

2. A cantilever of length \( L \) is loaded with uniform load \( w \) per unit length over the whole span. It is propped at the free end. Calculate the prop reaction if (a) the prop is rigid or (b) prop yields an amount \( \lambda \) under unit load.

3. For a uniformly loaded beam \( AB \) with built-in ends, determine the end moments by using theorem of Castigliano.

4. A cantilever \( AB \), loaded at the end \( B \), is supported by a short cantilever \( CD \) of the cross-section as cantilever \( AB \) as shown in Fig. 12.21. Prove that the pressure \( X \) between the two beams at \( C \) is given by \( X = \frac{3P}{L} \left( \frac{L}{L} - \frac{1}{M} \right) \).

5. A frame shown in Fig. 12.22 consists of a column \( AB \), fixed at \( A \) and having rigid connection at \( B \) with a double cantilever. The frame carries a point load \( P \) at \( C \). Calculate the vertical deflection of \( C \) and \( D \). EI is constant for the whole structure.
6. A structure shown in Fig. 12-23 consists of an upright cantilever \( AB \) of length 3\( R \) and a semicircular portion \( BC \) of radius \( R \). The flexural rigidity is constant throughout. It carries a load \( W \), acting vertically at \( C \). Determine the vertical deflection of \( C \) and the horizontal deflection of \( B \).

Fig. 12-22.

7. A circular bar is bent into the shape of a half-ring and supported in a vertical plane as shown in Fig. 12-24. Determine the horizontal movement of point \( C \) and the vertical movement of point \( B \).

Fig. 12-24.

8. Using the method of minimum strain energy, analyse the portal frame shown in Fig. 12-25. Plot the B.M. diagram. \( EI \) is constant.

Fig. 12-25.

9. Solve problem 9 (Fig. 11-21) of chapter 11 by minimum strain energy.

Answers

1. \( \frac{WL^3}{6EI} \)
2. (a) \( \frac{3}{8} wL \) (b) \( \frac{3}{8} \frac{wL^3}{L^2+3EI} \)
3. \( M_A = M_B = \frac{wL^2}{12} \)
4. \( s_C = \frac{56w}{3EI} \uparrow \); \( s_D = \frac{8w}{EI} \uparrow \)
5. \( s_{CW} = 16.71 \frac{WR}{EI} \); \( s_{BH} = \frac{9WR}{EI} \)
6. \( s_{CH} = \frac{PR}{2EI} \); \( s_{BH} = \frac{PR}{2EI} \)
7. \( M_B = 0.78 \); \( M_C = 0.39 \); \( H_A = 0.39 \uparrow \)
8. \( H_D = 0.39 \); \( R_A = 2 \); \( R_D = 2.8 \).

Deflection of Perfect Frames

13.1 GENERAL

An articulated structure or a truss is composed of a number of bars or members connected by frictionless pins, forming geometrical figures which are usually triangles. A truss is said to be statically determinate internally if it has the members given by the equation

\[ m = 2j - r \]

where

- \( m = \) Total number of members
- \( j = \) Total number of joints.
- \( r = \) Total number of condition equations available.

In the above equation, the value of \( r \) is usually 3 but if there is an additional hinge separating the structure in two parts, \( r = 3 + 1 = 4 \), and if there is a link somewhere in the structure, \( r = 3 + 2 = 5 \). The frame is said to be perfect if the number \( m \) is equal to right hand side of the equation.

When external loads are applied on the truss, at the joints, the members carry internal forces, usually called the stresses which may be either tensile or compressive. According to Hooke's law, if any axial member of length \( L \) carries a force \( P \), it will be deformed by an amount \( \frac{PL}{AE} \) where \( A \) is the area of cross-section of the members.

In the truss, therefore, the members carrying tension will be elongated while those carrying compression will be shortened. Thus, all the joints will move from their initial position, and will occupy their final equilibrium position. The axial deformation in the members of the truss may also be due to temperature changes or due to errors in fabrication or lack of fit of some members, and the joints may move from their original position. This movement of each joint is defined as the deflection of the joint in the direction of movement. The resolved part of the movement in the vertical direction is called vertical deflection of the joint, and that in the horizontal direction is
known as the horizontal deflection of the joint. In general, each joint has both vertical as well as horizontal deflection, unless constrained to move in a given direction.

There are various methods of computing the joint deflection of a perfect frame. We shall, however, discuss the following methods:

1. The unit load method.
2. Deflection by Castigliano’s first theorem.

The first two methods are analytical, and require the full knowledge of the methods of finding the stress in plane frames under static loading (see Author’s Vol. I).

13.2. THE UNIT LOAD METHOD

To develop the method, let us consider a perfect frame as shown in Fig. 13.1, where \( W_1, W_2, W_3 \ldots W_n \) etc., are external loads.

![Fig. 13.1.](image)

Let \( P_1, P_2, P_3 \ldots P_n \) etc., be the forces or stresses in the members 1, 2, 3, \ldots n etc., due to the external loading.

Let us find the vertical displacement (\( \delta v \)) of the joint \( F \) due to the external system of loads shown.

Apply gradually an infinitely small load \( \delta W \) at the joint \( F \) in the vertical direction, and let \( \delta v \) be the deflection of the joint. The work by \( \delta W \) will be

\[
\frac{1}{2} \delta W \cdot \delta v \quad (2)
\]

If \( u_1, u_2, u_3, \ldots, u_n \) are the forces in the various members due to unit vertical load at the joint \( F \), the forces in these members due to a load \( \delta W \) at \( F \) will be \( u_1 \delta W, u_2 \delta W, u_3 \delta W \ldots u_n \delta W \), respectively.
2. Remove the external loads and apply the unit vertical load at the joint if the vertical deflection of the joint is required, and find the stress \( u_1, u_2, \ldots, u_n \) in all the members. (If horizontal deflection is required, apply unit horizontal load there and find \( u'_1, u'_2, \ldots, u'_n \).)

3. Apply equation (13.1) for vertical deflection and (13.2) for horizontal deflection of the joint.

### 13.3. Joint Deflection If Linear Deformation of All the Members Are Known

If, in the place of external loads, the deformations \( \Delta_1, \Delta_2, \ldots, \Delta_n \) etc. of all the members are known, the deflection \( \delta \) can be calculated as follows:

Equation (13.1) can be rewritten as

\[
\delta_v = \sum_{i=1}^{n} u_i \frac{PL}{AE}
\]

But \( \frac{PL}{AE} \) = deformation of the member \( \Delta \) (according to Hooke’s law).

\[
\delta_v = \sum_{i=1}^{n} u_i \Delta
\]

Similarly,

\[
\delta_h = \sum_{i=1}^{n} u'_i \Delta
\]

Hence, in order to find the deflection \( \delta \) in such cases apply unit load at the joint, in the direction the deflection is required, and apply Eq. (13.4) or (13.5). See example 13.9 for illustration.

### Deflection of a Joint Due to Temperature Variation

Let \( \Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n \) be the changes in the lengths of various members of a perfect frame due to temperature variation. To find the deflection of an unloaded frame due to temperature variations apply the unit load at the joint and calculate \( u_1, u_2, \ldots \) and apply equation (13.4) (or 13.5) for the joint deflection.

Thus,

\[
\delta = \sum_{i=1}^{n} u_i \Delta_i = u_1 \Delta_1 + u_2 \Delta_2 \ldots + u_n \Delta_n
\]

If the change in length (\( \Delta \)) of certain members is zero, the product \( u_i \Delta_i \) for those members will be substituted as zero in the above equation. If, for example, there is only one member in which there is change in length \( \Delta_1 \), the deflection of a particular joint will be equal to \( u \Delta_1 \), where \( u \) is the stress in that member due to the unit load at the joint under consideration. See example 13.10.

### Deflection of a Joint Due to Lack of Fit of Certain Members

Let \( \Delta_1, \Delta_2, \ldots, \Delta_n \) be the lack of fit in the members. The joint deflection can be found by equation (13.4) or (13.5), i.e.

\[
\delta = \sum_{i=1}^{n} u_i \Delta_1 = u_1 \Delta_1 + u_2 \Delta_2 + \ldots + u_n \Delta_n
\]

If there is only one member having lack of fit \( \Delta_1 \), the deflection of a particular joint will be equal to \( u \Delta_1 \), where \( u \) is the stress in that member due to unit load at the joint under consideration.

**Example 13.1** Determine the vertical and horizontal displacements of the joint \( C \) of the pin-joined frame shown in Fig. 13.12(a). The cross-sectional area of \( AB \) is 100 sq. mm. and of \( AC \) and \( BC \) 150 sq mm each. \( E = 2 \times 10^8 \) N/mm².

**Solution**

The vertical and horizontal deflections of the joint \( C \) are given by,

\[
\delta_v = \sum_{i=1}^{n} \frac{P_u L}{AE}
\]

and

\[
\delta_h = \sum_{i=1}^{n} \frac{P_u L}{AE}
\]

Let us now find \( P, u \) and \( u' \) in each member.

(a) **Stresses due to external loading**

\( AC = \sqrt{3^2 + 4^2} = 5 \) m

\[
\sin \theta = \frac{3}{5} = 0.6; \quad \cos \theta = \frac{4}{5} = 0.8
\]

Resolving at the joint \( C \), we get

\[
6 = P_{AC} \sin \theta + P_{BC} \sin \theta
\]
Resolving horizontally, \( P_{AC} = P_{BC} \)
\[
2P_{AC} \sin \theta = 6, \text{ from which}
\]
\[
P_{AC} = P_{BC} = \frac{6}{2 \sin \theta} = \frac{6}{2 \times 0.8} = +5 \text{ kN (tension)}
\]
(Use + sign for tension and - sign for compression)

Resolving horizontally at \( A \),
\[
P_{AB} = P_{AC} \cos \theta \cdot 5 \times 0.8 = 4 \text{ kN (comp.)}
\]

(b) Stresses due to unit vertical load at \( C \)

Apply unit vertical load at \( C \).

The stresses in each member will be \( \frac{1}{6} \) th of those obtained above:

Thus
\[
\sigma_{AC} = \sigma_{BC} = +5 \text{ kN}
\]
and
\[
\sigma_{AB} = -\frac{4}{6} = -\frac{2}{3}
\]

(c) Stresses due to unit horizontal load at \( C \)

Assuming that the horizontal movement of joint \( C \) is to the left, apply a unit horizontal load at \( C \) as shown in Fig. 13.2(b), along with the reactions.

Resolving vertically at joint \( C \), we get
\[
\sigma'_{CA} = \sigma'_{CB} \text{ (numerically)}
\]
Resolving horizontally,
\[
\sigma'_{CB} \cos \theta + \sigma'_{CA} \cos \theta = 1
\]
or
\[
\sigma'_{CB} = \sigma'_{CA} = \frac{1}{2 \cos \theta} = \frac{1}{2 \times 0.8} = \frac{5}{8} \text{ kN}
\]
\[
\therefore \, \sigma'_{CA} = \frac{5}{8} \text{ kN} \, ; \, \sigma'_{CB} = -\frac{5}{8} \text{ kN}
\]
Resolving horizontally at \( B \), we get
\[
\sigma'_{AB} = \sigma'_{BC} \cos \theta = \frac{5}{8} \times 0.8 = 0.5 \text{ kN (comp.)}
\]
To calculate \( \frac{P_{UL}}{A} \) and \( \frac{P_{UL}}{A} \), the results are tabulated below.
where \( u \) is the stress due to unit vertical load at C.

The horizontal deflection of D is given by

\[
\delta_{DH} = \frac{n}{AE} \sum P u' L
\]

where \( u' \) is the stress due to unit horizontal load at D.

---

**Fig. 13-3**

(a) **Stresses due to external loading**

\[ \cos 60^\circ = \frac{1}{2}; \cos 30^\circ = \frac{\sqrt{3}}{2} \]

Resolving horizontally at joint C,

\[ P_{CB} = P_{CD} \]

Resolving vertically at joint C,

\[ 2P_{CB} \cos 60^\circ = W \]

\[ P_{BC} = W \text{ (tension)} \]

and

\[ P_{CD} = W \text{ (comp.)} \]

Resolving horizontally at B,

\[ P_{BA} = P_{BC} = W \text{ (tension)} \]

Resolving vertically at B,

\[ P_{BD} = 2P_{AB} \cos 60^\circ = W \text{ (comp.)} \]

Resolving horizontally at D,

\[ P_{DA} = P_{DC} = W \text{ (comp.)} \]

(b) **Stresses due to unit vertical load at C**

To find \( u \) in all members, put \( W = 1 \) in the expressions found above.

(c) **Stresses due to unit vertical load at D**

The find the horizontal deflection of D, apply unit horizontal load at D, in the direction shown in Fig. 13-3.

---

**DEFORMATION OF FRAMES**

By resolution at joint C,

\[ u'_{BC}=u'_{CD}=0 \]  
(Since there is no load at B)

Resolving horizontally at B,

\[ u'_{BA}=u'_{BC}=0 \]

Hence \[ u'_{BD}=0 \]

Resolving horizontally at D,

\[ u_{DA} \cos 30^\circ = 1 \]

or

\[ u_{DA} = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}} \text{ (comp.)} \]

The results are tabulated below:

<table>
<thead>
<tr>
<th>Member</th>
<th>Length</th>
<th>Area</th>
<th>( P )</th>
<th>( u )</th>
<th>( u' )</th>
<th>( \frac{P u L}{AE} )</th>
<th>( \frac{P u' L}{AE} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>L</td>
<td>a</td>
<td>+W</td>
<td>+1</td>
<td>0</td>
<td>+ ( \frac{W L}{a E} )</td>
<td>0</td>
</tr>
<tr>
<td>BC</td>
<td>L</td>
<td>a</td>
<td>+W</td>
<td>+1</td>
<td>0</td>
<td>+ ( \frac{W L}{a E} )</td>
<td>0</td>
</tr>
<tr>
<td>CD</td>
<td>L</td>
<td>2a</td>
<td>-W</td>
<td>-1</td>
<td>0</td>
<td>+ ( \frac{W L}{2a E} )</td>
<td>0</td>
</tr>
<tr>
<td>AD</td>
<td>L</td>
<td>2a</td>
<td>-W</td>
<td>-1</td>
<td>(-\frac{2}{\sqrt{3}})</td>
<td>+ ( \frac{W L}{2a E} ) + ( \frac{W L}{\sqrt{3}a E} )</td>
<td></td>
</tr>
<tr>
<td>BD</td>
<td>L</td>
<td>2a</td>
<td>-W</td>
<td>-1</td>
<td>0</td>
<td>+ ( \frac{W L}{2a E} )</td>
<td>0</td>
</tr>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>+ ( \frac{7W L}{2a E} )</td>
<td>+ ( \frac{W L}{\sqrt{3}a E} )</td>
</tr>
</tbody>
</table>

Hence

\[
\delta_{CV} = \frac{7W L}{2a E} \]

and

\[
\delta_{DH} = \frac{W L}{\sqrt{3}a E} \]

**Example 133.** Fig. 13-4 represents a crane structure attached to a vertical wall and carrying a vertical load of 20 kN at C.

All tension members are stressed to 80 N/mm² and all compression members to 50 N/mm². Determine the horizontal and vertical deflection of the end C. Take \( E = 2 \times 10^5 \text{ N/mm}^2 \). All members, except CD, have a length of 2 m. \( AE = 2 \text{ m.} \)
Solution

The horizontal and vertical deflections of \( C \) are given by

\[
\delta_v = \sum \frac{p u L}{E} = \Sigma \frac{p u L}{E} \tag{1}
\]

and

\[
\delta_H = \sum \frac{p u' L}{E} = \Sigma \frac{p u' L}{E} \tag{2}
\]

where \( p \) = intensity of stress in each member, and is known.

(a) Calculation of stresses due to unit vertical load at \( C \).

To calculate \( u \) in all members, apply a unit vertical load at \( C \).

Resolving perpendicular to \( BC \) at \( C \),

\[
ucd \sin 30^\circ = 1 \times \sin 60^\circ
\]

\[
ucd = \frac{\sin 60^\circ}{\sin 30^\circ} = \sqrt{3} \text{ (comp.)}
\]

\[
u_{CD} = \frac{u_{CD}}{\cos 30^\circ} = 1 \text{ (tension)}
\]

\[
u_{AB} = \nu_{BC} = 1 \text{ (tension)}
\]

\[
u_{BD} = 0 \text{ (by resolving perpendicular to } AB \text{ at } B)
\]

Resolving perpendicular to \( ED \) at \( D \),

\[
u_{AD} \sin 60^\circ = \nu_{DC} \sin 30^\circ
\]

or

\[
u_{AD} = \sqrt{3} \times \frac{1}{2} = \frac{1}{2} \times \sqrt{3} = 1 \text{ (tension)}
\]

Resolving along \( ED \),

\[
u_{ED} = \nu_{AD} \cos 60^\circ + \nu_{DC} \cos 30^\circ = 0 \text{ (comp.)}
\]

(b) Calculation of stresses due to unit horizontal load at \( C \).

Apply unit horizontal load at \( C \), as shown.

Resolving perpendicular to \( BC \) at \( C \),

\[
u_{CD} \sin 30^\circ = \nu_{CD} \sin 30^\circ
\]

\[
u_{CD} = 1 \text{ (comp.)}
\]

Resolving along \( BC \), at \( C \),

\[
u_{CB} = \nu_{CD} \cos 30^\circ + 1 \times \cos 30^\circ
\]

\[
= (1 \times \sqrt{3}) + (1 \times \frac{\sqrt{3}}{2}) = \sqrt{3} \text{ (tension)}
\]

The results are tabulated below:

<table>
<thead>
<tr>
<th>Member</th>
<th>( L ) (mm)</th>
<th>( p ) (N/mm²)</th>
<th>( u )</th>
<th>( u' )</th>
<th>( puL )</th>
<th>( pu' L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( AB )</td>
<td>2000</td>
<td>+80</td>
<td>+1-0</td>
<td>( +\sqrt{3} )</td>
<td>+16 \times 10⁴</td>
<td>+16\sqrt{3} \times 10⁴</td>
</tr>
<tr>
<td>( BC )</td>
<td>2000</td>
<td>+80</td>
<td>+1-0</td>
<td>( +\sqrt{3} )</td>
<td>+16 \times 10⁴</td>
<td>+16\sqrt{3} \times 10⁴</td>
</tr>
<tr>
<td>( AD )</td>
<td>2000</td>
<td>+80</td>
<td>+1-0</td>
<td>( +\sqrt{3} )</td>
<td>+16 \times 10⁴</td>
<td>+16\sqrt{3} \times 10⁴</td>
</tr>
<tr>
<td>( BD )</td>
<td>2000</td>
<td>0</td>
<td>+1-0</td>
<td>( +\sqrt{3} )</td>
<td>+16 \times 10⁴</td>
<td>+16\sqrt{3} \times 10⁴</td>
</tr>
<tr>
<td>( ED )</td>
<td>2000</td>
<td>-50</td>
<td>-2-0</td>
<td>( -\frac{2}{\sqrt{3}} )</td>
<td>+20 \times 10⁴</td>
<td>+20\sqrt{3} \times 10⁴</td>
</tr>
<tr>
<td>( DC )</td>
<td>2000 \sqrt{3}</td>
<td>-50</td>
<td>-\sqrt{3}</td>
<td>-1-0</td>
<td>+30 \times 10⁴</td>
<td>+10\sqrt{3} \times 10⁴</td>
</tr>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Sum</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>+98 \times 10⁴</td>
<td>+93\sqrt{3} \times 10⁴</td>
</tr>
</tbody>
</table>

Example 13-4. The steel truss shown in Fig. 13-5 is anchored at \( A \) and supported on rollers at \( B \). If the truss is so designed that, under the given loading, all tension members are stressed to 100 N/mm² and all compression members to 80 N/mm², find the vertical deflection of the point \( C \). Take \( E = 2 \times 10^5 \) N/mm².

Find also the lateral displacement of the end \( B \).

Solution

The vertical deflection is given by

\[
\delta_v = \sum \frac{p u L}{E} = \frac{98 \times 10^4}{2 \times 10^5} = 0.49 \text{ mm (↓)}
\]

and

\[
\delta_H = \sum \frac{n p u' L}{E} = \frac{93.5 \times 10^4}{2 \times 10^5} = 0.467 \text{ cm. (→)}
\]
To find the values of \( u \), apply a unit vertical load at \( C \), and analyse the frame. The results are tabulated below. (The students are advised to work out the stresses themselves).

![Diagram](image)

**Fig. 13-5**

(+ for tension; — for compression)

<table>
<thead>
<tr>
<th>Member</th>
<th>Length (mm)</th>
<th>( p ) (N/mm²)</th>
<th>( u )</th>
<th>( puL )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4000</td>
<td>-80</td>
<td>-4/9</td>
<td>+128/9 × 10⁴</td>
</tr>
<tr>
<td>2</td>
<td>4000</td>
<td>-80</td>
<td>-4/9</td>
<td>+128/9 × 10⁴</td>
</tr>
<tr>
<td>3</td>
<td>4000</td>
<td>-80</td>
<td>-8/9</td>
<td>+256/9 × 10⁴</td>
</tr>
<tr>
<td>4</td>
<td>5000</td>
<td>+100</td>
<td>+5/9</td>
<td>+250/9 × 10⁴</td>
</tr>
<tr>
<td>5</td>
<td>3000</td>
<td>-80</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>5000</td>
<td>0</td>
<td>-5/9</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>4000</td>
<td>-100</td>
<td>+8/9</td>
<td>+320/9 × 10⁴</td>
</tr>
<tr>
<td>8</td>
<td>3000</td>
<td>-80</td>
<td>+1/3</td>
<td>-8 × 10⁴</td>
</tr>
<tr>
<td>9</td>
<td>5000</td>
<td>+100</td>
<td>+10/9</td>
<td>+500/9 × 10⁴</td>
</tr>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td></td>
<td>+168 × 10⁴</td>
</tr>
</tbody>
</table>

\[ \delta_{CV} = \sum \frac{puL}{E} = \frac{168 \times 10^4}{2 \times 10^5} = 8.4 \text{ mm} \downarrow \]

**Example 13-5.** The frame shown in Fig. 13-6 consists of four panels each 2.5 m wide, and the cross-sectional areas of the members are such that, when the frame carries equal loads at the panel points of the lower chord, the stress in all the tension members is \( f \) N/mm², and the stress in all the compression members is \( 0.8f \) N/mm². Determine the value of \( f \) if the ratio of the maximum deflection to span is \( 1/900 \).

Take \( E = 2.0 \times 10^5 \) N/mm².

**Solution**

![Diagram](image)

By inspection, it can be seen that for the loads at the panel points of the lower panel, the top chord members will be in compression, and the bottom chord members, verticals and diagonals, will be in tension.

Due to symmetrical loading, the maximum deflection occurs at \( C \). Apply unit load at \( C \) to find \( u \) in all the members. All the members have been numbered 1, 2, ... etc. in Fig. 13-6.

By inspection, \( u_1 = 0 \); \( u_8 = 0 \); \( u_{10} = 0 \).

**Reaction**

\[ R_A = R_B = \frac{1}{2} \times 45^\circ; \cos 0 = \sin 0 = \frac{1}{\sqrt{2}} \]
Example 13.6. Fig. 13.7(a) shows the outline of a load which is distributed as 2 kN on each of the four points Q, R, S and T. The members PQ, PR, PS and PT, each have an area of 65 sq. mm and the members QR, RS and ST each have of 130 sq. mm. Determine the vertical deflection of Q and R relative to support P.

\[ \sigma = \frac{EPuL}{E} = 10^6 \text{ N/mm}^2. \]

Solution

Since the loading and truss are symmetrical, consider half the truss only. For the equilibrium of the half truss, the horizontal reaction \( R \) will be supplied by the other half of the truss as shown in Fig. 13.7(b). Thus the half truss of Fig. 13.7(b) is obtained by assuming the truss having cut by a vertical section through \( P \), and the basic system of Fig. 13.7(a) does not change. Fig. 13.7(c) shows the half truss under unit vertical load at \( Q \), and Fig. 13.7(d) shows the half truss under unit vertical load at \( R \).

The vertical deflection of \( Q \) and \( R \) are given by

\[ \delta_q = \frac{PUL}{AE} \]

(1) The summation being made for half the truss only.
where \( u \) is the stress in any member due to unit vertical load at \( Q \), and \( u' \) is the stress due to unit vertical load at \( R \). The members have been numbered 1, 2, 3, etc.

(a) Calculation of stresses due to external loading

\[
\begin{align*}
P_1 \sin 30^\circ &= 2; & P_1 &= 4 \text{ (tension)} \\
P_2 &= P_1 \cos 30^\circ = \frac{4\sqrt{3}}{2} = 2\sqrt{2} \text{ (comp.)} \\
P_3 \sin 60^\circ &= 2; & P_3 &= \frac{2 \times 2}{\sqrt{3}} = \frac{4\sqrt{3}}{3} \text{ (tension)} \\
P_4 &= P_3 \cos 60^\circ + P_2 = \frac{4\sqrt{3}}{3} \times \frac{1}{2} + \frac{4\sqrt{3}}{2} = \frac{8\sqrt{3}}{3} \text{ (comp.)}
\end{align*}
\]

(b) Calculation of stresses due to unit vertical load at \( Q \)

By inspection,

\[
\begin{align*}
u &= 0; & u_1 &= 2 \text{ (tension)} \\
u_2 &= u_2 \cos 30^\circ = \frac{2\sqrt{3}}{2} = \sqrt{3} \text{ (comp.)} \\
u_3 &= u_3 = \sqrt{3} \text{ (comp.)}
\end{align*}
\]

(c) Calculation of stresses due to unit vertical load at \( R \)

Since there is no load at \( Q \).

\[
\begin{align*}
u' &= 0 \\
u_1' &= 0 \\
u_2' &= \frac{2\sqrt{3}}{3} \text{ (tension)} \\
u_3' &= \frac{2\sqrt{3}}{3} \text{ (tension)} \\
u_4' &= \frac{2\sqrt{3}}{3} \times \frac{1}{2} = \frac{\sqrt{3}}{3} \text{ (comp.)}
\end{align*}
\]

The results are tabulated below:

<table>
<thead>
<tr>
<th>Member</th>
<th>( L ) (mm)</th>
<th>( A ) (mm²)</th>
<th>( u )</th>
<th>( u' )</th>
<th>( \frac{PAL}{A} )</th>
<th>( \frac{PA L}{A} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2000\sqrt{3}</td>
<td>65</td>
<td>+4</td>
<td>+2</td>
<td>0</td>
<td>+426</td>
</tr>
<tr>
<td>2</td>
<td>2000</td>
<td>130</td>
<td>-2\sqrt{3}</td>
<td>-\sqrt{3}</td>
<td>0</td>
<td>+923</td>
</tr>
<tr>
<td>3</td>
<td>2000</td>
<td>65</td>
<td>+4\sqrt{3}/3</td>
<td>0</td>
<td>+4\sqrt{3}/3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1000</td>
<td>130</td>
<td>-8\sqrt{3}/3</td>
<td>-\sqrt{3}</td>
<td>-\sqrt{3}/3</td>
<td>+61.5</td>
</tr>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>+579.8</td>
<td>+102.5</td>
</tr>
</tbody>
</table>

\( E = 2 \times 10^8 \text{ N/mm}^2 = 200 \text{ kN/mm}^2 \)

Example 13.7. The roof truss shown in Fig. 13.8 has members with cross-sectional areas such that when the loading is shown, all members are subjected to the same intensity of stress either tensile or compressive. If the vertical deflection of joint \( C \) is 15 mm, determine the change in the span of the truss.

Solution.

![Fig. 13.8.](image)

To find the vertical deflection of point \( C \), apply a unit vertical load at \( C \), as shown in Fig. 13.8. Let \( \pm \rho \) be the intensity of stress in the members due to external loading. The members have been numbered 1, 2, 3 etc.

\[
\begin{align*}
L_1 &= L_2 = L_3 = L_4 = L_5 = \sqrt{(2.5)^2 + \left(\frac{2.5}{2}\right)^2} = 2.5 \sqrt{5} \text{ m} \\
L_6 &= 2.5 \text{ m}; \ L_7 = L_8 = 5 \text{ m} \\
\sin \theta &= \frac{2.5}{2.5 \sqrt{5}} = \frac{1}{\sqrt{5}} \\
\cos \theta &= \frac{5}{2.5 \sqrt{5}} = \frac{2}{\sqrt{5}} \\
u_1 \sin \theta &= \frac{1}{2} \text{ or } u_1 = \frac{1}{2} \sqrt{5} = \frac{\sqrt{5}}{2} = u_4 \text{ (comp.)} \\
u_8 = u_4 \cos \theta = \frac{\sqrt{5}}{2} \times \frac{2}{\sqrt{2}} = 1 = u_8 \text{ (tension)}
\end{align*}
\]
The results are tabulated below, giving correct signs to ±p obtained by inspection of Eq. 13.8 under the actual loads.

(+ for tension; — for compression)

<table>
<thead>
<tr>
<th>Member</th>
<th>Length</th>
<th>Stress</th>
<th>u</th>
<th>puL</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1250√5</td>
<td>—p</td>
<td>—√5/2</td>
<td>3125p</td>
</tr>
<tr>
<td>2</td>
<td>1250√5</td>
<td>—p</td>
<td>—√5/2</td>
<td>3125p</td>
</tr>
<tr>
<td>3</td>
<td>1250√5</td>
<td>—p</td>
<td>—√5/2</td>
<td>3125p</td>
</tr>
<tr>
<td>4</td>
<td>1250√5</td>
<td>—p</td>
<td>—√5/2</td>
<td>3125p</td>
</tr>
<tr>
<td>5</td>
<td>1250√5</td>
<td>—p</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2500</td>
<td>—p</td>
<td>+1</td>
<td>2500p</td>
</tr>
<tr>
<td>7</td>
<td>1250√5</td>
<td>—p</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>5000</td>
<td>+p</td>
<td>+1</td>
<td>5000p</td>
</tr>
<tr>
<td>9</td>
<td>5000</td>
<td>—p</td>
<td>+1</td>
<td>5000p</td>
</tr>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td></td>
<td>+2500p</td>
</tr>
</tbody>
</table>

\[ \delta_c = \sum \frac{p_u L}{E} = \frac{2500p}{E} \text{ mm} \]  

But \[ \delta_c = 15 \text{ mm} \text{ (given) } \]  

Equating (1) and (2),

\[ 25000 \frac{p}{E} = 15 \]

\[ \frac{p}{E} = \frac{15}{25000} = \frac{3}{5000} \]

Now, change in the span \( AB = \Delta AC + \Delta CB \)

\[ = 2 \left\{ \frac{p \cdot (5000)}{E} \right\} \]

\[ = 10000 \frac{p}{E} \]

Substituting the value of \( \frac{p}{E} \), we get

\[ \text{Change in the span } AB = 10000 \times 3 = 30 = 6 \text{ mm.} \]

**Example 13.8.** The frame shown in Fig. 13.9 consists of four panels each 2.5 m wide, and cross-sectional areas of the members are such that when the frame carries equal loads at the panel joints of the lower chord, the stresses in all tension members is 100 N/mm² and the stress in all the compression members is 80 N/mm². Determine the relative movement between the joints C and K in the direction CK. Take \( E = 2 \times 10^5 \text{ N/mm²} \)

**Solution**

To find the relative movement between joints C and K, apply unit loads at C and K in the direction CK. The movement \( \delta \) of the joints C and K towards each other is then given by

\[ \delta = \sum \frac{p_u L}{E} \]

There will be no reaction at A and F due to the unit loads. Hence

\( u_1 = 0; \ u_2 = 0; \ u_3 = 0; \ u_4 = 0; \ u_5 = 0; \ u_6 = 0 \) and \( u_7 = 0 \)

Resolving at joint C,

\( u_{11} = 1 \sin \theta = \frac{1}{\sqrt{2}} \text{ (comp.)} \]

\( u_2 = 1 \cos \theta = \frac{1}{\sqrt{2}} \text{ (comp.)} \)

Resolving at joint K

\( u_8 = 1 \cos \theta = \frac{1}{\sqrt{2}} \text{ (comp.)} \)

\( u_{12} = 1 \sin \theta = \frac{1}{\sqrt{2}} \text{ (comp.)} \)
Resolving at $D$,

$$u_{12} \cos \theta = u_{13}$$

$$\therefore \quad u_{12} = \frac{-u_{13}}{\cos \theta} = \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{1} = 1 \text{ (tension)}$$

By inspection, it can be seen that for the loads at the panel points of the lower panel, the top chord members will be in compression, and the bottom chord members, verticals and diagonals will be in tension. Member $CJ$ does not carry any stress.

Thus, $p_1 = p_2 = p_3 = p_6 = 80 \text{ N/mm}^2$ (comp.)

$p_4 = p_5 = p_7 = p_9 = 100 \text{ N/mm}^2$ (tension)

$p_8 = p_10 = 100 \text{ N/mm}^2$ (tension)

$p_{11} = 0$.

The results are tabulated below. However, since $\delta$ is the function of the product of $p$ and $u$, those members having $u = 0$ or $p = 0$ (and hence $p.u = 0$) have not been included in the table.

<table>
<thead>
<tr>
<th>Member</th>
<th>Length (L mm)</th>
<th>$p$</th>
<th>$u$</th>
<th>$puL$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_6$</td>
<td>2500</td>
<td>-80</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
<td>$+14.14 \times 10^4$</td>
</tr>
<tr>
<td>$u_7$</td>
<td>2500</td>
<td>+100</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$-17.66 \times 10^4$</td>
</tr>
<tr>
<td>$u_{12}$</td>
<td>2500$\sqrt{2}$</td>
<td>+100</td>
<td>$+1$</td>
<td>$+35.32 \times 10^4$</td>
</tr>
<tr>
<td>$u_{13}$</td>
<td>2500</td>
<td>+100</td>
<td>$\frac{1}{2}$</td>
<td>$-17.66 \times 10^4$</td>
</tr>
</tbody>
</table>

| Sum    |               |     |     | $+14.14 \times 10^4$ |

$$E = 20 \times 10^6 \text{ N/mm}^2$$

$$\delta = \frac{14.14 \times 10^4}{2.0 \times 10^9} \text{ mm} = 0.0706 \text{ mm}$$

**Example 13'9.** Fig. 13'10(a) shows a pin-jointed frame which is hinged to rigid-supports $A$ and $D$ which are at the same level. All members have the same length and the span $AD$ is the same as the length of the members. Due to a certain loading, the changes in length of the members are estimated as $AB: +0.185 \text{ in}$, $BC: +0.240 \text{ in}$, $BD: -0.020 \text{ in}$, $CD: -0.365 \text{ in}$ (+ denotes extension).

Find the horizontal and vertical movements of $C$. (U.L.)

**Solution**

Fig. 13'10.

The horizontal and vertical deflections of $C$ are given by

$$\delta_v = \sum \frac{n}{AE} \frac{puL}{H} = \sum \frac{n}{AE} u \Delta$$

$$\delta_H = \sum \frac{n}{AE} \frac{pu'L}{H} = \sum \frac{n}{AE} u' \Delta$$

(2)

where $u$ is the force in any member due to unit vertical load at $C$, and $u'$ is the force due to unit horizontal load at $C$.

To find the values of $u$, apply unit vertical load at $C$. Fig. 13'10(b) shows the induced stresses in the members. Similarly, Fig. 13'10(c) shows the stresses $u'$ due to unit horizontal load at $C$. (Students are advised to work out the stresses independently).
The results are tabulated below.

(+ for tension and extension; − for compression and contraction)

<table>
<thead>
<tr>
<th>Member</th>
<th>Deformation ( \Delta ) (in.)</th>
<th>( u )</th>
<th>( u' )</th>
<th>( u\Delta )</th>
<th>( u'\Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( AB )</td>
<td>+0.185</td>
<td>(-\frac{1}{\sqrt{3}})</td>
<td>+1</td>
<td>-0.107</td>
<td>+0.185</td>
</tr>
<tr>
<td>( BC )</td>
<td>+0.240</td>
<td>(-\frac{1}{\sqrt{3}})</td>
<td>+1</td>
<td>-0.139</td>
<td>+0.240</td>
</tr>
<tr>
<td>( CD )</td>
<td>-0.365</td>
<td>(+\frac{2}{\sqrt{3}})</td>
<td>0</td>
<td>-0.422</td>
<td>0</td>
</tr>
<tr>
<td>( BD )</td>
<td>-0.200</td>
<td>(+\frac{1}{\sqrt{3}})</td>
<td>-1</td>
<td>-0.116</td>
<td>+0.200</td>
</tr>
</tbody>
</table>

\[ \delta v = \frac{h}{n} = -0.748 \text{ in.}, \text{i.e.} \ 0.748 \text{ in.} \]

\[ \delta H = \frac{h'}{n} = +0.625 \text{ in.} \ (\rightarrow). \]

**Example 13.10.** Determine the vertical deflection of the joint \( C \) of the frame shown in Fig. 13.11, due to temperature rise of 6°F in the upper chords only. The coefficient of expansion is \( 6.0 \times 10^{-6} \) per \( 1 \text{ °F} \) and \( E = 2 \times 10^{6} \text{ kg/cm}^{2} \).

**Solution**

\[ \delta v = \frac{h}{n} = -0.748 \text{ in.}, \text{i.e.} \ 0.748 \text{ in.} \]

To find \( u \), apply unit vertical load at \( C \). Since the change in length (\( \Delta \)) occurs only in the three top chord members, stresses in these members only need be found out.

- Reaction at \( A = \frac{4}{12} = \frac{1}{3} \)
- Reaction at \( B = \frac{8}{12} = \frac{2}{3} \)

Passing a section cutting members 1 and 4, and taking moments at \( D \), we get

\[ u_{1} = \left( \frac{1}{3} \times 4 \right) \frac{1}{3} = \frac{4}{9} \text{ (comp.)} \]

Similarly, passing a section cutting members 3 and 9, and taking moments at \( C \), we get

\[ u_{2} = \left( \frac{2}{3} \times 4 \right) \frac{1}{3} = \frac{8}{9} \text{ (comp.)} \]

Also

\[ u_{3} = u_{4} = \frac{4}{9} \text{ (comp.)} \]

\[ \delta v = u_{1} \Delta_{1} + u_{2} \Delta_{2} + u_{3} \Delta_{3} = \left\{ \left( -\frac{4}{9} \right) + \left( -\frac{4}{9} \right) + \left( -\frac{8}{9} \right) \right\} \times (0.0144) = -0.256 \text{ cm}, \text{i.e.,} 0.256 \text{ cm} \]

**Example 13.11.** Determine the horizontal and vertical deflection of the joint \( C \) of the frame shown in Fig. 13.12, if the member \( DF \) has a lack of fit of 1 cm (long).

**Solution**

Let \( u \) be the stress in member \( DF \) due to unit vertical load at \( C \), [Fig. 13.12 (a)] and \( u' \) be the stress in it due to unit horizontal load at \( C \), [Fig. 13.12 (b)]. Since only one member has lack of fit, we have

\[ \delta_{CV} = u_{4} \Delta_{e} = u_{4} \times (+1) = u_{4} \]

\[ \delta_{hv} = u_{4} \Delta_{e} = u_{4} \times (+1) = u_{4} \]

Refer Fig. 13.12 (a). Reactions at \( A \) and \( B \) will be \( \frac{1}{3} \) and \( \frac{2}{3} \) respectively. Pass a section to cut the members 2, 6 and 7.
Then force in member 6 = (shear in the panel) \times \left( \frac{\csc \theta}{3} \right)

\csc \theta = \frac{5}{3}

\begin{array}{c}
\begin{array}{c}
\text{Fig. 13.12.}
\end{array}
\end{array}

\therefore u_6' = \frac{1}{3} \times \frac{5}{3} = \frac{5}{9} (\text{comp.}) = - \frac{5}{9}

(since compression has negative sign)

\therefore \delta e_Y = u_6' = - \frac{5}{9} \text{ cm, i.e. } \frac{5}{9} \text{ cm} \uparrow .

Similarly, when a unit horizontal load is applied at C, [Fig. 13.12 (b)], the horizontal reaction at A = \pm 1 (\pm). Since the unit horizontal load at C and the reaction at A produce a couple in the anticlockwise direction, equal and opposite vertical reactions of magnitude \( \frac{1 \times 3}{12} = \frac{1}{4} \) will be induced at A and B as shown.

To find the force \( u_6' \) in FD, pass a section to cut the members 2, 6 and 7.

Then \( u_6' = (\text{shear in the panel}) \times (\csc \theta) \)

\[ u_6' = \frac{1}{4} \times \frac{5}{3} = \frac{5}{12} \text{ (comp.)} = - \frac{5}{12} \]

Hence \( \delta e_Y = u_6' = - \frac{5}{12} \text{ cm, i.e. } \frac{5}{12} \text{ cm} \downarrow \).

13.4. DEFLECTION BY CASTIGLIANO’S FIRST THEOREM

In chapter 12, it has been proved that the partial derivative of total strain energy with respect to a force gives the deflection in the direction of the force. This is Castigliano’s first theorem, and its application for finding the deflection of beams etc. has already been studied earlier.

In articulated structures, the loads are applied at panel points only, and hence the members carry axial forces, (either tension or compression) only. Thus the strain energy will be due to direct forces and is given by

\[ U = \frac{n}{2} \sum P_i L_i ^2 \]

where \( P \) is force in any member due to external loading. If \( W \) is an external load acting at a joint and it is required to find the deflection of the joint in the direction of the application of the load, we have, according to Castigliano’s first theorem.

\[ \delta = \frac{\partial U}{\partial W} = \frac{n}{1} \sum \frac{P_i L_i \partial P_i}{AE} \]

This is the expression for the deflection of a joint. If, however, it is required to find the deflection of a joint where \( W \) is not acting, a fictitious load \( W \) is applied there in the required direction. The fictitious load \( W \) is then equated to zero. A similar method may be employed for the joint where external load is acting, but the deflection is required to be found in some other direction.

Procedure for Computing Deflection

1. Apply a fictitious load \( W \) at the joint in the direction in which the deflection is required, if no such external load is acting.

2. Find the force \( P \) in all members. The force \( P \) will be a function of \( W \) and the external load. Thus, in general,

\[ P = a + bW \]

where \( a \) and \( b \) are the constants depending upon the geometry of the truss, position of the load, the position of the member and the system of external loading. In some cases, either \( a \) may be zero, or \( b \) may be zero, or both \( a \) and \( b \) may zero.

3. Find the value \( \frac{\partial P}{\partial W} \) (= \( b \)).

4. Calculate \( \frac{PL}{AE} \times \frac{\partial P}{\partial W} \) for each member. If \( W \) were a fictitious load, equate it to zero.

5. \( \sum \frac{PL}{AE} \times \frac{\partial P}{\partial W} \) gives the required deflection.

Comparison with Unit Load Method

The deflection by the unit load method is given by

\[ \delta = \sum \frac{P_i L_i u_i}{1} = \sum \frac{P_i L_i}{AE} \cdot \frac{u_i}{1} = \sum \frac{P_i}{AE} \cdot u = \sum \frac{P_i}{AE} \cdot \frac{u}{1} \]

(1)
The deflection by Castigliano's theorem is given by
\[ \delta = \sum \frac{PL}{AE} \cdot \frac{\partial P}{\partial W} \]  
(2)

If expressions (1) and (2) are compared, it is evident that \( \frac{\partial P}{\partial W} = u \). Thus, both the expressions are the same. However, due to different form, the procedure for computation is different. In the unit load method, one has to analyse the frame twice for finding \( P \) and \( u \) in each member while in the latter method, only one analysis is needed. However, the expressions for \( P \), by the Castigliano's method, are sometimes long and cumbersome.

**Example 13'12.** Solve example 13'2 by Castigliano's first theorem.

Solution. (Refer 13'3).

(a) Vertical deflection of \( C \)

An external load \( W \) is already acting vertically at \( C \). Hence
\[ \delta_{VC} = \sum \frac{PL}{AE} \cdot \frac{\partial P}{\partial W} \]

The value \( P \) in various members have already been calculated in example 13'2. The values of \( P \) and \( \frac{\partial P}{\partial W} \) etc. have been entered, and computations done in a tabular form below:

<table>
<thead>
<tr>
<th>Member</th>
<th>Length</th>
<th>Area</th>
<th>( P )</th>
<th>( \frac{\partial P}{\partial W} )</th>
<th>( \frac{PL}{AE} \cdot \frac{\partial P}{\partial W} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( AB )</td>
<td>( L )</td>
<td>( a )</td>
<td>( +W )</td>
<td>( +1 )</td>
<td>( +\frac{WL}{a} )</td>
</tr>
<tr>
<td>( BC )</td>
<td>( L )</td>
<td>( a )</td>
<td>( +W )</td>
<td>( +1 )</td>
<td>( +\frac{WL}{a} )</td>
</tr>
<tr>
<td>( AD )</td>
<td>( L )</td>
<td>( 2a )</td>
<td>( -W )</td>
<td>( -1 )</td>
<td>( +\frac{WL}{2a} )</td>
</tr>
<tr>
<td>( CD )</td>
<td>( L )</td>
<td>( 2a )</td>
<td>( -W )</td>
<td>( -1 )</td>
<td>( +\frac{WL}{2a} )</td>
</tr>
<tr>
<td>( BD )</td>
<td>( L )</td>
<td>( 2a )</td>
<td>( -W )</td>
<td>( -1 )</td>
<td>( +\frac{WL}{2a} )</td>
</tr>
</tbody>
</table>

Hence
\[ \delta_{VC} = \sum \frac{PL}{AE} \cdot \frac{\partial P}{\partial W} = \frac{7WL}{2a} \]

(b) Horizontal deflection of \( C \)

Apply a horizontal load \( H \) at the joint \( O \) (in the dotted direction shown in Fig. 13'3), with the external load \( W \) still acting on the frame. The horizontal deflection of \( C \) is given by
\[ \delta_{DH} = \sum \frac{PL}{AE} \cdot \frac{\partial P}{\partial H} \]

It will be seen that the forces in all members except \( AD \) will be the same as in the previous case (i.e., when only \( W \) is acting and \( H = 0 \)). The force in \( AD \) can be found by resolving horizontally at \( A \). As the horizontal reaction at \( A \) is \( H \), we have
\[ P_{AD} \cos 30^\circ = H + P_{AB} \cos 30^\circ = H + W \cos 30^\circ \]

\[ \therefore \quad P_{AD} = H - \frac{2}{\sqrt{3}} + W \]

and
\[ \frac{\partial P_{AD}}{\partial H} = \frac{2}{\sqrt{3}} \]

The forces in the four members are as below:
\[ P_{AB} = +W; \quad \frac{\partial P_{AB}}{\partial H} = 0 \]
\[ P_{BC} = +W; \quad \frac{\partial P_{BC}}{\partial H} = 0 \]
\[ P_{CD} = -W; \quad \frac{\partial P_{CD}}{\partial H} = 0 \]
\[ P_{BD} = -W; \quad \frac{\partial P_{BD}}{\partial H} = 0 \]

Hence
\[ \delta_{DH} = \sum \frac{PL}{AE} \cdot \frac{\partial P}{\partial H} = \frac{WL}{\sqrt{3}aE} \]

by putting \( H = 0 \).

**Example 13'13.** A pin joined frame shown in Fig. 13'13 is hinged to a rigid wall at \( A \) and is free to slide vertically at \( E \). The frame carries a vertical load \( W \) at \( B \). The area of each tension member is \( a \) and of each compression member \( 2a \) and the length \( AE \) is \( L \). Obtain an expression for the vertical displacement of \( C \) (U.L.)
Solution

Due to external loading, member AB, AD and AE will carry tension, and the area of each of these members is therefore, a. Members BD and DE carry compression and hence their area is 2a. Members BC and CD carry zero stress.

To find the vertical deflection of C, apply a fictitious vertical load Fig. 13.13.

\[ \delta_{cv} = \frac{\pi}{1} \frac{PL}{AE} \cdot \frac{\partial P}{\partial Q} \]

where \( P = \) stress in any member due to both external and fictitious load

Vertical Reaction at E = 0 (roller)

Vertical Reaction at A = (W + Q)

The stresses in BC, CD and AB will be in terms of Q only, and hence \( \frac{\partial P}{\partial Q} \) will be zero when the value of Q is put zero. Similarly stresses in BD and AD will be the function of W only and hence \( \frac{\partial P}{\partial Q} \) and \( P \cdot \frac{\partial P}{\partial Q} \) will be zero for these two members. Thus \( \Sigma \frac{PL}{AE} \cdot \frac{\partial P}{\partial Q} \) is required for members AE and DE only.

Pass a section to cut AB, AD and ED. Taking moments about A, we get

\[ P_{DE} = \frac{(W \times AB) + (Q \times AC)}{L \sin 60^\circ} = \frac{WL \cos 30^\circ + 2QL \cos 30^\circ}{L \sin 60^\circ} = (W + 2Q) \text{ (compression)} \]

Resolving vertically at E,

\[ P_{AE} = P_{DE} \cos 60^\circ = \frac{1}{2} (W + 2Q) \text{ tension} \]

To make it more clear how \( P \cdot \frac{\partial P}{\partial Q} \) is zero for the five members, value of P and \( \frac{\partial P}{\partial Q} \) are tabulated below for all the members.

<table>
<thead>
<tr>
<th>Member</th>
<th>Length L</th>
<th>Area A</th>
<th>P</th>
<th>( \frac{\partial P}{\partial Q} )</th>
<th>( \frac{PL}{AE} \cdot \frac{\partial P}{\partial Q} ) after putting Q = 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>( \frac{\sqrt{3}}{2} L )</td>
<td>a</td>
<td>( \frac{4Q}{\sqrt{3}} )</td>
<td>( \frac{4}{3} \times \frac{\sqrt{3} QL}{aE} = 0 )</td>
<td></td>
</tr>
<tr>
<td>BD</td>
<td>( \frac{L}{2} )</td>
<td>2a</td>
<td>( -W )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>CD</td>
<td>( \frac{\sqrt{3}}{2} L ) (say)</td>
<td>a (say)</td>
<td>( \frac{4Q}{\sqrt{3}} )</td>
<td>( \frac{4}{3} \times \frac{\sqrt{3} QL}{aE} = 0 )</td>
<td></td>
</tr>
<tr>
<td>AD</td>
<td>L</td>
<td>a</td>
<td>( +W )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>AE</td>
<td>L</td>
<td>a</td>
<td>( \frac{1}{2} (W + 2Q) )</td>
<td>( \frac{L(W + 2Q)}{2aE} = \frac{WL}{2aE} )</td>
<td></td>
</tr>
<tr>
<td>DE</td>
<td>L</td>
<td>2a</td>
<td>( -(W + 2Q) )</td>
<td>( \frac{2(W + 2Q)L}{2aE} = \frac{WL}{2aE} )</td>
<td></td>
</tr>
</tbody>
</table>

Sum \( \frac{3WL}{2aE} \)

13.5. MAXWELL’S RECIPROCAL THEOREM APPLIED TO FRAMES

As applied to the deflection of articulated structures, Maxwell’s theorem of reciprocal deflection has the following statement:

“\text{In a perfect frame under equilibrium, the deflection of any joint A due to a load at the joint B is equal to the deflection of the joint B due to same load at the joint A}.”
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Expressed mathematically,

$$\delta_A = \frac{1}{AE} \sum P u L$$

where

$$\delta_A = \text{deflection of } A \text{ due to load at } B$$

Thus in Fig. 13.1, let \( W \) be the load at \( B \). According to unit load method, the deflection of the joint \( A \) is given by

$$\delta_A = \frac{1}{AE} \sum \frac{P u L}{1}$$

where

\( P = \text{Stress in any member due to } W \text{ at } B \)

$$u = \text{Stress in any member due to unit load at } A$$

$$u' = \text{Stress in any member due to unit load at } B$$

Now remove the load from \( B \), and apply it at the joint \( A \). Then the deflection of joint \( A \) is given by

$$\delta_A = \frac{1}{AE} \sum \frac{n P' u' L}{1}$$

where

\( P' = \text{Stress in any member due to } W \text{ at } A \)

$$= n W$$

and

$$u' = \text{Stress in any member due to unit load at } B$$

Substituting these values of \( P' \) and \( u' \) in (2), we get

$$\delta_A = \frac{1}{AE} \sum \frac{n L}{1} \left( \frac{P}{W} \right) = \frac{1}{AE} \sum \frac{n P u L}{1}$$

Comparing (1) and (3), we get

$$\delta_A = \delta_B$$

which proves the statement.

**Example 13.14.** Determine the vertical displacement of both lower points \( C \) and \( D \) for the pin jointed frame shown in Fig. 13.15. The cross-sectional area of all members is 130 sq. mm. and the modulus of elasticity is 200 kN/mm². Determine the magnitude of an additional vertical load \( W \) placed at \( D \) necessary to increase the deflection at \( C \) by 50%.

**Solution**

From Castigliano's first theorem, the deflections of \( C \) and \( D \) due to external load is given by

$$\delta_C = \frac{1}{AE} \sum \frac{n P u L}{1} \text{ and } \delta_D = \frac{1}{AE} \sum \frac{n P u L}{1}$$

where

\( P = \text{force in any member due to the load of 9 kN acting at } C \).

$$u = \text{force in any member due to unit load at } C$$

$$u' = \text{force in any member due to unit load at } D$$

The calculations of \( P, u_1 \) and \( u_2 \) are presented in the tabular form below:

### Table

<table>
<thead>
<tr>
<th>Member</th>
<th>( L ) (mm)</th>
<th>( P )</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( P u_1 L )</th>
<th>( P u_2 L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2500( \sqrt{2} )</td>
<td>-6( \sqrt{2} )</td>
<td>-2( \sqrt{2} )</td>
<td>( -\sqrt{2} )</td>
<td>4</td>
<td>+2-828 \times 10^4</td>
</tr>
<tr>
<td>2</td>
<td>2500</td>
<td>-6</td>
<td>-2( \sqrt{2} )</td>
<td>( -\sqrt{2} )</td>
<td>4</td>
<td>+1-0 \times 10^4</td>
</tr>
<tr>
<td>3</td>
<td>2500( \sqrt{2} )</td>
<td>-3( \sqrt{2} )</td>
<td>-1( \sqrt{3} )</td>
<td>( -2\sqrt{2} )</td>
<td>3</td>
<td>+0-707 \times 10^4</td>
</tr>
<tr>
<td>4</td>
<td>2500</td>
<td>+3</td>
<td>+1( \sqrt{2} )</td>
<td>( +\sqrt{2} )</td>
<td>3</td>
<td>+0-250 \times 10^4</td>
</tr>
<tr>
<td>5</td>
<td>2500</td>
<td>+3</td>
<td>+1( \sqrt{2} )</td>
<td>( +\sqrt{2} )</td>
<td>3</td>
<td>+0-250 \times 10^4</td>
</tr>
<tr>
<td>6</td>
<td>2500</td>
<td>+6</td>
<td>+2( \sqrt{3} )</td>
<td>( +\sqrt{3} )</td>
<td>3</td>
<td>+1-0 \times 10^4</td>
</tr>
<tr>
<td>7</td>
<td>2500</td>
<td>+6</td>
<td>+2( \sqrt{3} )</td>
<td>( +\sqrt{3} )</td>
<td>3</td>
<td>+1-0 \times 10^4</td>
</tr>
<tr>
<td>8</td>
<td>2500( \sqrt{2} )</td>
<td>+3( \sqrt{2} )</td>
<td>+1( \sqrt{3} )</td>
<td>( -\sqrt{2} )</td>
<td>4</td>
<td>-0-707 \times 10^4</td>
</tr>
<tr>
<td>9</td>
<td>2500</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Fig. 13.15.**

(+ for tension ; — for compression)

The sum of the entries in the last column gives the total deflection at \( C \).

**Diagram:**

```
Fig. 13.15
```

```
<table>
<thead>
<tr>
<th>( L ) (mm)</th>
<th>( P )</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( P u_1 L )</th>
<th>( P u_2 L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2500</td>
<td>-6</td>
<td>-2( \sqrt{2} )</td>
<td>( -\sqrt{2} )</td>
<td>4</td>
<td>+2-828 \times 10^4</td>
</tr>
<tr>
<td>2500</td>
<td>-6</td>
<td>-2( \sqrt{2} )</td>
<td>( -\sqrt{2} )</td>
<td>4</td>
<td>+1-0 \times 10^4</td>
</tr>
<tr>
<td>2500( \sqrt{2} )</td>
<td>-3( \sqrt{2} )</td>
<td>-1( \sqrt{3} )</td>
<td>( -2\sqrt{2} )</td>
<td>3</td>
<td>+0-707 \times 10^4</td>
</tr>
<tr>
<td>2500</td>
<td>+3</td>
<td>+1( \sqrt{2} )</td>
<td>( +\sqrt{2} )</td>
<td>3</td>
<td>+0-250 \times 10^4</td>
</tr>
<tr>
<td>2500</td>
<td>+3</td>
<td>+1( \sqrt{2} )</td>
<td>( +\sqrt{2} )</td>
<td>3</td>
<td>+0-250 \times 10^4</td>
</tr>
<tr>
<td>2500</td>
<td>+6</td>
<td>+2( \sqrt{3} )</td>
<td>( +\sqrt{3} )</td>
<td>3</td>
<td>+1-0 \times 10^4</td>
</tr>
<tr>
<td>2500</td>
<td>+6</td>
<td>+2( \sqrt{3} )</td>
<td>( +\sqrt{3} )</td>
<td>3</td>
<td>+1-0 \times 10^4</td>
</tr>
<tr>
<td>2500( \sqrt{2} )</td>
<td>+3( \sqrt{2} )</td>
<td>+1( \sqrt{3} )</td>
<td>( -\sqrt{2} )</td>
<td>4</td>
<td>-0-707 \times 10^4</td>
</tr>
<tr>
<td>2500</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
```

Sum: 7-742 \times 10^4 | 4-621 \times 10^4
432

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\[
\delta_C = \sum \frac{P_u L}{AE} = \frac{7.742 \times 10^4}{130 \times 200} = 2.98 \text{ mm} \quad (1)
\]

\[
\delta_D = \sum \frac{P_u L}{AE} = \frac{4.621 \times 10^4}{130 \times 200} = 1.78 \text{ mm} \quad (2)
\]

Let the additional load at D be W.
We want \( \frac{2.98}{2} = 1.49 \) mm.

According to Maxwell's reciprocal theorem
\[
c\delta_D = d\delta_C
\]
Hence \( L = 1.49 \) mm

When a load of 9 kN is at C, \( \delta_D = 1.78 \) mm

Hence when a load of W is at C, \( c\delta_D = \frac{1.78}{9} W \)

Equating (3) and (4), we get
\[
\frac{1.78}{9} W = 1.49
\]
or
\[
W = \frac{1.49 \times 9}{1.78} = 7.53 \text{ kN.}
\]

13.6. GRAPHICAL METHOD

(a) Williot Diagram

In the graphical method, the extension or the contraction of each member is first calculated by Hooke's law (i.e., \( \Delta = \frac{PL}{AE} \)).

These stretching or shortening of the members are then plotted to get the position of the joints and the deflections.

![Diagram](image)

Fig. 13.16.

To start with, let us take the simple triangular truss of example 13.1, which carries a point load of 6 tonnes at the point C. The members \( AC \) and \( BC \) carry a tensile force of 5 kN each, and \( AB \)
carries a compressive force of 4 kN. Hence the extensions or contractions of various members are:

\[
\Delta_AC = \frac{5 \times 1000 \times 5000}{150 \times 2 \times 10^6} = 0.834 \text{ mm (i.e., extension)}
\]

and

\[
\Delta_AB = \frac{4 \times 1000 \times 8000}{100 \times 2 \times 10^6} = -1.6 \text{ mm (i.e., contraction)}
\]

The deformed shape of the frame can now be plotted by using the changed lengths as the sides and plotting the triangle \( ABC \).

However, since \( \Delta AC \) and \( \Delta AB \) are extremely small, the deformed shape of the frame will practically coincide with the original shape, and thus accurate measurements for the deflection of the joint \( C \) cannot be made. For this reason, bigger scale is used for plotting the changes in the lengths. For the truss shown in Fig. 13.16 (b), the point A is position fixed. Also the movement of the roller at B is horizontal, and hence the direction \( AB \) is also fixed. Thus, when \( AB \) carries compression under the given system of loading, \( B \) will move towards \( A \), by an amount \( BB' = \Delta AB \). This contraction \( \Delta AB (=BB') \) has been marked on \( BA \), to the left of \( B \), on a bigger scale than that of the original diagram. From \( B' \), a line \( B'P \) is drawn parallel to \( BC \) and equal in length to \( BC \).

Now on \( AC \), \( CC \) is made equal to \( \Delta AC \) (extension) on the enlarged scale. Similarly on \( B'P \). \( PC \) is made equal to \( \Delta AC \) (extension). Then by striking arcs from centres \( A \) and \( B' \) with radii \( AC \) and \( B'C \) respectively, the intersection gives \( C' \), the new position of point \( C \). For every small changes in lengths, the angle \( CC'C \) and \( PC'C \) are right angles. Hence right angles are set off at \( C \) and \( C' \), intersecting at \( C' \). We thus get the figure \( PCC'C \), which is of much larger scale than that of the original diagram \( ABC \). \( C' \) is the final position of the joint \( C \), and the perpendicular distance of \( C' \) from \( PC \) gives the vertical deflection of the joint \( C \).

In the case of bigger frames, it is convenient to draw the deflection diagram (such as \( PCC'C \)), separately, as shown in Fig. 13.16 (b). Point \( A \) being fixed in position, it is chosen as the reference point, and the direction \( AB \) is the reference direction. Thus, in Fig. 13.16 (b), point \( A' \) is first chosen, and point \( B' \) is marked to the left of \( A' \) (since point \( B \) moves to the left relative to \( A \)), and by a magnitude \( A'B' = \Delta AB \). Line \( A'B' \) is evidently parallel to the direction \( AB \). Since \( AC \) extends, point \( C \) moves to the right relative to \( A \), and in the direction \( AC \). Hence the extension \( AC \)
The deflection of joint \( C' \), when scaled, comes out to be 2.5 mm. The diagram is known as the Williot diagram.

(b) Williot-Mohr Diagram

In the case illustrated above, the reference point \( A \) was fixed in position and the reference direction \( AB \) was also fixed. In some cases, however, there may not be even a single member which is either fixed in direction or carries zero stress. In such a case, the Williot diagram is first plotted with reference to the direction of any member. The necessary correction (known as Mohr's correction) is then made to the diagram. The final diagram so obtained is known as the Williot-Mohr diagram. The procedure for plotting the Williot-Mohr diagram is described in example 13:16.

Example 13.15. Solve example 13:9 by graphical method.

Solution.

In this frame, both points \( A \) and \( D \) are position fixed, and hence the direction \( AD \) is fixed. Thus there is no change in the length \( AD \) and \( \Delta AD = 0 \).

To draw the Williot diagram (Fig. 13:17(b)); points \( A' \) and \( D' \) coincide. To find the position \( B' \) (of the joint \( B \)), \( A'B_1 \) drawn parallel to \( AB \) and equal to \( \Delta AB \). Since \( \Delta AB \) is positive (i.e., extension), \( B \) moves away from \( A \), in the direction \( A'B_1 \). Similarly, since \( \Delta AB \) is negative (i.e., contraction), \( B \) moves towards \( D \). Hence \( D'B_2 \) is drawn parallel to \( BC \), in the direction \( B \) to \( D \), and equal to \( \Delta BD \). Perpendiculaxes are drawn at \( B_1 \) and \( B_2 \) to meet at \( B' \). Thus, \( B' \) is the location of the joint \( B \) relative to joints \( A \) and \( D \).

Similarly, to locate the position \( C' \) (of the joint \( C \)), draw \( B'C_1 \) parallel to \( BC \) (since \( \Delta BC \) is extension and \( C \) moves away from \( B \) in the direction \( B \) to \( C \)), and equal to \( \Delta BC \). Similarly draw \( D'C_2 \) parallel to \( CD \) (since \( \Delta CD \) is contraction and \( C \) moves towards \( D \) in the direction \( C \) to \( D \), and equal to \( \Delta CD \). Perpendiculaxes are then drawn at \( C_1 \) and \( C_2 \) to meet at \( C' \). \( C' \) is then the location of the joint \( C \) relative to joints \( A \) and \( D \).

From the Williot diagram (Fig. 13:17(b)), the vertical deflection of \( C = \delta CV = 0.78\) mm, the horizontal deflection of \( C = \delta H = 0.63\) mm and the vertical distance between \( A' \) and \( C' = 0.78\) mm and the horizontal distance between \( A' \) and \( C' = 0.63\) mm.
Example 13.16. The members of a Warren truss are subjected to the changes (in mm) in length, shown against each member in Fig. 13.18 due to a certain loading.

Draw the Williot-Mohr diagram, and find the vertical deflection of the joint C.

Solution.

\[ \text{Fig. 13.18.} \]

In this truss, joint A is position fixed, but neither AB nor AE is direction fixed. Also, there is no such member which carries zero stress, to act as a reference member. Hence, to start with, AB is arbitrarily chosen as the reference member, fixed in direction, and the Williot diagram is constructed as in the previous example. Correction is then applied to the Williot diagram, by constructing on it the Mohr's diagram. The combined diagram, known as the Williot-Mohr's diagram, gives the deflection of various joints.

To construct the Williot diagram, point A' is chosen, and A'B' is drawn parallel to AB, and equal to \( \Delta AB \). Since AB extends, point B moves to the right of A, and hence B' is drawn to the right of A'. Now, to plot E', the deflected position of point E relative to A'B', draw B'E', parallel to BE, and equal to \( \Delta BE \). Since B'E' is extended, E moves away from B, in direction B'E', and hence E is drawn along the direction B to E. Similarly, A'E is drawn parallel to EA, and equal to \( \Delta EA \). Since AE carries compression, E moves towards A in downward direction, and hence E is drawn from A' in the downward direction. At E and E', perpendiculars are drawn to meet at E', thus giving the deflected position of the joint E relative to AB. Similarly, deflected positions of point F, C, G, and D are plotted at F', C', G' and D' respectively, as shown in Fig. 13.19.

From the Williot diagram so obtained, the point D' is the deflected position of D relative to AB, and hence the vertical distance between A' and D' gives vertical deflection of D relative to A. Actually, since D is supported on rollers, both A and D are at the same elevation, and the vertical deflection of D relative to A must be zero. This discrepancy has crept due to the fact that the Williot diagram has been plotted with reference to the original axis of the member AB. Actually, AB is not fixed, and it rotates about the joint A.
The necessary correction is applied through the Mohr's diagram. The deformed truss should be rotated about the hinge \( A \), in clockwise direction, in such a way that the joint \( D \) comes in level with the joint \( A \). The vertical displacement given to the joint \( D \) is equal to the height of \( D' \) above \( A' \). The horizontal component of the displacement represented by \( D' \) still remains. Due to the rotation of the truss, other joints will also be proportionately displaced.

To draw the correction diagram (i.e., Mohr's diagram), draw \( D'D' \) parallel to \( BA \), to meet the vertical line through \( A' \) in \( D' \). The line \( A'D' \) represents the lower chord of the truss. The other joints of the lower chord, \( B \) and \( C \), will lie on \( A'D' \). Hence, on the base \( AD \) construct the figure \( A' B' C' D' G' F' E' \) similar to that of the truss \( ABCDGE \), as shown by chain-dotted lines in Fig. 13-19. The figure \( A' B' C' D' G' F' E' \) is known as the Mohr's diagram, and must be constructed on such side of the line \( A'D' \) that after rotating it by 90°, it becomes parallel and similar to the original truss diagram. In the combined diagram, the following points must be carefully noted.

(i) The firm lines represent the deformation of various members.
(ii) The dotted lines represent the perpendicular drawn to locate the deflected position of the joints.
(iii) The letters with 'dash' (i.e., \( B' \), \( C' \) etc.) represent the deflected position of the corresponding joints, relative to \( A \), taking \( AP \) as the fixed reference direction.
(iv) The displacement of various joints \( B, C, D, E, F, G \) are represented by the vectors \( B'B' \), \( C'C' \), \( D'D' \), \( E'E' \), \( F'F' \), and \( G'G' \). Thus the vertical displacement of the joint \( D \) is the vertical component of the vector \( D'D' \), and is evidently zero. The horizontal displacement of \( D \) is equal to the horizontal component of the vector \( D'D' \), and is evidently equal to the length \( D'D' \).

From the Williot-Mohr diagram, 
\[ \delta_{CV} = \text{vertical component of vector } C'C' = 10.5 \text{ mm.} \]

**PROBLEMS**

1. Fig. 13-21 shows a jointed plane frame hinged to a rigid wall at \( C \) and \( D \), and carrying a vertical load \( W \) at \( A \). The area of each tension member is \( 'a' \) and that of each compression member is \( '2a' \). The length \( AD \) is.

2. A crane structure is shown in Fig. 13-21. The length of the member \( AD \) is \( 2L \) and all other members are of length \( L \). The cross-sectional area of \( AD \) is \( 2a' \) and that of all other members is \( 'a' \). Determine the horizontal and vertical deflection of the joint \( F \) due to a vertical load \( W \) at that joint.

3. A pin jointed structure shown in Fig. 13-23 is pinned to an abutment at \( J \) and rests on rollers at \( G \). A load \( W \) is applied horizontally at \( D \). Determine the horizontal movement of \( D \), if the area of all tension members is \( 'a' \) and that of all compression members \( '2a' \). The panels are all equilateral and of side length \( L \).

4. Determine the vertical deflection of the load in the structure shown in Fig. 13-23. The tension members are stressed to \( 124 \text{ N/mm}^2 \) and the compression members are stressed to \( 92 \text{ N/mm}^2 \). \( E=2.01 \times 10^5 \text{ N/mm}^2 \).
5. The frame shown in Fig. 13:24 carries load which produce deformations (in mm) as shown against each of the members. A positive sign denoting an extension. Determine, preferably by Williot diagram, the vertical and horizontal displacement of $F$.

6. Fig. 13:25 shows a small pin-jointed frame which is hinged at $B$ and supported by a roller at $C$. Bar $BC$ is horizontal and the lengths of the members are:

- $AB = 30$ cm
- $BC = 50$ cm
- $AC = 45$ cm

The lengths of the bars are adjustable. Find the vertical and horizontal movements of $A$ due to the following changes in lengths.

- $AB : +2.5$ mm
- $AC : -1.0$ mm
- $BC : -1.5$ mm

7. Determine the horizontal deflection of the roller support $C$ of the frame shown in Fig. 13:26 due to applied load of 8 tonnes at $B$. Members $AB$, $BC$ and $BD$ are each of 800 sq. mm area and $AD$ and $CD$ are each of 1600 sq. mm area. $E = 2.06 \times 10^6$ N/mm$^2$.

8. The truss shown in Fig. 13:27 carries vertical loading uniformly divided between the panel points of the lower chord while the cross-section of the members are such that all loaded ties are stressed to 135 N/mm$^2$ and all struts to 90 N/mm$^2$ under the loading. All joints are pin-joints and the value of $E$ for the material of the truss is $2.02 \times 10^6$ N/mm$^2$. Find the vertical deflection of the point $A$.

---

**Answers**

1. $\delta_Y = 2.57 \frac{WL}{aE}$; $\delta_H = 0.29 \frac{WL}{aE}$.
2. $\delta_Y = 5.67 \frac{WL}{3aE}$; $\delta_H = 4.04 \frac{WL}{aE}$.
3. $\Delta = 12.88 \frac{WL}{aE}$.
4. 14.8 mm.
5. 24.2 mm; 14.7 mm.
6. $\delta_Y = 0.212$; $\delta_H = 0.133 \rightarrow$
7. $\delta_H = 16.35$ mm.
8. 8.33 mm.
Redundant Frames

14. DEGREE OF REDUNDANCY

A frame is said to be statically indeterminate when the number of unknown reactions or stress components exceed the total number of condition equations of equilibrium. If the number of unknowns are equal to the number of condition equations available, the frame is said to be a perfect frame. The total degree of indeterminacy or redundancy of a frame is therefore equal to the number by which the unknowns (i.e., reaction components as well as stress components) exceed the condition equations of equilibrium. The excess restraints or members are described as redundant.

To find the total degree of indeterminacy, the structure may be, somewhat arbitrarily classified as statically indeterminate externally, internally or both. In case of externally redundant structures, there are redundant reactive restraints. The degree of external indeterminacy or redundancy is given by

\[ E = R - r \]  

(14'1)

where

- \( E \) = external redundancy,
- \( R \) = total number of reaction components
  (one for a roller, two for a hinge, and three for a fixed support),
- \( r \) = total number of condition equations available
  (i.e., minimum number of reaction components required for the stability of the frame).

The structure is said to be internally indeterminate if it has redundant members, and are therefore over stiff. The degree of internal redundancy (\( I \)) is given by

\[ I = m - (2j - r) \]  

(14'2)

where

- \( I \) = degree of internal redundancy
- \( m \) = total number of members
- \( j \) = total number of joints.
- \( r \) = minimum number of reaction components required for the stability of the structure.

A structure may be redundant both internally as well as externally (i.e., it has redundant reaction components as well as redundant members). In this case, the total redundancy (\( T \)) is given by:

\[ T = E + I = (R - r) + m - 2j + r = R + m - 2j \]

\[ = m - (2j - R) \]  

(14'3)

Difference between equations 14'2 and 14'3 must be carefully noticed. In equation 14'2, \( r \) is the minimum number of reaction components required for the stability of the structure, while in equation 14'3, \( R \) is the actual number of the reaction components present in the structure.

14.2. APPLICATION OF CASTIGLIANO'S THEOREM OF MINIMUM STRAIN ENERGY

The principle of Least Work is a statement of the practical fact that if an elastic structure is in a state of stable equilibrium under any forces whatsoever, then the work stored is smallest amount possible. The theorem of minimum strain energy can, therefore, be used for analysing the redundant frames. To use this method, the redundant members are replaced by the unknown forces (\( T_1, T_2 \) etc.) acting at the joints. The statically determinate system which results from the removal of the redundant members is called the base or principal or perfect system. Then, by Castigliano's theorem of minimum strain energy, we get

\[ \frac{\partial U}{\partial T_1} = 0, \quad \frac{\partial U}{\partial T_2} = 0, \text{ etc.} \]

where \( U \) is the total strain energy (inclusive of that in the redundant members) of the frame. The number of equations will be the same as the number of unknowns.

As an illustration, consider a frame shown in Fig. 14'1 (a). The total number of reaction components \( R = (2+1) = 3 \): minimum reaction components required is \( r = 3 \) (from statical equilibrium).

Hence \( E = R - r = 3 - 3 = 0 \), and the frame is externally determinate.

Now \( m = 8 \) and \( j = 5 \); \( r = 3 \)
Hence \( I = m - (2j - r) = 8 - (2 \times 5 - 3) = 1 \)

Thus frame is internally indeterminate to single degree i.e., it has one redundant member. Considering member AB as the redundant, replace it by a force \( T \) at the corners B and C, as shown in Fig. 14.1 (b). The base system is thus obtained, and stresses in various members can be calculated in terms of external loads and the redundant force \( T \). If \( U \) is the total strain energy stored in the frame (inclusive of that in BC), we have

\[
\frac{\partial U}{\partial T} = 0
\]

Now strain energy in any member carrying axial force is \( \frac{PL}{2AE} \)

Hence

\[
U = \sum \frac{PL}{2AE}
\]

and

\[
\frac{\partial U}{\partial T} = \sum \frac{P}{AE} L = 0
\]  

(14.4)

The force \( P \) in any member will be a function of the external load \( W \) and the redundant force \( T \), i.e.

\[
P = aT + bW
\]

\[
\frac{\partial P}{\partial T} = a
\]

where \( a \) may be zero for some remote members.

For the frame of Fig. 14.1(c), \( m = 11 \), and \( j = 6 \), \( r = 3 \)

\[
I = 11 - (6 \times 2 - 3) = 2
\]

Hence the frame is redundant to second degree, i.e., it has two redundant members. Choosing \( AB \) and \( CD \) as the redundant members they can be replaced by the forces \( T_1 \) and \( T_2 \) at the appropriate corners to obtain the base system as shown in Fig. 14.1(d). The forces \( T \) in the various members can now be calculated. In general, \( F = uT + sW \). If \( U \) is the total strain energy stored in the system (inclusive of that in the redundant members), we have, from the theorem of minimum strain energy:

\[
\frac{\partial U}{\partial T} = \sum \frac{P}{AE} L = 0
\]

and

\[
T = \sum \frac{P}{AE} L = 0
\]  

(14.5) (a)

(14.5) (b)

The simultaneous solution of equations 14.5(a) and 14.5(b) gives the values of the redundant forces \( T_1 \) and \( T_2 \) in the members \( AB \) and \( CD \) respectively. If the value of \( F \) or \( T_1 \) or \( T_2 \) comes out to be negative, the actual force in the redundant member will be of the reverse sign, i.e., if the original assumption is a compressive force designated by \( -T_1 \) at the joints, the actual force will be tensile, and vice versa.

Examples will now follow to illustrate the application of the principle of minimum strain energy.

**Example 14.1.** Find the force in the member BC of the frame loaded as shown in Fig. 14.2. All the members have the same cross-sectional area.

**Solution**

The frame is redundant to single degree, since \( I = m - (2j - r) = 6 - (2 \times 4 - 3) = 1 \). Treating BC as the redundant, and assuming that it carries a tensile force \( T \), apply forces \( T \) at joints B and C as shown, and remove the member. The forces in various members can now be found as under:

\[
\sin \theta = \frac{3}{5} = 0.6 \quad \cos \theta = \frac{4}{5} = 0.8
\]

At B, resolving vertically,

\[ P_{BD} = T \sin \theta = 0.6T \text{ (comp.)} \]

At A, resolving horizontally

\[ P_{AB} = T \cos \theta = 0.8T \text{ (comp.)} \]

Resolving horizontally,

\[ P_{AB} = \frac{1}{\cos \theta} (10 - P_{AB}) = 12.5 \text{ (comp.)} \]

Resolving vertically,

\[ P_{AC} = P_{AB} \sin \theta = 0.6(12.5 - T) \]

**Fig. 14.2.**
STRENGTH OF MATERIALS AND THEORY OF STRUCTURES

Treating member \( AC \) to be redundant, replace it with tensile force \( T \) at the joints \( A \) and \( C \) as shown.

\[
\frac{\partial U}{\partial T} = \frac{n}{1} \frac{\partial P}{\partial T} \cdot \frac{L}{AE} = 0
\]

Since \( A \) and \( E \) are the same for all the members, we have

\[
\frac{\partial U}{\partial T} = \frac{n}{1} \frac{\partial P}{\partial T} \cdot L = 0
\]

Calculation of stresses in the members.

Let the length of the side of the square = \( L \):

Lenth of the diagonal = \( L\sqrt{2} \).

Resolving horizontally at \( A \),

\( P_{AB} = P_{AD} \)

Resolving vertically,

\[ 2P_{AB} \cos \theta + T = 10 \]

or

\[ P_{AB} = P_{AD} = \frac{\sqrt{2}}{2} (10 - T) = \frac{1}{\sqrt{2}} (10 - T) \], tension

Reaction at \( C = 10 \) kN ↑. Hence \( P_{BC} = P_{CD} = \frac{1}{\sqrt{2}} (10 - T) \), tension.

Resolving at \( B \),

\[ P_{BD} = P_{BC} \cos \theta + P_{AB} \cos \theta \]

\[
= \frac{2}{\sqrt{2}} \left\{ \frac{1}{\sqrt{2}} (10 - T) \right\}
\]

\[ = 10 - T, \text{ compression} \]

The result may now be tabulated as shown on next page from the table we get

\[
\frac{n}{1} \frac{\partial P}{\partial T} \cdot L = (4 \cdot 828 T - 34 \cdot 14) L = 0
\]

which gives \( T = +7.07 \) kN.

The + sign indicates that the sign of the assumed stress in \( AC \) is correct (i.e., it carries tension). The value of \( T \) can now be substituted in column (3), and the stresses computed as shown in column (6).
Example 14.3. In the frame work shown in Fig. 14.4, the member AB, BC and CA have area of cross-section ‘2a’ and the member DA, DB and DC have area of cross-section ‘a’. Find the force in the member DA due to a load of 10 tonnes applied horizontally at A.

Solution

\[ m = 6; j = 4; r = 3 \]

\[ I = 6 - (2 \times 4 - 3) = 1 \]

Hence the frame is indeterminate to single degree. Treating AD as the redundant, replace it by tensile force \( T \) at the joints A and D. Then

\[ \frac{\partial U}{\partial T} = \sum_{1}^{n} \frac{\partial P}{\partial T} \cdot \frac{L}{A} = 0 \]

\( E \) being same for all members).

At the joint D, resolving horizontally, \( P_{DB} = P_{DC} \)

Resolving vertically,

\[ P_{BD} \cos 45^\circ + P_{DC} \cos 45^\circ = T \]

\[ \therefore P_{BD} = P_{DC} = \frac{T}{\sqrt{2}} \] (tension).

At the joint A, resolving vertically, \( P_{AB} \sin \theta = T + P_{AC} \sin \theta \) \( \ldots (1) \)

Resolving horizontally,

\[ P_{AB} \cos \theta + P_{AC} \cos \theta = 10 \] (2)

where

\[ \sin \theta = \frac{3}{\sqrt{10}} = 0.948; \]

\[ \cos \theta = \frac{1}{\sqrt{10}} = 0.316 \]

Solving (1) and (2), we get

\[ P_{AB} = 0.527 \cdot T + 15.85 \] (comp.)

\[ P_{AC} = 15.85 - 0.527 \cdot T \] (tension)

Resolving horizontally at B,

\[ P_{BC} = P_{AB} \cos \theta - P_{BD} \cos 45^\circ \]

\[ = (0.527 \cdot T + 15.85) \cdot 0.948 - \frac{T}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \]

\[ = -0.334 \cdot T + 5 \] (tension)

The values of \( P \) and \( \frac{\partial P}{\partial T} \) etc. are tabulated below:

\( (+ \text{ for tension} ; - \text{for compression}) \)
Example 14.4. The frame shown in Fig. 14.5 is pin-jointed to a rigid support at A and B and the joints C and D are also pinned. The diagonals AD and BC act independently and the members are all of the same cross-section and material. ABC and BCD are equilateral triangles.

Initially there is no load in any of the members which may be assumed weightless.

If a load of 5 kN is hung at D, calculate the forces in all of the members.

Solution

Both the hinges are essential for the equilibrium of the frame,

Hence \( m = 5; j = 4 \)

Thus the frame is indeterminate to single degree.

Treating AD as the redundant member it may be replaced by compressive force \( R \) at the joints A and D as shown. The stresses in various members can now be calculated as under:

![Diagram](image)

Fig. 14.5.

At the joint D

Resolving vertically, \( P_{BD} = \frac{\sqrt{3}}{2} R = 5 \)

or \( P_{BD} = \frac{10 + R}{\sqrt{3}} \) (tension)

Resolving horizontally, \( P_{CD} = P_{BD} \cos 60^\circ - R \cos 30^\circ \)

Thus

\[
\frac{10 + R}{\sqrt{3}} \times \frac{1}{2} - \frac{\sqrt{3}}{2} R
\]

and

\[
\frac{5 - R}{\sqrt{3}} \text{ (compression)}
\]

At the joint C

Since members CA and CB are equally inclined to vertical,

\( P_{CB} = P_{CA} \)

Resolving horizontally, \( 2P_{AC} \cos 60^\circ = P_{CD} = \frac{5 - R}{\sqrt{3}} \)

Thus

\( P_{AC} = \frac{5 - R}{\sqrt{3}} \) (compression)

and

\( P_{BD} = \frac{5 - R}{\sqrt{3}} \) (tension)

The calculation for \( P = \frac{dU}{dR} \cdot L \) is done in the tabular form below:

<table>
<thead>
<tr>
<th>Member</th>
<th>Length</th>
<th>( P )</th>
<th>( \frac{dP}{dR} )</th>
<th>( P \cdot \frac{dP}{dR} \cdot L )</th>
<th>Final Force</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC</td>
<td>( L )</td>
<td>( \frac{5 - R}{\sqrt{3}} )</td>
<td>( \frac{1}{\sqrt{3}} )</td>
<td>( \frac{L}{3} (R - 5) )</td>
<td>(-2.57 )</td>
</tr>
<tr>
<td>BC</td>
<td>( L )</td>
<td>( \frac{5 - R}{\sqrt{3}} )</td>
<td>( -\frac{1}{\sqrt{3}} )</td>
<td>( \frac{L}{3} (R - 5) )</td>
<td>(+2.57 )</td>
</tr>
<tr>
<td>CD</td>
<td>( L )</td>
<td>( \frac{5 - R}{\sqrt{3}} )</td>
<td>( \frac{1}{\sqrt{3}} )</td>
<td>( \frac{L}{3} (R - 5) )</td>
<td>(-2.57 )</td>
</tr>
<tr>
<td>DA</td>
<td>( \sqrt{3}L )</td>
<td>( -R )</td>
<td>(-1 \sqrt{3} RL )</td>
<td>(-0.545 )</td>
<td></td>
</tr>
<tr>
<td>BD</td>
<td>( L )</td>
<td>( \frac{10 + R}{\sqrt{3}} )</td>
<td>( \frac{1}{\sqrt{3}} )</td>
<td>( \frac{L}{3} (10 - R) )</td>
<td>(+6.07 )</td>
</tr>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td></td>
<td>( \frac{L}{3} (9.19 R - 5) )</td>
<td></td>
</tr>
</tbody>
</table>

\[ \sum P \frac{dP}{dR} \frac{L}{A} = 0 = \frac{L}{3} (9.19 R - 5) \]

which gives \( R = 0.545 \) kN (compression).
Substituting the value of \( R \) in column 3, the force in each member can be calculated, and tabulated as shown in column (6).

**Example 14.5.** The pin jointed frame shown in Fig. 14.6 is of the shape of a regular hexagon. All the members have the same area of cross-section. Calculate the force in the member CH.

**Solution**

There are 3 reaction components necessary for the equilibrium of the whole frame, and hence \( r = 3 \).

\[
m = 12 - (7 \times 2 - 3) = 1
\]

Hence the frame is indeterminate to single degree. Treating member CH to be redundant, replace it by tensile force \( T \) at the joints C and H as shown.

**At the joint C**

Resolving vertically, \( P_{CD} = T \) (compression)

Resolving horizontally, \( P_{CB} = W - P_{CD} \cos 60^\circ - T \cos 60^\circ = (W - T) \), tension

**At the joint B**

Resolving vertically, \( P_{BA} = P_{BH} \)

Resolving horizontally, \( P = P_{BA} \cos 60^\circ + P_{BH} \cos 60^\circ = P_{BC} = W - T \)

\[ P_{BA} = (W - T), \text{ tension} \]

\[ P_{BH} = (W - T), \text{ compression} \]

**At the joint D**

Resolving vertically \( P_{DF} = P_{CD} = T \) (comp.)

Resolving horizontally, \( P_{DH} = 2P_{CD} \cos 60^\circ = T \) (tension)

**At the joint F**

Since there is roller at \( F \), it can take only the vertical reaction \( V_F \). Thus, at the hinge \( A \), horizontal reaction is \( H_A = W \). Let \( V_A \) and \( V_F \) be the vertical reactions at \( A \) and \( F \) respectively. Taking moments about \( A \), we get

\[
V_F \left( L + \frac{L}{2} \right) = \frac{WL}{2} + W \cdot \frac{L}{2} \cdot \frac{3}{2} \]

From which \( V_F = 0.91 W \)

\[ V_A = W - 0.91 W = 0.09 W \]

Now resolving vertically at \( F \),

\[ P_{FH} \cos 30^\circ + P_{FD} \cos 30^\circ = V_F = 0.91 W \]

\[ P_{FH} = 0.91 W \times \frac{2}{\sqrt{3}} - T = (1.05 W - T) \text{ compression.} \]

Resolving horizontally,

\[ P_{FG} = P_{DF} \cos 60^\circ - P_{FH} \cos 60^\circ = \frac{T}{2} - \frac{1}{2} (1.05 W - T) \]

\[ = (T - 0.525 W), \text{ compression.} \]

**At the joint G**

Resolving horizontally,

\[ P_{GA} \cos 60^\circ + P_{GF} = P_{GH} \cos 60^\circ \]

or \[ \frac{1}{2} P_{GA} + (T - 0.525 W) = \frac{1}{2} (1.155 W - P_{GA}) \]

From which \( P_{GA} = (1.103 W - T) \), tension.

Resolving vertically,

\[ P_{GH} + P_{GA} = W \sec 30^\circ = 1.155 W \]

\[ P_{GH} = 1.155 W - (1.103 W - T) \]

\[ = (0.052 W + T), \text{ tension.} \]

**At the joint A**

Resolving horizontally,

\[ P_{AH} + W = P_{AH} \cos 60^\circ + P_{AG} \cos 60^\circ \]

\[ P_{AH} = \frac{1}{2} (W - T) + \frac{1}{2} (1.103 W - T) - W \]

\[ = (0.052 W - T), \text{ compression.} \]

The summation may now be carried out in the tabular form below. \( \frac{L}{AE} \) is same for all the members, and has not been included in the table.

\[
\frac{\partial U}{\partial T} = \sum \frac{\partial P}{\partial T} \cdot \frac{L}{AE} = 0
\]

or

\[ 12 \frac{T - 5.678 W}{0} \]

or

\[ T = 0.437 W \text{ (tension).} \]
Example 14.6. Find the axial force in the member BC of the frame shown in Fig. 14.7. The figures in brackets indicate the cross-sectional area in cm². The members are all of the same material.

<table>
<thead>
<tr>
<th>Member</th>
<th>$P$</th>
<th>$P \overline{T}$</th>
<th>$P \overline{T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>$(W-T)$</td>
<td>-1</td>
<td>$T-W$</td>
</tr>
<tr>
<td>BC</td>
<td>$(W-T)$</td>
<td>-1</td>
<td>$T-W$</td>
</tr>
<tr>
<td>GD</td>
<td>$-T$</td>
<td>-1</td>
<td>$T$</td>
</tr>
<tr>
<td>DF</td>
<td>$-T$</td>
<td>-1</td>
<td>$T$</td>
</tr>
<tr>
<td>FG</td>
<td>$(T-0.525W)$</td>
<td>-1</td>
<td>$T-0.525W$</td>
</tr>
<tr>
<td>GA</td>
<td>$(1.103W-T)$</td>
<td>-1</td>
<td>$T-0.103W$</td>
</tr>
<tr>
<td>AH</td>
<td>$(0.052W-T)$</td>
<td>-1</td>
<td>$T-0.052W$</td>
</tr>
<tr>
<td>BH</td>
<td>$(W-T)$</td>
<td>-1</td>
<td>$T$</td>
</tr>
<tr>
<td>CH</td>
<td>$+T$</td>
<td>+1</td>
<td>$T$</td>
</tr>
<tr>
<td>DB</td>
<td>$+T$</td>
<td>+1</td>
<td>$T$</td>
</tr>
<tr>
<td>FH</td>
<td>$(T-0.05W-T)$</td>
<td>-1</td>
<td>$T-0.05W$</td>
</tr>
<tr>
<td>GH</td>
<td>$(0.052W+T)$</td>
<td>-1</td>
<td>$T+0.052W$</td>
</tr>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td>$12T-5.678W$</td>
</tr>
</tbody>
</table>

Solution

There are 3 reaction components necessary for the equilibrium of the whole frame, and hence $r=3$

$m=8$; $j=5$

$l=8-(5+2-3)=1$

Hence the frame is indeterminate to first degree. Treating member BC to be redundant, replace it by a compressive force $R$ at the joints B and C as shown.

$\sin \theta = \frac{200}{\sqrt{(200)^2+(150)^2}} = \frac{200}{250} = 0.8$

$\cos \theta = \frac{150}{250} = 0.6$

At the joint $E$

$P_{BE} = W \csc \theta = 1.25W$ (tension)

$P_{DE} = P_{BE} \cos \theta = 1.25W \cdot 0.6 = 0.75W$ (comp.)

At the joint $B$

Resolving vertically,

$P_{BD} = (1.25W \cdot 0.8) - (R \cdot 0.8) = W - 0.8R$ (comp.)

Resolving horizontally

$P_{AB} = (1.25W \cdot 0.6) + (R \cdot 0.6) = 0.75W + 0.6R$ (tension)

At the joint $D$

Resolving vertically,

$P_{AD} = \frac{1}{0.8} \times P_{BD} = 1.25W - R$ (tension)

Resolving horizontally,

$P_{CD} = 0.73W + 0.6(1.25W - R) = 1.5W - 0.6R$ (comp.)

At the joint $C$

Since there is a roller at C, vertical reaction is zero. Hence, resolving vertically,

$P_{IC} = R \times 0.8 = 0.8R$ (tension)

The result may now be tabulated as below.

Now

$\frac{\partial U}{\partial R} = \sum P \frac{\partial P}{\partial R} = \frac{L}{AE} = 0$

:. From the table

$\frac{1}{a}(837R - 472.5W) = 0$

From which $R = 0.564W$ (compression).
Example 14.7. Find the force in the member AC of the frame shown in Fig. 14.8. The quantity AE is constant for all the members.

Solution
A close study of the frame will reveal that all the three hinged supports are essential for the stability of the frame. Thus the reaction component necessary for the equilibrium of the frame is

\[ r = 3 \times 2 = 6 \]

Thus the frame is indeterminate to single degree. Considering the member AC to be redundant, replace it by tensile forces \( T \) at the joints A and C, as shown.

\[
\begin{align*}
\Sigma P &= 0 \\
\Sigma M_A &= 0 \\
\Sigma M_C &= 0
\end{align*}
\]

Resolving horizontally at A,
\[ P_{AB} \cos 45^\circ + P_{AD} \cos 60^\circ = 1000 \]  
(1)

Resolving vertically at A,
\[ P_{AB} \cos 45^\circ + T = P_{AD} \sin 60^\circ \]

From (1) and (2), we get
\[ P_{AB} = 896 - 0.516 T \text { (tension)} \]
\[ P_{AD} = 0.733 T + 733 \text { (comp.)} \]

For the whole frame,
\[ \frac{\partial U}{\partial T} = 0 = n \sum \frac{AP}{AR} \cdot \frac{L}{AE} \]

The summation may be carried in the tabular form below:

<table>
<thead>
<tr>
<th>Member</th>
<th>L (cm)</th>
<th>P</th>
<th>( \frac{AP}{AR} )</th>
<th>( \frac{L}{AE} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>2\sqrt{2}L</td>
<td>896 - 0.516 ( T )</td>
<td>-0.516</td>
<td>(-656 + 0.576 ( T )) ( L )</td>
</tr>
<tr>
<td>AD</td>
<td>( \frac{2}{3}L )</td>
<td>-0.733 (T + 733)</td>
<td>-0.733</td>
<td>(0.620 ( T + 620 )) ( T )</td>
</tr>
<tr>
<td>AC</td>
<td>L</td>
<td>+T</td>
<td>+1</td>
<td>( T )</td>
</tr>
</tbody>
</table>

Sum

\[ 1996T - 36 = 0 \]

or

\[ T = 18 \text{ N (tension)} \]

Example 14.8. The bars of the pin jointed frame shown in Fig. 14.9 are of the same material and have the same cross-sectional area. Show that the forces in AB and CD are compressive and tensile respectively of magnitude \( \frac{W}{2} \left( \frac{1+\sqrt{2}}{3+4\sqrt{2}} \right) \).

Solution

\[
\begin{align*}
\text{Resolving horizontally at } A, \\
P_{AB} \cos 45^\circ + P_{AD} \cos 60^\circ &= 1000 \\
\text{Resolving vertically at } A, \\
P_{AB} \cos 45^\circ + T &= P_{AD} \sin 60^\circ
\end{align*}
\]

From (1) and (2), we get
\[ P_{AB} = 896 - 0.516 T \text { (tension)} \]
\[ P_{AD} = 0.733 T + 733 \text { (comp.)} \]

For the whole frame,
\[ \frac{\partial U}{\partial T} = 0 = n \sum \frac{AP}{AR} \cdot \frac{L}{AE} \]

The summation may be carried in the tabular form below:

<table>
<thead>
<tr>
<th>Member</th>
<th>L</th>
<th>P</th>
<th>( \frac{AP}{AR} )</th>
<th>( \frac{L}{AE} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>2\sqrt{2}L</td>
<td>896 - 0.516 ( T )</td>
<td>-0.516</td>
<td>(-656 + 0.576 ( T )) ( L )</td>
</tr>
<tr>
<td>AD</td>
<td>( \frac{2}{3}L )</td>
<td>-0.733 (T + 733)</td>
<td>-0.733</td>
<td>(0.620 ( T + 620 )) ( T )</td>
</tr>
<tr>
<td>AC</td>
<td>L</td>
<td>+T</td>
<td>+1</td>
<td>( T )</td>
</tr>
</tbody>
</table>

Sum

\[ 1996T - 36 = 0 \]

or

\[ T = 18 \text{ N (tension)} \]
From the frame, \( \begin{align*} r &= 3 \\ m &= 15 \\ j &= 8 \\ l &= 15 - (2 \times 8 - 3) = 2. \end{align*} \)

Hence the frame is indeterminate to second degree. Treating \( AB \) and \( CD \) as redundants, replace them with the forces, \( R_1 \) and \( R_2 \) at the joints \( (A, B) \) and \( (C, D) \) respectively, as shown. It is assumed that \( AB \) carries a compressive force \( (R_1) \) and \( CD \) carries a tensile force \( (R_2) \).

Reaction at \( E = \frac{2W \times 3}{4} = \frac{3}{2} W \) \( \uparrow \).

Reaction at \( H = \frac{2W \times 1}{4} = \frac{W}{2} \) \( \uparrow \).

At the joint \( E \)

\[ P_{EA} = \frac{3}{2} W \cos 45^\circ = \frac{3}{2} \sqrt{2} W \text{ (comp.)} \]

\[ P_{EB} = P_{EA} \cos 45^\circ = \frac{3}{2} W \times \frac{1}{\sqrt{2}} = \frac{3}{2} W \text{ (tension)} \]

At the joint \( A \)

Resolving vertically

\[ P_{AF} = \frac{W}{\sqrt{2}} = \frac{W}{2} + R_1 - 2W \]

\[ P_{AF} = \frac{3}{\sqrt{2}} W + \sqrt{2} R_1 - 2\sqrt{2} W \]

\[ = \left( R_1 - \frac{W}{2} \right) \sqrt{2} \text{ (tension)} \]

Resolving horizontally,

\[ P_{AG} = \frac{3}{\sqrt{2}} W \times \frac{1}{\sqrt{2}} + \left( R_1 - \frac{W}{2} \right) \sqrt{2} = (W + R_1) \text{ (comp.)} \]

At the joint \( B \)

\[ P_{BG} = P_{AB} \cos 45^\circ = \sqrt{2} R_1 \text{ (tension)} \]

and \( P_{BF} = P_{EB} - P_{BG} \cos 45^\circ = \frac{3}{2} W - \sqrt{2} R_1 \frac{1}{\sqrt{2}} = \left( \frac{3}{2} W - R_1 \right) \text{ (tension)} \)

At the joint \( H \)

\[ P_{CH} = \frac{W}{2} \cos 45^\circ = \frac{W}{2} \sqrt{2} \text{ (compression)} \]

\[ P_{DH} = P_{CH} \cos 45^\circ = \frac{W}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{W}{2} \text{ (tension)} \]

The summation is carried out in tabular form on next page.

From the Table,

\[ \Sigma \frac{\partial P}{\partial R_1} \cdot L = (4 + 4\sqrt{2}) R_1 - R_2 - \frac{W}{2} (1 + 2\sqrt{2}) = 0 \] (1)

and

\[ \Sigma P = \frac{\partial P}{\partial R_1} \cdot L = (4 + 4\sqrt{2}) R_1 - R_2 - \frac{W}{2} (1 + 2\sqrt{2}) = 0 \] (2)
The simultaneous solution of (1) and (2) gives

\[ R_1 = \frac{W}{2} \left( \frac{1+2\sqrt{2}}{3+4\sqrt{2}} \right), \] compression

and

\[ R_2 = \frac{W}{2} \left( \frac{1+2\sqrt{2}}{3+4\sqrt{2}} \right), \] tension.

<table>
<thead>
<tr>
<th>Member</th>
<th>( L )</th>
<th>( P )</th>
<th>( \frac{2P}{3R_1} )</th>
<th>( \frac{3P}{3R_1} )</th>
<th>( \frac{2P}{3R_1} \times L )</th>
<th>( \frac{3P}{3R_1} \times L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>EA</td>
<td>( \sqrt{2} )</td>
<td>( \frac{3}{\sqrt{2}} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>AG</td>
<td>1</td>
<td>( W+R_1 )</td>
<td>-1</td>
<td>0</td>
<td>( W+R_1 )</td>
<td>0</td>
</tr>
<tr>
<td>GC</td>
<td>2</td>
<td>( W-R_4 )</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>-( W+R_2 )</td>
</tr>
<tr>
<td>CH</td>
<td>( \sqrt{2} )</td>
<td>( \frac{W}{\sqrt{2}} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>EB</td>
<td>1</td>
<td>( \frac{3}{2} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>BF</td>
<td>1</td>
<td>( \left( \frac{3}{2} \right) - R_1 )</td>
<td>-1</td>
<td>0</td>
<td>-( \frac{3}{2} ) ( W+R_1 )</td>
<td>0</td>
</tr>
<tr>
<td>FD</td>
<td>1</td>
<td>( \left( \frac{W}{2} + R_4 \right) )</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>( \frac{W}{2} + R_2 )</td>
</tr>
<tr>
<td>DH</td>
<td>1</td>
<td>( \frac{W}{2} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>AB</td>
<td>1</td>
<td>( -R_1 )</td>
<td>-1</td>
<td>0</td>
<td>( R_1 )</td>
<td>0</td>
</tr>
<tr>
<td>BG</td>
<td>( \sqrt{2} )</td>
<td>( \sqrt{2} R_4 )</td>
<td>+( \sqrt{2} )</td>
<td>0</td>
<td>2( \sqrt{2} R_1 )</td>
<td>0</td>
</tr>
<tr>
<td>AF</td>
<td>( \sqrt{2} )</td>
<td>( \left( \frac{R_1 - \frac{W}{2}}{\sqrt{2}} \right) \sqrt{2} )</td>
<td>+( \sqrt{2} )</td>
<td>0</td>
<td>2( \sqrt{2} R_1 ) - ( \sqrt{2} W )</td>
<td>0</td>
</tr>
<tr>
<td>GF</td>
<td>1</td>
<td>( R_1 - R_3 )</td>
<td>-1</td>
<td>+1</td>
<td>( R_1 - R_3 )</td>
<td>( R_1 - R_3 )</td>
</tr>
<tr>
<td>GD</td>
<td>( \sqrt{2} )</td>
<td>( -\sqrt{2} R_1 )</td>
<td>0</td>
<td>-( \sqrt{2} )</td>
<td>0</td>
<td>( 2\sqrt{2} R_1 )</td>
</tr>
<tr>
<td>FC</td>
<td>( \sqrt{2} )</td>
<td>( \frac{W}{\sqrt{2}} - \sqrt{2} R_1 )</td>
<td>0</td>
<td>-( \sqrt{2} )</td>
<td>0</td>
<td>( \sqrt{2} R_2 )</td>
</tr>
<tr>
<td>CD</td>
<td>1</td>
<td>( R_1 )</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>( R_1 )</td>
</tr>
</tbody>
</table>

**Example 14.9.** Find the forces in all the members of the frame shown in Fig. 14.10. All the bars are of same area of cross-section and are of same material.

**Solution**

All the four hinged supports are necessary for the equilibrium of the frame. If any of the hinged support is replaced with a roller support, the frame becomes unstable.

Hence \( r = 4 \times 2 = 8 \)
\( m = 4 \)
\( f = 5 \)
\( I = 4 - (2 \times 5 - 8) = 2 \)

Thus the frame is indeterminate to second degree. Considering \( AC \) and \( AD \) as redundants, replace them with forces \( R_1 \) and \( R_2 \) at \( A \). Considering \( AC \) carries a tensile force \( R_1 \) and \( AD \) carries a compressive force \( R_2 \).

Consider the equilibrium of the joint \( A \).

Resolving horizontally, we get
\( P_{AB} \cos 45^\circ + R_1 \cos 75^\circ + R_2 \cos 75^\circ + P_{AE} \cos 45^\circ = 10 \)
\( 0.707 P_{AB} + 0.259 (R_1 + R_2) + 0.707 P_{AE} = 10 \) \hspace{1cm} (1)

Resolving vertically, we get
\( P_{AB} \sin 45^\circ + R_1 \sin 75^\circ + P_{AE} \sin 45^\circ + R_2 \sin 75^\circ = 0 \)
\( 0.707 P_{AB} + 0.966 (R_1 - R_2) - 0.707 P_{AE} = 0 \) \hspace{1cm} (2)

Solving (1) and (2), we get
\( P_{AB} = 0.5 R_3 - 0.866 R_1 + 7.07 \) (tension)
\( P_{AE} = 0.5 R_3 - 0.866 R_2 + 7.07 \) (compression)

Let the height of the frame = \( L \)

: Length of \( AB \) and \( AE = L \) cosec \( 45^\circ = 1.414 L \)
Length of \( AC \) and \( AD = L \) cosec \( 75^\circ = 1.035 L \)
The result may now he tabulated.

<table>
<thead>
<tr>
<th>Member</th>
<th>L</th>
<th>P</th>
<th>(P_{\frac{\partial P}{\partial R}})</th>
<th>(P_{\frac{\partial P}{\partial R}})</th>
<th>(P_{\frac{\partial P}{\partial R}})</th>
<th>(P_{\frac{\partial P}{\partial R}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>1:414 L</td>
<td>0.5 R, 0.612 R, 0</td>
<td>0.612 R, 0.612 R, 0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>AC</td>
<td>1:033 L</td>
<td>+R, 0.612 R, 0.612 R, 0</td>
<td>0.612 R, 0.612 R, 0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>AD</td>
<td>1:033 L</td>
<td>0, 0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>AE</td>
<td>1:414 L</td>
<td>0.612 R, 0.612 R, 0</td>
<td>0.612 R, 0.612 R, 0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
\frac{\partial U}{\partial R} = 0 = \sum_{n=1}^{P_{\frac{\partial P}{\partial R}}} L = (2.449 R_1 - 1.224 R_2 - 3.66) L \quad (1) \\
\frac{\partial U}{\partial R} = 0 = \sum_{n=1}^{P_{\frac{\partial P}{\partial R}}} L = (2.449 R_2 - 1.224 R_1 - 3.66) L \quad (2)
\]

Solving (1) and (2), we get:

- \(R_1 = 3 \text{ kN} \) (tension)
- \(R_2 = 3 \text{ kN} \) (comp.)

Hence, \(P_{AB} = 6 \text{ kN} \) (tension)
- \(P_{AE} = 9 \text{ kN} \) (comp.)

### 14.3. Maxwell's Method

The stresses in redundant frames can also be evaluated by the method given by Clerk Maxwell.

Fig. 14'11 (a) shows a redundant frame. Treating \(AC\) as the redundant, replace it by tensile forces \(T\) at \(A\) and \(C\), as shown in Fig. 14'11 (b).

![Fig. 14'11](image)

Due to the application of forces \(T\), the joints \(A\) and \(C\) will move towards each other, thus shortening the length \(AC\).

Let \(P\) = Force in any member due to external load acting on the perfect frame obtained by removing the redundant member.

\(u\) = Force in any member due to unit loads at \(A\) and \(C\).

\(uT\) = Force in any member due to force \(T\) at \(A\) and \(C\).

Hence the total force in any member, due to external loading as well as force \(T\) at \(A\) and \(C\) is \((P + uT)\).

Hence from Eq. 13'1, the deflection of joints \(A\) and \(C\) towards each other is:

\[
\frac{\sum_{n=1}^{P_{\frac{\partial P}{\partial R}}} (P + uT)uL}{AE} = \frac{uTL'}{AE'}
\]

(The summation being done for all members except \(AC\)).

Since member \(AC\) carries tensile force \(T\), it is elongated by the amount \(TL'\), where \(L', A', E'\) stand for \(AC\).

Since the deformation of the member \(AC\) is consistent with the movement of joints \(A\) and \(C\), we get

\[
\frac{\sum_{n=1}^{P_{\frac{\partial P}{\partial R}}} (P + uT)uL}{AE} = \frac{uTL'}{AE'}
\]

Thus \(T\) can be found by the solution of the above equation.

Rewriting the above equation, we get

\[
\frac{\sum_{n=1}^{P_{\frac{\partial P}{\partial R}}} PuL}{AE} + \frac{\sum_{n=1}^{P_{\frac{\partial P}{\partial R}}} u^2TL'}{AE} = \frac{TL'}{AE'}
\]

In the above expression, \(u\) is the force in any member due to unit load at the joints \(A\) and \(C\). Hence, for the member \(AC\) itself, \(u = 1\). Therefore the term \(\frac{uTL'}{AE'}\) can also be written as \(\frac{TTL'}{AE'}\).

Thus, we get

\[
\frac{\sum_{n=1}^{P_{\frac{\partial P}{\partial R}}} PuL}{AE} + \frac{\sum_{n=1}^{P_{\frac{\partial P}{\partial R}}} u^2TL'}{AE} = \frac{TTL'}{AE'}
\]

or

\[
\frac{\sum_{n=1}^{P_{\frac{\partial P}{\partial R}}} PuL}{AE} = \frac{TTL'}{AE'} - \frac{u^2L}{AE}
\]

or

\[
\frac{\sum_{n=1}^{P_{\frac{\partial P}{\partial R}}} PuL}{AE} = \frac{TTL'}{AE'} - T \frac{u^2L}{AE}
\]
From which \[ T = - \frac{n-1}{1} \sum_{i=1}^{n} \frac{P_i L_i}{A E} \]

(14.7)

(where \( n \) is the total number of members)

Notes:

1. In the above equation, \( \Sigma \) is the summation for the whole frame except the redundant member, while \( \sum \) is the summation for all members including the redundant member.

(2) Comparing Maxwell's method with that of Castigliano's method, it will be seen that in the Maxwell's method the stress in each member has to be calculated twice, first due to the external loads (in the absence of the redundant member) and second by unit force applied at the ends of the redundant member.

(3) Equations 14.6 and 14.7 are valid only for tensile force in the redundant members. This is an important point to note. Thus to start with, the redundant member is assumed to carry tensile force.

If at the end of the solution, a positive sign is obtained with the numerical value of \( T \), it will be tensile, as assumed earlier. If negative sign is obtained with the numerical value of \( T \), it will be compressive.

Procedure:

1. Remove the redundant member completely.
2. Find the force \( P \) in each member due to external loading.
3. Apply unit load (\( \rightarrow \), \( \rightarrow \)) at the end joints of redundant member, and find unit force (\( u_i \)) in each member.
4. Apply Eq. (14.6) or (14.7) to find the force \( T \).

Frames with two or more redundant members

Let the frame be redundant to second degree. Remove the two redundant members, thus making the frame perfect.

Let \( P \) = Force in any member due to external loading, obtained after making the frame 'perfect' by removing the redundant members.

\( u_i \) = Force in any member due to unit pulls at the joints of the first redundant member.

\( \theta \) = Force in any member due to unit pulls at the joints of the second redundant member.

\[ \theta = \text{Force in any member due to unit pulls at the joints of the redundant member} \]

Hence the force in any member \( \theta = (P + u_i T_i + u_i T_i) \)

Movement of joints (towards each other)

\[ \sum_{i=1}^{n-2} (P + u_i T_i + u_i T_i) = \frac{T_i L'}{A E} \]

(14.8)

The extension of the first redundant bar \( \theta = \frac{T_i L'}{A E} \)

Since the movement of the joints is consistent with the deformation of the redundant member, we get

\[ \sum_{i=1}^{n-2} (P + u_i T_i + u_i T_i) = \frac{T_i L'}{A E} \]

(14.8a)

Similarly, for the second redundant member.

\[ \sum_{i=1}^{n-2} (P + u_i T_i + u_i T_i) = \frac{T_i L''}{A E} \]

(14.8b)

The simultaneous solution of Eqs. 14.8(a) and 14.8(b) will give the values of the redundant forces \( T_i \) and \( T_j \).

Example 14.10. Solve example 14.1 by Maxwell's method.

Solution. (Ref. Fig. 14'12).

(a) BC REMOVED

(b) UNIT LOAD

Fig. 14'12.

(a) Calculation of \( P \) : \( \sin 60° = 0.83, \cos 60° = 0.8 \)

Remove the redundant member, as shown in Fig. 14'12(a).

Since there is no external load at the joint \( B \),

\[ P_{AB} = 0 \]

\[ P_{BD} = 0 \]

\[ u_i = \text{Force in any member due to unit pulls at the joints of the redundant member} \]
Resolving horizontally at joint A,

\[ P_{AB} \cos \theta = 10 \]

\[ P_{AD} = \frac{10}{0.8} = 12.5 \text{ kN (comp.)} \]

Resolving vertically at joint A,

\[ P_{AC} = P_{AD} \sin \theta = 12.5 \times 0.6 = 7.5 \text{ kN (tension)} \]

\[ P_{CD} = \text{horizontal reaction at } C = 10 \text{ kN (tension)} \]

(Alternately, \( P_{CD} = P_{AD} \cos \theta = 12.5 \times 0.8 = 10 \text{ kN} \))

(b) Calculation of \( u \) [Fig. 14.12 (b)]: Apply unit loads at C and D.

Reaction at C and D will be zero since unit loads are equal and opposite.

Resolving at joint C,

\[ u_{AC} = \sin \theta = 0.6 \text{ (comp.)} \]

\[ u_{CD} = \cos \theta = 0.8 \text{ (comp.)} \]

Resolving at the joint A,

\[ u_{AD} = \frac{0.6}{0.6} = 1 \text{ (tension)} \]

Resolving at the joint B,

\[ u_{AB} = 1 \cos \theta = 0.8 \text{ (comp.)} \]

\[ u_{BD} = 1 \sin \theta = 0.6 \text{ (comp.)} \]

The summation may be carried out in the tabular form below:

<table>
<thead>
<tr>
<th>Member</th>
<th>( L ) (m)</th>
<th>( P )</th>
<th>( u )</th>
<th>( PuL )</th>
<th>( u^2L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( AB )</td>
<td>4</td>
<td>0</td>
<td>-0.8</td>
<td>0</td>
<td>2.56</td>
</tr>
<tr>
<td>( AC )</td>
<td>3</td>
<td>+7.5</td>
<td>-0.6</td>
<td>-13.5</td>
<td>1.08</td>
</tr>
<tr>
<td>( CD )</td>
<td>4</td>
<td>+10.0</td>
<td>-0.8</td>
<td>-32.0</td>
<td>2.56</td>
</tr>
<tr>
<td>( BD )</td>
<td>3</td>
<td>0</td>
<td>-0.6</td>
<td>0</td>
<td>1.08</td>
</tr>
<tr>
<td>( AD )</td>
<td>5</td>
<td>-12.5</td>
<td>+1.0</td>
<td>-62.5</td>
<td>5.00</td>
</tr>
<tr>
<td>( BC )</td>
<td>5</td>
<td>-</td>
<td>+1.0</td>
<td>-</td>
<td>5.00</td>
</tr>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td>-108.0</td>
<td></td>
<td>17.28</td>
</tr>
</tbody>
</table>

The redundant frames are determined from equation 14.7,

\[ \sum_{n=1}^{n-1} \frac{PuL}{AE} \]

\[ T = \frac{1}{\sum_{n=1}^{n-1} \frac{u^2L}{AE}} \]

\[ = \frac{-108}{17.28} = -6.25 \text{ kN (tension)} \]

(Plus sign indicates that the redundant member carries tension.)

Example 14.11. In the plane braced frame work shown in Fig. 14.13 (a) all the members have the same cross sectional area and are made of the same material. Determine the forces in all bars when loads \( W \) are applied as shown.

(U.L.)

Solution
Let us treat \( FE \) as the redundant member, carrying a tensile force \( T \). Since the frame is indeterminate to single degree, we have

\[ \sum_{n=1}^{n-1} \frac{PuL}{AE} = \frac{\sum_{n=1}^{n-1} u^2L}{AE} \]

\[ T = \frac{1}{\sum_{n=1}^{n-1} \frac{u^2L}{AE}} \]

(1)

(Since \( AE \) is constant for all members.)

(a) Calculation of \( P \)
To calculate \( P \) in each member due to external loading, remove the redundant member \( FE \), and make the frame statically determinate (or perfect), as shown in Fig. 14.13 (b).

By inspection, all the inclined members and diagonals are inclined at 45° to the horizontal.

At the joint \( A : P_{AF} = P_{AB} \)

and \[ 2P_{AF} \frac{1}{\sqrt{2}} = W \]

\[ : \quad P_{AF} = P_{BD} = \frac{W}{\sqrt{2}} = \frac{W}{\sqrt{2}} \text{ (tension)} \]

\[ : \quad P_{AB} = P_{CD} = \frac{W}{\sqrt{2}} \text{ (comp)} \]

At the joint \( F : P_{FC} = P_{AF} = \frac{W}{\sqrt{2}} \text{ (tension)} = P_{FB} \)

\[ P_{FB} = 2P_{FA} \frac{1}{\sqrt{2}} = 2 \cdot \frac{W}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = W \text{ (comp.)} \]

\[ P_{EC} = W \text{ (comp.), by symmetry.} \]
At the joint B.

\[ P_{BC} = (P_{AB} + P_{BC}) \frac{1}{\sqrt{2}} = \left( \frac{W}{\sqrt{2}} + \frac{W}{\sqrt{2}} \right) \frac{1}{\sqrt{2}} = W \text{ (comp.)} \]

(b) Calculation of \( u \)

Remove the external load and apply unit pulls at \( F \) and \( E \), as shown in Fig. 14.13(c). Since the applied unit pulls are equal and opposite, and act along the same direction, the reactions at \( B \) and \( C \) are each zero.

Since no external load is acting at joints \( A \) and \( D \),

\( u_{AF} = 0; \quad u_{AB} = 0; \quad u_{AD} = 0; \quad u_{CD} = 0. \)

At the joint \( F \),

\[ u_{FC} = 1/\sqrt{2} = \sqrt{2} \text{ (comp.)} = u_{BE} \]

\[ u_{FB} = u_{FC} \times \frac{1}{\sqrt{2}} = 1 \text{ (tension)} = u_{CE} \]

At the joint \( R \),

\[ u_{BC} = u_{BE} \times \frac{1}{\sqrt{2}} = 1 \text{ (tension)} \]

The result may now be tabulated below.

From the Table

\[ T = -\frac{1}{n} \sum_{i=1}^{n} \frac{P_i L_i}{u_i} \]

\[ T = -\frac{1}{n} \frac{11.636W_{a}}{19.312a} = +0.6035 W \]

14.4. STRESSES DUE TO ERROR IN LENGTH

If any one member of a redundant frame has lack of fit, stresses will be induced in all the members of the redundant frame when that member is forced in position. Let a member of redundant frame be short in length by an amount \( \lambda \). When this member is forced into position, it will exert pull \( T \) at the joints (or ends) of the member. Thus, the two joints will have a tendency to move towards each other while the member (having lack of fit) will be subjected to a tensile force \( T \). In the equilibrium position, by compatibility of deformation, we have:

\[ \text{INWARD MOVEMENT OF JOINTS} + \text{EXTENSION OF THE MEMBER} = 0. \]
From chapter 13, the inward movement of the joints due to a force \( F \) acting at the joints is:

\[
 u = \sum_{i=1}^{n-1} \frac{(uF)uL}{AE}
\]

Where \( u \) = force in any member due to unit pulls at the two joints

\[
P = \text{force in any member due to pulls} \ F \ \text{at the two joints}
\]

Also the extension of the member having lack of fit, due to the tensile force:

\[
T = \frac{TL'}{AE}
\]

(Where \( L' \) and \( A' \) stand for that member)

Substituting the values in (1), we get:

\[
\frac{n-1}{1} \frac{(uF)uL}{AE} + \frac{TL'}{AE} = \lambda
\]

As in the previous article, writing \( TL' \frac{AE}{AE} = u^2TL' \frac{AE}{AE} \), we get:

\[
\frac{n-1}{1} \frac{(uF)uL}{AE} + \frac{u^2TL'}{AE} = \lambda
\]

or

\[
\frac{n}{1} \frac{u^2TL}{AE} = \lambda
\]

Hence:

\[
T = + \frac{\lambda}{\frac{n}{1} \frac{u^2L}{AE}}
\]

(14.9)

where \( \lambda \) is taken to be positive if the member is short in length (so as to exert pull \( T \) at the joints), and negative if the member is excess in length (so as to apply push at the joints).

Analysis by Castigliano's Theorem:

The stresses in members of a redundant frame due to lack of fit of a member can also be found by Castigliano's theorem as under:

Let the member carry a force \( T \) when forced in position. All the members of frame will be strained, and the partial derivative of the total strain energy with respect to the force \( T \) will, according to Castigliano's second theorem, be equal to the lack of fit.

\[
\frac{\partial U}{\partial T} = \sum_{i=1}^{n} P \frac{\partial P}{\partial T} \frac{L}{AE} = \lambda
\]

Example 14'12. Find the forces in all the members of the frame shown in Fig. 14'14(a), if the member BC is short in length by 10 mm and is forced into position. Take \( E = 2 \times 10^5 \) N/mm². All members have same area of cross-section of 100 mm².

Solution

[Diagram of the frame is shown with labels for joints and members.]
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(For tension: — for compression)

<table>
<thead>
<tr>
<th>Member</th>
<th>$L$ (mm)</th>
<th>$u$</th>
<th>$uL$</th>
<th>Actual stress $P=uT$ (kN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>4000</td>
<td>-0.8</td>
<td>2560</td>
<td>-9.26</td>
</tr>
<tr>
<td>$AC$</td>
<td>3000</td>
<td>-0.6</td>
<td>1080</td>
<td>-6.94</td>
</tr>
<tr>
<td>$CD$</td>
<td>4000</td>
<td>-0.8</td>
<td>2560</td>
<td>-9.26</td>
</tr>
<tr>
<td>$BD$</td>
<td>3000</td>
<td>-0.6</td>
<td>1080</td>
<td>-6.94</td>
</tr>
<tr>
<td>$AD$</td>
<td>5000</td>
<td>+1.0</td>
<td>5000</td>
<td>+11.57</td>
</tr>
<tr>
<td>$BC$</td>
<td>5000</td>
<td>+1.0</td>
<td>5000</td>
<td>+11.57</td>
</tr>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td>17280</td>
<td></td>
</tr>
</tbody>
</table>

4.5. COMBINED STRESSES DUE TO EXTERNAL LOAD AND ERROR IN LENGTH

Let us now discuss the case of combined stresses in the members of a redundant frame having error in length of one member and subjected to external loads. The actual or final stress in any member can be found by analysing the two effects separately and then superimposing them. However, the labour can be very much reduced by computing the stresses in one operation by Castigliano's second theorem as discussed below.

Let a member of the redundant frame be short in length by an amount $\lambda$.

By Castigliano's second theorem, if $U$ is total strain energy stored in the frame, both due to forcing the member in position, as well as due to external load, we have

$$\frac{\partial U}{\partial T} - \frac{\partial P}{\partial T} \frac{L}{AE} = \lambda$$

where

$T$ = tensile force in the member having lack of fit.
$\lambda$ = amount by which the member is short.
$P$ = actual force in any member (= $F+uT$).
$u =$ stress in any member due to unit pulls at the joints (or ends) of the member having lack of fit.
$uT =$ stress in any member due to pull $T$ at the joints (or ends) of the member having lack of fit.
$F =$ stress in any member due to external loads alone when the member (having lack of fit) is not put in position (i.e., when the frame is perfect one).

Now

$$P = F + uT$$

Substituting in (1), we get

$$\sum_{i=1}^{n} (F+uT)u \frac{L}{AE} = \lambda$$

or

$$\frac{nF_uL}{AE} + \sum_{i=1}^{n} \frac{u^2TL}{AE} = \lambda$$

(14.10)

The above equation can also be expressed in the form:

$$T = \frac{-\sum_{i=1}^{n} \frac{F_uL}{AE}}{\sum_{i=1}^{n} \frac{u^2L}{AE}}$$

(14.11)

Special Cases:

1. If the member does not have and lack of fit, i.e., when $\lambda = 0$, we get

$$T = -\frac{\sum_{i=1}^{n} \frac{F_uL}{AE}}{\sum_{i=1}^{n} \frac{u^2L}{AE}}$$

If should be noted that $F$ is to be found when the redundant member is removed. Hence $F$ for the redundant member is nil and $\frac{nF_uL}{AE}$ is the same as $\sum_{i=1}^{n-1} \frac{F_uL}{AE}$ Thus the above equation can also be
The above equation is the same as equation 14.7 derived earlier.

(2) If there is no external load, but there is lack of fit \( \lambda \) (short), we have \( F = 0 \) in each member, and hence \( \sum F = 0 \). Thus

\[
T = \frac{\sum \frac{n F u L}{AE}}{\sum \frac{n u^2 L}{AE}}
\]  

[14.11(b)]

The above equation is the same as equation 14.9 derived earlier.

Note: In all these equations, \( \lambda \) is taken positive when the member is short in length and negative when it is excess in length.

Procedure for Computation

(1) Remove the members having lack of fit, and calculate \( F_1 \), \( F_2 \), etc. in the member, due to external loading.

(2) Remove the external load and apply unit pulls at the ends of the redundant member (i.e., member having lack of fit), and calculate \( u_1 \) and \( u_2 \), etc. in members.

(3) Calculate \( T \) from equation 14.10 after substituting the various quantities with their proper algebraic sign, i.e.

(i) Tensile force is positive and compressive force negative.

(ii) \( \lambda \) is positive if the member is short in length and negative if excess in length.

**Example 14.13.** In the pin-jointed framework shown in Fig. 14.15(a), all the members have the same cross sectional area = 2 in\(^2\), and \( E = 1300 \) t/in\(^2\). The support at A is hinged and it may be assumed that E is supported on a roller. During construction, member EB was made \( \frac{1}{8} \) in. too long and was forced into place.

Determine the resultant force in members EB and AD when the frame supports the 8 ton load as shown.  

\[ (A.M.I. \ Struct. \ E) \]
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\[ \lambda = - \frac{1}{8} = 1.125 \] (since \( BE \) is excess in length).

\[ A = 2 \text{ in}^2. \]

\[ E = 13000 \text{ t/in}^2. \]

\[ \Sigma F_L = + (201 \times 12) \text{ ton}^3 \text{-in}. \]

\[ \Sigma M_L = + (34.6 \times 12) \text{ ton}^3 \text{-in}. \]

Substituting the values in equation 14.10, we get

\[ \frac{(201 \times 12)}{2 \times 13000} + \frac{34.6 \times 12}{2 \times 13000} = -0.125 \]

or

\[ (201 + 34.6) = -271 \]

From which \( T = \frac{472}{34.6} = -13.6 \) tons

Hence force in \( EB = 13.6 \) tons (compression)

Force in \( AD = (F + uT) \)

\[ = +13.3 + (10)(-13.6) = -0.3 \]

\[ = 0.3 \text{ ton (compression)}. \]

14.6. EXTERNALLY INDETERMINATE FRAMES

Structures may be somewhat arbitrarily classified as statically indeterminate externally, internally or both. Externally statically indeterminate structures are those which have redundant reaction restraints. The degree of external indeterminacy is given by the expression,

\[ E = R - r \]

where

- \( E \) = degree of external indeterminacy.
- \( R \) = total number of reaction-components (one for a roller, two for a hinge and three for fixed support).
- \( r \) = total number of reaction components actually necessary for the stability of the structure (or total number of condition equations available).

Solution by Castigliano's Theorem:

If the structure has one redundant reaction component (say \( H \)) we have, from Castigliano's theorem of minimum strain energy,

\[ \frac{\partial U}{\partial H} = 0 = \sum \frac{\partial P}{\partial H} \cdot \frac{L}{AE} \]

(14.12)

If, however, the frame has two redundant reaction components \( H_1 \) and \( H_2 \), we have

\[ \frac{\partial U}{\partial H_1} = 0 = \sum \frac{\partial P}{\partial H_1} \cdot \frac{L}{AE} \]

(14.13a)

and

\[ \frac{\partial U}{\partial H_2} = 0 = \sum \frac{\partial P}{\partial H_2} \cdot \frac{L}{AE} \]

(14.13b)

where \( n \) is the total number of members in the frame.

To calculate the value of \( P \) in the members the redundant reaction restraint is removed, and in its place an external force \( (H) \) is applied in the appropriate direction.

Solution by Maxwell's Method

The redundant reaction \( (H) \) is given by the equation

\[ H = -\frac{1}{AE} \sum \frac{P \cdot L}{AE} \]

(14.14)

where

- \( H \) = redundant reaction component
- \( P \) = force in any member due to external loading, after removing the redundant reaction and making the structure statically determinate
- \( u \) = force in any member due to unit force applied at the support in the direction of the redundant reaction.

The actual force in any member will be equal to \( (P + uH) \) and can be calculated after knowing \( H \).

Yielding of support:

If the support yields by an amount \( \lambda \) in the direction of \( H \), we have, by Castigliano's theorem,

\[ \frac{\partial U}{\partial H} = 0 = \sum \frac{\partial P}{\partial H} \cdot \frac{L}{AE} \]

(14.15)

and

\[ H = \frac{\sum \frac{P \cdot L}{AE}}{\frac{\lambda}{AE}} \]

by Maxwell's method. (14.16)

Example 14.14. Determine the reaction at \( B \), taking it as a redundant, and forces in the members of the truss shown in Fig. 14.16(a).

The value of \( \frac{L}{AE} \) is constant for all members.

Solution

The total number of reaction components (\( R \)) = \( 2 + 1 + 1 = 4 \).
Actual reaction components necessary for the stability of the frame = \(2 + 1 = 3\) (i.e., even if the roller at B is removed, the frame will be stable).

Thus, \(E = R - r = 4 - 3 = 1\)

i.e. the frame is externally indeterminate to single degree.

To get the values of \(P\) in all the members, make the frame statically determinate by removing the redundant reaction at B, shown in Fig. 14.16(b).

To get the values of \(u\) in all the members, apply unit load at B in the direction as the reaction as shown in Fig. 14.16(c).

\((b)\) Calculation of \(P\) [Fig. 14.16(b)]

By inspection, all the inclined members are at 45° to the horizontal. Hence \(\sin \theta = \cos \theta = \frac{1}{\sqrt{2}}\).

Since there is no vertical load at B, 

\[P_{BC} = 0; \quad P_{AB} = 0.\]

Hence 

\[P_{CF} = 0; \quad P_{GC} = 0; \quad P_{DG} = 0\]

At the joint G, \(P_{GF} = W\) (comp.)

At the joint F, \(P_{DF} = P_{AF} \frac{1}{\sqrt{2}} = \frac{W}{\sqrt{2}}\) (tension)

\[P_{FA} = P_{GF} \frac{1}{\sqrt{2}} = \frac{W}{\sqrt{2}}\] (comp.)

At the joint A, \(P_{AD} = P_{AF} \frac{1}{\sqrt{2}} = \frac{W}{\sqrt{2}}\) (tension)

At the joint B, \(P_{BC} = 1\) (comp.)

At the joint C, \(P_{CF} = P_{BC} \tan \theta = \sqrt{2}\) (tension)

\[P_{CG} = P_{CF} \cos \theta = \frac{1}{\sqrt{2}}\] (tension.)

At the joint G, \(P_{DG} = P_{CG} = 1\) (comp.)

At the joint F, \(P_{DF} = 0\)

At the joint A, \(P_{AD} = P_{AF} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{W}{\sqrt{2}} = 1\) (comp.)

The calculations may be done in the tabular form below:

\(+\) for tension; \(-\) for compression

<table>
<thead>
<tr>
<th>Member</th>
<th>(P)</th>
<th>(u)</th>
<th>(P.u)</th>
<th>(u^2)</th>
<th>Actual stress ((P+uV_B))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(AB)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(BC)</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-0.1875 (W)</td>
</tr>
<tr>
<td>(CG)</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-0.1875 (W)</td>
</tr>
<tr>
<td>(GD)</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-0.1875 (W)</td>
</tr>
<tr>
<td>(DA)</td>
<td>+(W/2)</td>
<td>-1</td>
<td>-(W/2)</td>
<td>1</td>
<td>+0.3125 (W)</td>
</tr>
<tr>
<td>(AF)</td>
<td>-(W/\sqrt{2})</td>
<td>+(\sqrt{2})</td>
<td>-(W)</td>
<td>2</td>
<td>-0.442 (W)</td>
</tr>
<tr>
<td>(FG)</td>
<td>0</td>
<td>+(\sqrt{2})</td>
<td>0</td>
<td>2</td>
<td>+0.265 (W)</td>
</tr>
<tr>
<td>(FD)</td>
<td>+(W/\sqrt{2})</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+0.707 (W)</td>
</tr>
</tbody>
</table>

The vertical reaction \(V_B\) at the roller at B is given by

\[V_B = \frac{\Sigma P.u L}{\Sigma P.u^2} \frac{n}{1 + AE} \frac{n}{1 + AE} \quad \text{since} \quad L/\text{AE}\text{is constant.}\]
Hence \( V_B = -\frac{n \sum P_u}{\sum u^2} = -\frac{3W}{8} = -\frac{3}{16}W \)

\( V_B = 0.1875W \) (↑).

The actual stress each member will be \((P + uV_B)\) and has been tabulated in the last column of the table above.

**Example 14.15.** Fig. 14.17 (a) shows a pin joined frame of uniform material supported on a roller at B and hinged at A and C. The ratio of length to cross-sectional area is constant for each member of frame. Find the forces in all members.

**Solution**

![Diagram](image)

\( R = \text{total reaction components} = 2 + 2 + 1 = 5 \)

\( r = \text{Actual reaction components necessary} = 2 + 2 = 4 \)

\( E = R - r = 5 - 4 = 1 \)

Thus the frame is indeterminate to single degree. We shall solve this example by treating member No. 12 as the redundant member carrying a force \( T \) assumed tensile to start with.

Then, by Maxwell's method,

\[
T = -\frac{\sum P_u L}{\sum AE} = -\frac{\sum Pu}{\sum \frac{u^2}{AE}}
\]

\[
\text{(Since } \frac{L}{AE} \text{ is constant)}
\]

\( a) \) Calculation of \( P \)

To calculate \( P \) in all the members due to external loads, remove member No. 12, as shown in Fig. 14.17(b). The stresses \( P \) have been marked against each member in the diagram.

(b) Calculation of \( u \)

To calculate \( u \), remove the external load, and apply unit pulls at the ends (or joints) of the redundant member, as shown in Fig. 14.17(c). The stresses \( u \) have been marked in the diagram.

The values may now be tabulated as under:

\[
T = -\frac{\sum Pu}{\sum \frac{u^2}{AE}} = -\frac{16}{\frac{\sqrt{2}}{73}W} = -2.78W
\]

The minus sign indicates that member No. 12 carries a compressive force of 2.78 \( W \).

The actual stress in any member will be \((P + Tu)\) and have been entered in the last column of the table.
Example 14.16. A pin-jointed frame of uniform material has loads and dimensions as shown in Fig. 14.18(a). The cross-sectional area is 100 mm² for each member of the frame. The truss is hinged to rigid supports at A and B.

Calculate (a) Reactions at A and B.
(b) Horizontal reaction at B if it moves (or yields) by 10 mm horizontally to the right.
(c) The maximum horizontal yielding of the support B to have zero horizontal reaction there.

\[ P_u = \frac{Pu}{AE} \]

\[ H = -\frac{PuL}{AE} \sum \frac{u^2L}{AE} \]

To calculate \( P_u \), make the structure statically determinate by providing a roller support at B, as shown in Fig. 14.18(b), where the stresses due to external loading have been marked.

To calculate \( u \), remove the external loads and apply unit pull at the joint B as shown in Fig. 14.18(c). The stresses have been marked on the diagram.
The computations are done in the tabular form below:

<table>
<thead>
<tr>
<th>Member</th>
<th>Length L (mm)</th>
<th>P (kN)</th>
<th>u</th>
<th>PuL</th>
<th>uL</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC</td>
<td>4470</td>
<td>-8.95</td>
<td>+2.23</td>
<td>-89400</td>
<td>22210</td>
</tr>
<tr>
<td>CD</td>
<td>2000</td>
<td>-6.0</td>
<td>+1.00</td>
<td>-12000</td>
<td>2000</td>
</tr>
<tr>
<td>DE</td>
<td>2000</td>
<td>-6.0</td>
<td>+1.00</td>
<td>-12000</td>
<td>22210</td>
</tr>
<tr>
<td>EB</td>
<td>4470</td>
<td>-8.95</td>
<td>+2.23</td>
<td>-89400</td>
<td>22210</td>
</tr>
<tr>
<td>GH</td>
<td>2000</td>
<td>+4.0</td>
<td>-2.00</td>
<td>-16000</td>
<td>8000</td>
</tr>
<tr>
<td>FG</td>
<td>2000</td>
<td>+4.0</td>
<td>-2.00</td>
<td>-16000</td>
<td>8000</td>
</tr>
<tr>
<td>FA</td>
<td>2830</td>
<td>+5.7</td>
<td>-2.83</td>
<td>-45700</td>
<td>22640</td>
</tr>
<tr>
<td>HB</td>
<td>2830</td>
<td>+5.7</td>
<td>-2.83</td>
<td>-45700</td>
<td>22640</td>
</tr>
<tr>
<td>CF</td>
<td>2000</td>
<td>+4.0</td>
<td>-2.00</td>
<td>-16000</td>
<td>8000</td>
</tr>
<tr>
<td>CG</td>
<td>2830</td>
<td>+2.83</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>DG</td>
<td>2000</td>
<td>-4.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>EG</td>
<td>2830</td>
<td>+2.83</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>EH</td>
<td>2000</td>
<td>+4.0</td>
<td>-2.00</td>
<td>-16000</td>
<td>8000</td>
</tr>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td></td>
<td>-358200</td>
<td>125700</td>
</tr>
</tbody>
</table>

Substituting the values in (1), we get

\[ H = \frac{-358200}{125700} = 2.85 \text{ kN (}) \]

or

\[ R_A = R_B = \sqrt{4^2 + (2.85)^2} = 4.91 \text{ kN}. \]

(b) The general expression for the horizontal reaction at B is

\[ H = \frac{\sum PuL}{AE} \sum uL \] (14.16)

where \( \lambda \) = horizontal movement of B towards A.

For the present case \( \lambda = -10 \text{ mm} \) (since B moves away from A or opposite to \( H \)).

\[ A = 100 \text{ mm}^2 \]
\[ E = 2 \times 10^6 \text{ N/mm}^2 = 200 \text{ kN/mm}^2 \]

(c) For this case \( H = 0 \).

Substituting the values in equation 14.16, we have

\[ H = 0 = \frac{-358200}{100 \times 200} = \frac{125700}{100 \times 200} \]

or

\[ \frac{125700}{100 \times 200} = -17.91 \text{ mm} \]

Horizontal yielding of \( B = 17.91 \text{ mm} \).
By the provision of ties and struts, a trussed beam is a statically indeterminate structure, and can be analysed by Castigliano's theorem of minimum strain energy. Thus, in Fig. 14.19 the beam is statically indeterminate to single degree. Let the thrust (R) in CD be redundant.

Strain energy in the beam due to bending = \( \int_0^L \frac{M^2}{2EI} \, dx \)

Strain energy in all the four members due to axial forces

\( \frac{n \cdot P \cdot L}{2AE} \)

From Castigliano's theorem of minimum strain energy,

\[ \frac{\partial U}{\partial R} = 0 = \int_0^L M \frac{\partial M}{\partial R} \frac{dx}{EI} + \sum P \frac{\partial P}{\partial R} \frac{L}{AE} \]  \( \text{Eqn. 14.17} \)

where \( \sum \) is the summation for all the members of the trussed beam.

If, however, there are two redundant forces \( R_1 \) and \( R_2 \) as in Fig. 14.20, we have

\[ \frac{\partial U}{\partial R_1} = 0 = \int_0^L M \frac{\partial M}{\partial R_1} \frac{dx}{EI} + \sum P \frac{\partial P}{\partial R_1} \frac{L}{AE} \] \( \text{Eqn. 14.18 (a)} \)

and

\[ \frac{\partial U}{\partial R_2} = 0 = \int_0^L M \frac{\partial M}{\partial R_2} \frac{dx}{EI} + \sum P \frac{\partial P}{\partial R_2} \frac{L}{AE} \] \( \text{Eqn. 14.18 (b)} \)

The solutions of these equations gives the values of the redundant forces. After having known the redundant forces, the forces in the ties bars etc. can be calculated, and the B.M. and S.F. diagram can be plotted for the main beam.

**Example 14.17.** A trussed timber beam, 120 mm wide and 160 mm deep, is 4 m long and has a central C.I. strut 1 m long and 1000 mm\(^2\) area of cross-section. The tie rods are of steel and 500 mm\(^2\) area of cross-section. Calculate the thrust in the strut if the beam carries a uniformly distributed load of 10 kN/m. Take \( E \) for wood, C.I. and steel as \( 10^4 \) N/mm\(^2\) and \( 2 \times 10^5 \) N/mm\(^2\), respectively.

**Solution**

Let the redundant force in the strut CD be \( R \).

Then

\[ \frac{\partial U}{\partial R} = 0 \]

\[ \text{Redundant Forces} \]

Now \( \cos \theta = \frac{2}{\sqrt{3}} = 0.732 \), \( \sin \theta = \frac{1}{\sqrt{3}} = 0.577 \), \( \cosec \theta = \sqrt{3} = 1.732 \)

\[ P_{AD} = P_{BD} = \frac{R}{2} \cosec \theta = 1.732 \frac{R}{2} \text{ (tension)} \]

Reactions at \( A \) and \( B \) = \((10 \times 2) - P_{AD} \sin \theta\)

\[ = 20 - \frac{1}{2} R \]

(a) For the beam \( AB \)

Strain energy due to bending = \( U_{AB1} = 2 \int_0^L \frac{M^2}{2EI} \, dx \)

\[ \begin{align*}
\frac{\partial U_{AB1}}{\partial R} &= 2 \int_0^L M \frac{\partial M}{\partial R} \frac{dx}{EI} \\ (\text{Eqn. 14.18 (a)}) \end{align*} \]

At any section distant \( x \) from \( A \),

\[ M = - \left( 20 - \frac{1}{2} R \right) x + 5x^3 \]

\[ \frac{\partial M}{\partial R} = + \frac{1}{2} x \]

\[ \frac{\partial U_{AB1}}{\partial R} = 2 \int_0^L \left\{ - \left( 20 - \frac{1}{2} R \right) x + 5x^3 \right\} \frac{1}{2} x \, dx \]

\[ = \frac{1}{EI} \left[ \left( -20x^2 + \frac{1}{2} Rx^3 + 5x^4 \right) \frac{dx}{dx} \right] \]

\[ = 4R - 100 \left( \frac{3}{3} \right) \]

where \( EI \) is in kN-mm\(^2\) units.

\[ E \text{ for timber} = 1 \times 10^4 \text{ N/mm}\(^2\); E = 10 kN/mm\(^2\); E = 1 \times 10^7 \text{ kN/m}^2 \]

\[ EI_1 = 1 \times 10^7 \left[ \frac{0.732(10)^2}{12} \right] = 409.6 \text{ kN/mm}^2 \]

\[ \frac{\partial U_{AB1}}{\partial R} = \frac{4R - 100}{3 \times 409.6} \]
STRENGTH OF MATERIALS AND THEORY OF STRUCTURES

(b) Strain energy due to direct forces
The tension in tie bars $P_{AD} = P_{BD} = 1.12R$
Compression in the beam $P_{AD} \cos \theta = (1.12R)(0.894) = R$

The calculations are arranged in the tabular form below:

<table>
<thead>
<tr>
<th>Member</th>
<th>L (m)</th>
<th>A (mm$^2$)</th>
<th>E (kN/mm$^2$)</th>
<th>$P$ (kN)</th>
<th>$\frac{\partial P}{\partial R}$</th>
<th>$P \frac{\partial P}{\partial R}$</th>
<th>$\frac{L}{AE}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CD</td>
<td>1</td>
<td>1000</td>
<td>100</td>
<td>$-R$</td>
<td>$-1$</td>
<td>$R \times 1 \times \frac{1}{1000 \times 100}$</td>
<td>$R \times 10^{-4}$</td>
</tr>
<tr>
<td>AD</td>
<td>2.24</td>
<td>500</td>
<td>200</td>
<td>$+1.12R$</td>
<td>$+1.12$</td>
<td>$(1.12R) \times 0.00 \times 200$</td>
<td>$2.8R \times 10^{-4}$</td>
</tr>
<tr>
<td>BD</td>
<td>2.24</td>
<td>500</td>
<td>200</td>
<td>$+1.12R$</td>
<td>$+1.12$</td>
<td>$2.8R \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>AB</td>
<td>4</td>
<td>19200</td>
<td>10</td>
<td>$-R$</td>
<td>$-1$</td>
<td>$R \times 1 \times \frac{4}{19200 \times 10}$</td>
<td>$2.98R \times 10^{-4}$</td>
</tr>
</tbody>
</table>

$: \frac{\partial U}{\partial R} = 8.68R \times 10^{-5}$

Now $\frac{\partial U}{\partial R} = 0 = \int \frac{3}{EI} + \sum \frac{P}{EI} \frac{\partial P}{\partial R} \frac{L}{AE}$

$: 4R - 100 + 8.68R \times 10^{-5} = 0$

which gives $R = 24.35$ kN.

(Note. If the support CD were rigid, the value of $R$ would have been $\frac{5}{8}wL = \frac{5}{8} \times 10 \times 4 = 25$ kN).

PROBLEMS

1. The material and cross-sectional area of the bars of the frame shown in Fig. 14-22 are same. Show that force in BC is $0.707 W$ tensile. (U.L.)

2. Find the forces in the members of the frame work shown in Fig. 14-23. The quantity $AE$ is constant for all the members.

3. Determine the axial force in the members of the frame shown in Fig. 14-24. The cross-sectional area of bars $AB$ and $AC$ is $2a$ and that of other member is $a$.

4. A braced cantilever is loaded as shown in Fig. 14-25. All the members are of the same cross-sectional area. Find the axial force in the member BC.

5. Determine the forces in the members of the frame, shown in Fig. 14-26 which is pinned to supports $A$ and $D$ and carries loads of $W$ and $2W$ at $B$ and $C$ respectively. The members $AB$ and $CD$ are $3a$ and the remainder $a$ in cross-sectional area.
6. Find the force in the member AF of the pin-jointed frame-work shown in Fig. 14-27. All members have the same area of cross-section and are of the same material.

7. The three rods AD, BD and CD are pinned to each other at D and to a rigid member ABC at A, B and C respectively, as shown in Fig. 14-28. The rods AD and BD are each of 1000 sq. mm and CD of 1600 sq. mm. If a horizontal load of 100 kN is applied at D, find the loads in all the three members. Take \( E = 2.1 \times 10^8 \) N/mm\(^2\) (210 kN/mm\(^2\)).

8. A pin-jointed frame is loaded as shown in Fig. 14-29. The frame is hinged at A and supported on rollers at B. The ratio \( \frac{\text{Length}}{\text{Area}} \) for all the members is the same. Treating CF as the redundant member, obtain the forces in the central members CF and DE.

9. The frame work shown in Fig. 14-30 is made from bars all having the same extensibility AE. It is supported at B and C and carries loads at A and D.

10. Fig. 14-31 shows a pin jointed frame work carrying a vertical load of 10 kN at E and supported by vertical reactions at A and B. The dimensions of the figure are such that if the line CE is omitted, all the angles are either 30°, 60° or 90°. The members are all of the same material and cross-section. Find the load in the member CE.

11. A pin-jointed rectangular frame with two diagonals is built up as shown in Fig. 14-32. The bar BC is the last to be added and is short by 5 mm. Find the force in the member BC when it is forced into position. The cross-section area of each side bar is 1000 mm\(^2\) and of each diagonal 500 mm\(^2\). Take \( E = 2.1 \times 10^8 \) N/mm\(^2\).

12. In a pin-jointed frame shown in Fig. 14-33, the cross-sectional area of each member is 2 in\(^2\), and \( E = 13000 \) ton/in\(^2\). During construction the member EB was made 1/16 in. too long and was forced into plane. Determine the force developed in each of the diagonal members AD and EB when a vertical load of 6 tons is applied at C. (A.M.I. Struct. E)

13. The rectangular frame shown in Fig. 14-34 consists of five bars pinned together at A, B, C and D and suspended in a vertical plane from a beam. All the bars are of steel and 2 in\(^2\) in cross-sectional area. Find the force in the member AD due to the two loads at A and D. If the temperature is raised 20°C, the distance BC remaining unchanged, find the force in each of
the bars. Take $E=13200$ tons/in$^2$. Coefficient of linear expansion = $11 \times 10^{-6}$ per °C.

Fig. 14-34.

14. A horizontal cantilever of length $L$ has its free end attached to a vertical tie rod $1.5L$ long, which is initially unstrained. The moment of inertia of the section of the cantilever is $I$ and the area of the cross-section of the tie rod is $a$. Prove that the load $(R)$ taken by the tie rod due to U.D.L. of weight $w$ per unit length on the cantilever is $R = \frac{3waL^2}{4(2aL^2 + 9I)}$.

$E$ for both the rod and cantilever is the same.

15. A trussed timber beam, 200 mm wide and 300 mm deep, is 6 m long and has a central C.I. strut 1 m long and 80 mm dia. The tie rods are of steel and 30 mm in diameter. Calculate the thrust in the strut if the beam carries a uniformly distributed load of 30 kN/m.

$E$ for wood = $1 \times 10^6$ N/mm$^2$ (10 kN/mm$^2$)
$E$ for C.I. = $1 \times 10^6$ N/mm$^2$ (100 kN/mm$^2$)
$E$ for steel = $2 \times 10^6$ N/mm$^2$ (200 kN/mm$^2$)

16. In the pin-jointed frame work shown in Fig. 14-36, the three triangles

Fig. 14-35.

ABG, HCF and GDE are equilateral. The members AB, BC, CD, DE, BH, CG and DF have a cross-section $2a$, and the remaining members $a$. Treating the verticals BH and DF as redundant members shows that they are subjected to tension of magnitude $42+6\sqrt{3}$ W.

17. Computed the reaction and forces in the members of the truss shown in Fig. 14-37. $L/AE$ is constant for all members.

Fig. 14-37.

18. A pin-jointed frame of uniform material has the loads and dimensions shown in Fig. 14-38. The ratio of length to cross-sectional area is constant for each member of the frame. The frame is hinged to foundations at $A$ and $C$ and is supported by a horizontal free-roller bearing at $B$. Taking member $8$ as the redundant member, calculate the magnitude and kind of force it carries due to loads.

19. Calculate the force in the members of the frame shown in Fig. 14-39 which is supported on pins at $A$, $B$, and $C$ and carries a load of 10 kN at $E$. All the bars have equal areas.

Fig. 14-38.

Fig. 14-39.
20. The structure shown in Fig. 14-40 is hinged at A and F and subjected to a load W at C. Assuming the sectional area of all the members to be the same, determine the force in each of the members AB and DE.

**Answers:**

1. \( P_{AC} = +2.5 \text{ kN} ; P_{BC} = +3.33 \text{ kN} ; P_{CD} = +5.83 \text{ kN}. \)
2. \( P_{AB} = P_{AC} = -0.535 \text{ W}; P_{BC} = +0.33 \text{ W}; P_{BD} = P_{AD} = P_{CD} = -0.07 \text{ W} \)
3. \( P_{AC} = +0.87 \text{ W}; P_{AB} = +0.75 \text{ W}. \)
4. \( P_{AF} = +5.13 \text{ kN}. \)
5. \( P_{AC} = P_{BD} = +0.138 \text{ W}; P_{BC} = -0.108 \text{ W}; P_{DC} = +1.919 \text{ W}. \)
6. \( P_{AB} = -0.535 \text{ W}; P_{BC} = +0.33 \text{ W}; P_{DE} = -0.07 \text{ W}. \)
7. \( P_{AC} = -0.588 \text{ W}; P_{AD} = +0.42 \text{ W}; P_{CD} = +4.29 \text{ W}. \)
8. \( P_{AB} = P_{AC} = -0.052 \text{ W}; P_{BD} = +0.52 \text{ W}. \)
9. \( 98.7 \text{ kN}. \)
10. \( P_{AB} = P_{BC} = -0.86 \text{ kN}; P_{CD} = -1.894 \text{ kN}. \)
11. \( P_{DA} = +1.438 \text{ kN}; P_{DB} = -0.364 \text{ kN}. \)
12. \( 0.77 \text{ W} \) (comp.).
13. \( P_{AE} = -0.43 \text{ kN}; P_{BE} = +2.14 \text{ kN}; P_{DE} = -0.98 \text{ kN}. \)
14. \( P_{EE} = -0.58 \text{ kN}; P_{AD} = -0.81 \text{ kN}; P_{DE} = +0.81 \text{ kN}. \)
15. \( P_{AC} = -1.15 \text{ kN}. \)
16. \( P_{AB} = 0.299 \text{ W}; P_{DE} = 0.669 \text{ W}. \)

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**Cables and Suspension Bridges**

15.1 **INTRODUCTION**

Suspension bridges are used for highways, where the span of a bridge is more than 200 m. Essentially, a suspension bridge (Fig. 15.1) consists of the following elements: (i) the cable, (ii) suspenders, (iii) decking, including the stiffening girder, (iv) supporting tower, and (v) anchorage.

![Suspension Bridge Diagram](image)

Fig. 15.1. Elements of a suspension bridge.

The traffic load of the decking is transferred to main cable through the suspenders. Since the cable is the main load bearing member, the curvature of the cable of an unstiffened bridge changes as the load moves on the decking. To avoid this, the decking is...
stiffened by provision of either a three hinged or two hinged stiffening girder. The stiffening girder transfers a uniformly distributed or equal load to each suspender, irrespective of the load position on the deck. The suspension cable is supported on either side. There are two arrangements generally used. The suspension cable may either pass over a smooth frictionless pulley and anchored to the other side, or it may be attached to a saddle placed on rollers. In the former case, the tension on the cable on the two sides of the pulley are equal while in the latter case, the horizontal components of the tension on the two sides are equal since the cable cannot have a movement relative to the saddle. The cable consists of either wire rope, parallel wires joined with clips or eye bar links. The cable can carry direct tension only, and the bending moment at any point on the cable is zero. The suspenders consist of round rods or ropes with turn-buckles so that adjustment in their lengths may be done if required. The anchorage consists of huge mass of concrete, designed to resist the tension of the cable.

15.2. EQUILIBRIUM OF LIGHT CABLE : GENERAL CABLE THEOREM

Fig. 15-2 shows a light cord or cable, suspended from two points A and B and subjected to a number of point loads $W_1, W_2, \ldots, W_n$. Let $L$ be the horizontal span of the cable and $\alpha$ be the inclination of the line $AB$, with the horizontal. Evidently, the difference in elevation between the two supports A and B is equal to $L \tan \alpha$.

Let $V_A$ and $V_B$ be the vertical components of reactions at A and B. Since there is no horizontal loading on the cable, the horizontal reaction ($H$) at the ends A and B will be equal in magnitude but opposite in direction. Since the cable is in equilibrium, it will take the shape of a funicular polygon for the load system, and will, therefore, deform as shown.

In order to find the vertical reaction $V_A,$ take moments about $B$:

$$-V_A \cdot L - H \cdot L \tan \alpha + \Sigma M_B = 0$$

or

$$V_A = \frac{\Sigma M_B}{L} - H \tan \alpha \tag{15.1}$$

where $\Sigma M_B =$ sum of moments of all external loads about $B$.

Consider any point $X$ at a horizontal distance $x$ from $A$.

Evidently, $X_1X_2 = x \tan \alpha$

Assuming that the cable is perfectly flexible so that the bending moment at any point on the cable is zero, the sum of moments ($\Sigma M_x$) of all external forces to the left of point $X$ is zero.

$$-H(XX_2) - V_A \cdot x + \Sigma M_x = 0$$

or

$$H \cdot (x \tan \alpha - y) - V_A \cdot x + \Sigma M_x = 0$$

where $M_x =$ sum of moments of all forces to the left of $X$. $y = XX_2$

Substituting the value of $V_A$ from Eq. 15.1, we get

$$-H(x \tan \alpha - y) - \left\{ \frac{\Sigma M_B}{L} - H \tan \alpha \right\} x + \Sigma M_x = 0$$

or

$$Hy - \frac{x}{L} \Sigma M_B + \Sigma M_x = 0$$

or

$$Hy = \frac{x}{L} \Sigma M_B - \Sigma M_x \tag{15.2}$$

Eq. 15.2 is the general cable theorem.

15.3. UNIFORMLY LOADED CABLE

(a) EXPRESSION FOR HORIZONTAL REACTION

Fig. 15-3 shows a cable supporting a uniformly distributed load of intensity $p$ per unit length. From the general cable theorem derived in the previous article we have

$$Hy = \frac{x}{L} \Sigma M_B - \Sigma M_x$$

where $y = XX_2 =$ vertical ordinate between the line $AB$ and chord at the point $X$. 
Due to symmetry, $V_A = V_B = \rho \cdot \frac{L}{2}$.

Let $C$ be the lowest point of the cable, at its middle, where the dip is equal to $d$. Consider the equilibrium of the portion $CA$ to the left of $C$ [Fig. 15.4(b)]. This left portion is in equilibrium under these forces: (i) the cable tension $T$ at $A$ (i.e., the resultant of $H$ and $V_A$), (ii) the external load $pL/2$ acting at $L/4$ from $C$, and (iii) the horizontal cable tension $H$ at $C$. All these forces must meet at a point $E$, distant $L/4$ from $C$. Thus triangle $AA'E$ becomes a triangle of forces from which

$$H = \frac{pL}{2}, \quad \frac{A_E}{AA} = \frac{pL}{3}, \quad \frac{L}{4} \cdot \frac{1}{d} = \frac{pL^2}{8d}$$

which is the same as obtained earlier.

(b) EXPRESSION FOR CABLE TENSION AT THE ENDS

The cable tension $T$ at any end is the resultant of vertical and horizontal reactions at the end. Thus

$$T_A = \sqrt{V_A^2 + H^2}$$

$$T_B = \sqrt{V_B^2 + H^2}$$

Knowing $H$ from Eq. 15.4 and $V_A$ from Eq. 15.1, the cable tension $T_A$ or $T_B$ can be easily calculated. When the cable chord is horizontal, $V_A = V_B = \rho \cdot \frac{L}{2}$. Hence

$$T_A = T_B = T = \sqrt{\left(\frac{pL}{2}\right)^2 + \left(\frac{pL^2}{8d}\right)^2}$$

or

$$T = \frac{pL}{2} \sqrt{1 + \frac{L^2}{16d^2}}$$

or

$$T = H \sqrt{1 + \frac{16d^2}{L^2}}$$

The inclination $\delta$ of $T$ with the vertical is given by

$$\tan \delta = \frac{H}{pL} = \frac{pL^2}{8d} \cdot \frac{2}{pL} = \frac{L}{4d}$$

It should be remembered that the horizontal component of cable tension at any point will be equal to $H$. 

---

Fig. 15.4.
Cable chord horizontal.
(c) SHAPE OF THE CABLE

Let us now determine the shape of the cable under the uniformly distributed load. Substituting the value of \( H \) (Eq. 15.4) in Eq. 15.3, we get

\[
\left( \frac{9L^2}{8d} \right) y = \frac{pLx}{2} - \frac{px^2}{2}
\]

or

\[
y = \frac{4dx}{L^2} \left( L - x \right)
\]  

(13.9)

This is, thus, the equation of the curve with respect to the cable chord. The cable, thus, takes the form of a parabola when subjected to uniformly distributed load.

(d) LENGTH OF THE CABLE: BOTH ENDS AT THE SAME LEVEL

When both the ends of the cable are at the same level [Fig. 15.4 (a)], the equation of the parabola can be written, with \( C \) as the origin, as follows:

\[ y = kx^2 \]

At \( A \), \( x = \frac{L}{2} \) and \( y = d \)

\[
k = \frac{y}{x^2} = \frac{d}{(L/2)^2} = \frac{4d}{L^2}
\]

\[
y = \frac{4dx}{L^2} \frac{x}{x^2}
\]  

(15.10)

\[
dy = \frac{8dx}{L^2} x
\]

Consider an element of length \( ds \) of the curve, having co-ordinates \( x \) and \( y \). The total length \( s \) of the curve is given by

\[
s = \int_0^L ds = \int_2^{L/2} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} dx
\]

\[
= 2 \left[ L/2 \left( 1 + \frac{8dx}{L^2} x \right)^{1/2} \right]^{1/2} dx
\]

Expanding \( 1 + \left( \frac{8dx}{L^2} x \right)^{1/2} \) by Binomial theorem, and neglecting higher powers of \( \frac{dx}{L^2} x^2 \), we get

\[
s = 2 \left[ \frac{L/2}{2} \left( 1 + \frac{8dx}{L^2} x + \ldots \right) \right] dx
\]

\[
= 2 \left[ \frac{3L/2}{3} \left( \frac{8dx}{L^2} x \right)^{3/2} \right]^{1/2}
\]

\[
= \frac{L}{3} + \frac{8}{3} \frac{d^2}{L} x
\]  

(15.11)

(e) LENGTH OF THE CABLE: ENDS AT DIFFERENT LEVELS

Consider a cable \( AB \) with the supports \( A \) and \( B \) at different levels. Let \( C \) be the lowest point of the cable, such that the horizontal equivalent of \( AC \) is \( L_1 \) and that of \( CB \) is \( L_2 \).

Evidently,

\[ L_1 + L_2 = L \]  

(1)

Imagine the portion \( AC \) to be extended to a point \( A_1 \) such that \( A \) and \( A_1 \) are at the same level. Let \( d_1 \) be the dip of this hypothetical cable, below the chord \( AA_1 \). From Eq. 15.4, we have

\[
H = \frac{pL^3}{8d}, \text{ where } L = 2L_1 \text{ and } d = d_1
\]

\[
H = \frac{p}{8} \left( \frac{2L_1}{d_1} \right)^2 = \frac{pL_1^3}{2d_1}
\]  

(15.12)

Similarly, imagine the portion \( BC \) to be extended to a point \( B_1 \) such that \( B \) and \( B_1 \) are at the same level. Let \( d_2 \) be the dip of this hypothetical cable, below the chord \( BB_1 \). From Eq. 15.4

\[
H = \frac{pL^3}{8d}, \text{ where } L = 2L_2 \text{ and } d = d_2
\]

\[
H = \frac{p}{8} \left( \frac{2L_2}{d_2} \right)^2 = \frac{pL_2^3}{2d_2}
\]  

(15.12)

Since \( H \) is the same at \( C \) for both the portions of the cable, we get

\[
\frac{pL_1^3}{2d_1} = \frac{pL_2^3}{2d_2}
\]

or

\[
\frac{L_1}{L_2} = \sqrt{\frac{d_1}{d_2}}
\]  

(2)

Solving (1) and (2), the values of \( L_1 \) and \( L_2 \) can be known in terms of \( L \), \( d_1 \) and \( d_2 \).
In order to find the vertical reaction $V_A$ at $A$, take moments about $A$. Then
\[ V_A = \frac{1}{L} \left[ \frac{PL}{2} + H(d_1 - d_2) \right] \]
where $H = \frac{pL^2}{2d_1}$.

\[ = \frac{p}{2L} \left[ L^3 + \frac{L^3}{d_1} (d_1 - d_2) \right] = \frac{p}{2L} \left[ L^3 + \frac{L^3}{d_1} \frac{d_1}{d_1} \right] \]

\[ = \frac{p}{2L} \left[ L^3 + \frac{L^3}{L} \times \frac{L}{L} \right] \]

\[ = \frac{p}{2L} \left[ L^3 \times 1 + L^3 \times \frac{L}{L} \right] = \frac{p}{2L} \left[ 2L^3 + 2L_1 \right]. \]

Hence $V_A = \frac{pL_1}{2}$

Similarly, $V_B = \frac{pL_2}{2}$.

For the imaginary cable $ACA_1$, the length $s_1$ is given by

\[ s_1 = 2L_1 + \frac{8}{3} \frac{d_1^2}{L_1} = \frac{2L_1^3}{3} + \frac{4}{3} \frac{d_1^2}{L_1} \]

Similarly, the length of the cable $BCB_1$ is given by

\[ s_2 = 2L_2 + \frac{8}{3} \frac{d_2^2}{L_2} = 2L_2 + \frac{4}{3} \frac{d_2^2}{L_2} \]

Hence the total length of the actual cable $ABC$ is

\[ s = \frac{1}{2} (s_1 + s_2) \]

or

\[ s = \frac{1}{2} \left( \left( 2L_1 + \frac{4}{3} \frac{d_1^2}{L_1} \right) + \left( 2L_2 + \frac{4}{3} \frac{d_2^2}{L_2} \right) \right) \]

or

\[ s = L_1 + \frac{2}{3} \frac{d_1^2}{L_1} + L_2 + \frac{2}{3} \frac{d_2^2}{L_2} \]

or

\[ s = L_1 + \frac{2}{3} \frac{d_1^2}{L_1} + L_2 + \frac{2}{3} \frac{d_2^2}{L_2} \]

(15'13)

15'4. ANCHOR CABLES

The suspension cable is supported on either sides, on the supporting towers. The anchor cables transfer the tension of the suspension cable to the anchorage consisting of huge mass of concrete. There are generally two arrangements for this. The suspension cable can either be passed over the guide pulley for anchoring it to the other side or it can be attached to a saddle mounted on roller.

In the former case, when the suspension cable passes over the guide pulley and forms the part of the anchor cable to the other side, the tension $T$ in the cable is the same on both sides.

Let $\beta_1 =$ inclination of the suspension cable, with vertical.

$\beta_2 =$ inclination of the anchor cable, with vertical.

Pressure on the top of pier: $V_P = T \cos \beta_1 + T \cos \beta_2$,

$= T (\cos \beta_1 + \cos \beta_2)$ (15'14)

Horizontal force on the top of the pier

$= T \sin \beta_1 - T \sin \beta_2$,

$= T (\sin \beta_1 - \sin \beta_2)$ (15'15)

This horizontal force will cause bending moment in the tower.

If the cable is supported on a saddle mounted on rollers, as shown in Fig. 15'6 (b), the horizontal components of the tensions in the suspension cable and the anchor cable will be equal since the rollers do not have any horizontal reaction.

$T_1 \sin \beta_1 = T_2 \sin \beta_2 = H$ (15'16)

The vertical pressure on the pier is given by

$V_P = T_1 \cos \beta_1 + T_2 \cos \beta_2$ (15'17)

15'5. TEMPERATURE STRESSES IN SUSPENSION CABLE

Let $s =$ length of the cable

$\delta s =$ change in the length due to change in temperature

$\delta d =$ corresponding change in the dip.

From Eq. 13'11,

\[ s = L + \frac{8}{3} \frac{d^2}{L} \]

\[ \delta s = \frac{16}{3} \frac{d}{L} \delta d \]

or

\[ \delta d = \frac{3}{16} \frac{L}{d} \delta s \]

But

\[ \delta s = s - v \cdot t \]

where $a =$ coefficient of thermal expansion of cable

$t =$ change in the temperature.

\[ a = \frac{\delta s}{s - v \cdot t} \]
\[ \delta s = \alpha t \left( L + \frac{8}{3} \frac{d^2}{L} \right) \]

or

\[ \delta s = L \alpha t + \frac{8}{3} \frac{d^2}{L} \alpha t. \]

Neglecting \( \frac{8}{3} \frac{d^2}{L} \alpha t \) in comparison to \( L \alpha t \), we have

\[ \delta s = L \alpha t. \]  

Substituting this in (1), we get

\[ \delta d = \frac{3}{16} \frac{L}{d} \left( L \alpha t \right) = \frac{3}{16} \frac{L^2}{d} \alpha t \]  

(15'18)

It is to be noted that when the temperature rises, \( L \) will increase, and hence \( \delta d \) will increase. Similarly, when the temperature falls, \( L \) will decrease, and hence, \( \delta d \) will decrease. Let us now find the corresponding change in the value of \( H \) due to this change in \( d \).

\[ H = \frac{pL^2}{8d} \]

or

\[ H \propto \frac{1}{d}. \]

\[ \frac{\delta H}{H} = \frac{-\delta d}{d}. \]

If \( f \) is the stress in the cable,

\[ \frac{\delta f}{f} = \frac{\delta H}{H} = \frac{-\delta d}{d}. \]

(15'10)

**Example 15'1.** A light cable, 18 m long, is supported at two ends at the same level. The supports are 16 m apart. The cable supports three loads of 8, 10, and 12 N dividing the 16 m distance in four equal parts. Find the shape of the string and the tension in various portions.

**Solution.**

Let \( CC_1 = yc \); \( DD_1 = yd \) and \( EE_1 = ye \).

Since both the supports are at the same level,

\[ V_1 = \frac{1}{16} \left\{ (8 \times 12) + (10 \times 8) + (12 \times 4) \right\} = 14 \text{ N} \]

Since the cable is in equilibrium, the shape taken by it is that of a funicular polygon. The shape, thus, represents the bending moment diagram to some scale.

![Diagram](image)
and 

\[ y_E = \frac{8}{7}, \quad y_c = 2.86 \, \text{m} \]

Thus, with the known values of \( y_c, y_d \) and \( y_E \), the shape of the cable is determined.

In order to find the horizontal reaction \( H \), apply the general cable theorem (Eq. 15'2) at point \( C \).

\[ H_{yc} = \frac{4}{16} \sum M_B - \sum M_C \]

where \( \sum M_B \) = sum of moments of external loads, about \( B \) = \((8 \times 12) + (10 \times 8) + (12 \times 4) = 224 \)

\( \sum M_C \) = sum of moments of external loads, about \( C \) = 0.

\[ y_c = 2.50 \]

\[ 2.5H = \frac{4}{16} \left(224\right) = 56 \]

\[ H = \frac{56}{2.50} = 22.4 \]

Hence tension in the part \( AC \) is given by

\[ T_{ac} = \sqrt{V_{ac}^2 + H^2} = \sqrt{(45)^2 + (22.4)^2} = 26.4 \, \text{N} \]

The tensions in parts \( CD, DE \) and \( EB \) will be such that their horizontal component is equal to \( H = 22.4 \).

**Example 15'2.** A cable is used to support five equal and equidistant loads over a span of 39 metres. Find the length of the cable required and its sectional area if the safe tensile stress is 140 N/mm². The central dip of the cable is 2.5 m and loads are 5 kN each.

**Solution**

![Diagram](image)

**Fig. 15-8.**

Since the cable is in equilibrium, the shape of the cable will correspond to the funicular polygon for the load system. The ordinates at \( C, D, E, F \) and \( G \) will be proportional to the bending moments at these points. Since both the cables are at the same level, we have
Solution

Fig. 15.9.

Let $d =$ dip of the cable at $P$.

Taking moments about $B$, we get

$$(V_A \times 40) = (300 \times 30) + (1 \times 40 \times 20)$$

or

$$V_A = \frac{9800}{40} = 245 \text{ N}$$

and

$$V_A = 300 + 40 \times 1 - 245 = 95 \text{ N}$$

The maximum tension in the cable is $T_A = \sqrt{V_A^2 + H^2}$

or

$$1000 = \sqrt{(245)^2 + (H)^2}$$

or

$$H = 970 \text{ N}$$

Taking moments about $P$, we get

$$M_P = 0 = Hd - V_A \times 10$$

or

$$970d - 245 \times 10 = 0$$

or

$$d = \frac{2450}{970} = 2.52 \text{ m}$$

:. Height of $P$ above ground is $12 - 2.52 = 9.48 \text{ m}$.

Example 15.4. A wire of uniform material weighing 0.32 lb. per cu. in. hangs between two points 120 ft. apart horizontally, with one end 3 ft. above the other. The sag of the wire measured from the highest point is 5 ft. Calculate the maximum stress in the wire. (U.L.)

Solution

Let $p =$ weight of cable, per ft. length (in lbs).

From Eq. 15.12,

$$\frac{L_1}{L_2} = \sqrt{\frac{d_1}{d_2}}$$

where

$d_1 =$ maximum dip = 5'; $d_2 = 5 - 3 = 2'$

:. $$\frac{L_1}{L_2} = \sqrt{\frac{5}{2}} = 1.58$$

Example 15.5. A flexible suspension cable of weight $\frac{1}{2} \text{ N/m}$ hangs between two vertical walls 60 m apart, the left hand end being attached to the wall at point 10 m below the right hand end. A concentrated load of 100 N is attached to the cable in such a manner that the point of attachment of the load is 20 m horizontally from the left hand wall and 5 m below the left hand support. Show that the maximum resultant cable tension is at the right hand end and find its value. The cable weight may be taken as uniformly distributed horizontally.

Solution

To find the reactions $V_B$ and $H$, take moments about $A$:

$$60 V_B = 20 V + \frac{PL^2}{2} + 10 H$$

or

$$V_B = \frac{1}{60} \left( 20 \times 100 + \frac{3}{4} \times \frac{3600}{2} + 10 H \right)$$

Again, since the cable is flexible, $Mc = 0$.
Example 15.6. A suspension cable 160 m span and 16 m central dip carries a load of \( \frac{3}{4} \) kN per lineal horizontal metre. Calculate the maximum and minimum tensions in the cable. Find horizontal and vertical forces in each pier under the following alternative conditions:

(a) If the cable passes over frictionless rollers on the top of the piers.

(b) If the cable is firmly clamped to saddles carried on frictionless roller on the top of the piers.

Solution

In each case the back stay is inclined at \( 30^\circ \) to the horizontal.

The maximum tension \( T \) in the cable is always at its ends while the minimum tension in the cable is at its lowest point and is equal to \( H \).

From (1) and (2), we get

\[
H = 196 \text{ N} \quad \text{and} \quad V_B = 88.5 \text{ N}
\]

Since \( V_B > V_A \), \( T_B \) will be greater than \( T_A \).

\[
T_B = \sqrt{V_B^2 + H^2} = \sqrt{(88.5)^2 + (196)^2} = 215 \text{ N}
\]

(cables and suspension bridges)

From Eq. 15.7,

\[
T = T_{\text{max}} = H \sqrt{1 + \frac{d^2}{L^2}} = 100 \sqrt{1 + \frac{16 \times 16^2}{160^2}} = 108 \text{ kN}
\]

(b) When the cable passes over the frictionless pulley, the tension in the back stay is equal to the tension in the cable. Let the inclination of the cable be \( \phi \) with horizontal.

Then the load on pier = \( T \sin \phi + T \sin 30^\circ \)

\[
= 94 \text{ kN}
\]

Horizontal shear = \( T \cos \phi - T \cos 30^\circ \)

\[
= 647 \text{ kN}
\]

(a) Rollers permit no horizontal shear on towers. Hence the horizontal components of tension balance. Let \( T_S = \text{tension in the back stay} \).

\[
T_S \cos 30^\circ = T \cos \phi = H = 100
\]

\[
T_S = \frac{100}{\cos 30^\circ} = 0.866 = 115.5 \text{ kN}
\]

Total compression on pier

\[
= T_S \sin 30^\circ + T \sin \phi = T_S \sin 30^\circ + V
\]

\[
= 97.8 \text{ kN}.
\]
Example 15.7. A cable is swung between two points at the same level with a central dip of 12 m over a span of 120 m. The cable carries a uniformly distributed load of intensity 2 kN/m of horizontal length. Calculate the change in the horizontal tension if the temperature rises by 20°F from the original. Take \( a = 6 \times 10^{-6} \) per 1°F.

Solution

\[
H = \frac{pL^2}{8d} = \frac{2(120)^2}{8 \times 12} = 300 \text{ kN}
\]

From Eq. 3.19

\[
\frac{\delta H}{H} = \frac{3}{16} \left( \frac{L}{d} \right)^2 a = -\frac{3}{16} \left( \frac{120}{12} \right)^2 \times 6 \times 10^{-6} \times 20 = -\frac{9}{4} \times 10^{-8}
\]

\[
\delta H = \frac{9}{4} \times 10^{-8} \times 300 = -0.675 \text{ kN}
\]

i.e.

\[\delta H = 0.675 \text{ kN (decrease)}\]

15.6. THREE HINGED STIFFENING GIRDER

Since the cable of the suspension bridge is the main load-bearing member, the curvature of the cable of an unstiffened bridge changes as the load moves on the decking. To avoid this, the decking is stiffened by provision of either a three hinged stiffening girder or a two hinged stiffening girder. The stiffening girder transfers a uniform or equal load to each suspender, irrespective of the position of the load on the decking. Fig. 15.13 shows a suspension bridge with a stiffening girder hinged at the abutments \( D \) and \( F \) and also at the centre \( E \). We shall now consider the effect of a unit point load rolling on the decking and plot (i) bending moment diagram for fixed load position, (ii) influence line for horizontal reaction \( H \) of the cable (iii) influence line for bending moment at a section, (iv) maximum bending moment diagram due to a point load \( W \) and (v) maximum bending moment diagram due to a uniformly distributed load of intensity \( w \).

For the purposes of analysis of the above items, let us first consider the equilibrium of the cable as well as the stiffening girder separately.

1) EQUILIBRIUM OF THE CABLE

By the provision of a stiffening girder the suspenders carry a uniformly distributed load of intensity \( p \), irrespective of the nature.
and position of the load on the decking. Thus the cable is subjected to a downward uniform load of intensity \( p \) per unit length, as shown in the upper part of Fig. 15'13(a). The reactions at the support \( A \) and \( B \) are equal, the horizontal reaction being equal the \( H \) and the vertical reaction equal to \( \frac{pL}{2} \).

From Eq. 15'5

\[
H = \frac{pL^3}{8d}
\]

Since the cable is flexible, the bending moment at any point on it is equal to zero. Consider any point \( X' \) distant \( x \) from left support \( A \), Then

\[
M_x = 0 = H_y - \frac{pL}{2} x + \frac{px^2}{2}
\]

\[
H_y = \left( \frac{pL}{2} x - \frac{px^2}{2} \right)
\]

(2)

The equation of the parabola (cable), with \( A \) as the origin may be written as

\[
y = kx(L-x^2)
\]

at

\[
x = \frac{L}{2}, y = d
\]

\[
k = \frac{4d}{L^2}
\]

Hence

\[
y = \frac{4dx}{L^2} (L-x)
\]

(3)

(2) EQUILIBRIUM OF THE GIRDER

The lower part of Fig. 15'13(a) shows the equilibrium of the three hinged stiffening girder which is subjected to the following forces: (i) the external unit load acting at a distance of \( aL \) from left hand support, (ii) the simply supported reactions \( V_0=(1-a) \) and \( V_F=a \) respectively at points \( D \) and \( F \), due to the external load, (iii) the uniformly distributed pull \( p \) exerted by the suspenders, and (iv) downward reactions \( \frac{pL}{2} \) at \( D \) and \( F \), due to the pull \( p \) of the hingers.

(3) BENDING MOMENT DIAGRAM

Consider a point \( X \) distant \( x \) from the left hand support. The bending moment at \( X \) may be expressed as follows:

\[
M_x = \left[ -V_d.x + 1 \times (x-aL) \right] + \left[ \frac{pL}{2} x - \frac{px^2}{2} \right]
\]

(4)

Thus the bending moment at \( X \) consists of two parts. The first part, \( i.e. [-V_d.x + 1 \times (x-aL)] \) may be designated as \( \mu_x \), where \( \mu_x \) is the bending moment at the point \( X \) treating the girder as simply supported beam. The second part, \( i.e. \left[ \frac{pL}{2} x - \frac{px^2}{2} \right] \) is found to be equal to \( H.y \), from Eq. (2). Hence Eq. (4) may be re-written as:

\[
M_x = \mu_x + H \cdot y
\]

(15'20)

The bending moment diagram for the girder can thus be obtained by superimposing the \( \mu_x \) diagram over the \( H.y \) diagram. The \( \mu_x \) diagram for a simply supported beam is a triangle having an ordinate of \( \frac{Wab}{L} = \frac{1}{2} \times aL \times (1-a)L = -aL(1-a) \) under the point load, while the \( H.y \) diagram will be a parabola, since \( y \) varies parabolically while \( H \) is constant for a fixed load position. For the load position at \( aL \) from left support, \( H = \frac{aL}{2d} \) (see sub-Section 4 below). The maximum value of \( y \) is equal to \( d \) at the centre. Hence the \( H.y \) diagram is a parabola having a maximum ordinate of \( \frac{aL}{2d} \). \( d = \frac{aL}{2} \) under the centre of the cable. The final bending moment diagram is shown in Fig 15'13(b).

(4) INFLUENCE LINE FOR \( H \)

In order to find the value of \( H \) for the unit load position at a distance \( aL \), we apply Eq. 15'20 at the central hinge \( E \) of the girder where the bending moment is zero.

\[
M_E = 0 = \mu_E + H \cdot y
\]

\[
H = -\frac{\mu_E}{y}
\]

(15'21)

Now

\[
\mu_E = -a \cdot \frac{L}{2}, \text{ and } y = d \text{ at } E
\]

(15'22)

Eq. 15'22 suggests that for a given load position, \( H \) is constant. However, when the unit load changes its position, \( H \) also changes linearly as per Eq. 15'22. However, Eq. 15'22 is valid for load position between \( D \) and \( E \), i.e. for the range \( aL = 0 \) to \( aL = L/2 \).
When \( \alpha L = 0 \), \( H = 0 \)

\[ \alpha L = \frac{L}{2}, \quad H = \frac{L}{4d} \]

By symmetry, when the unit load is at \( F \), \( H = 0 \). Thus, the influence line diagram for \( H \) is a triangle having a maximum ordinate of \( \frac{L}{4d} \) under the central hinge, as shown in Fig. 15'13(c).

(5) INFLUENCE LINE FOR \( H \)

From Eq. 15'22

\[ H = \frac{8d}{L^3} \cdot \frac{aL}{2d} = \frac{4d}{L^3} \quad (15.23) \]

Thus, the load carried by the suspenders vary with the load position in the case of a three hinged stiffening girder. Eq. 15'24 is valid for the range of \( \alpha L = 0 \) to \( \alpha L = \frac{L}{2} \).

When the load is at \( D \), \( \alpha = 0 \) \( \therefore p = 0 \).

When the load is at the central hinge \( E \), \( \alpha L = \frac{L}{2} \), or \( \alpha = \frac{1}{2} \).

\[ p = \frac{4}{L} \cdot \frac{1}{2} = \frac{2}{L} \]

By symmetry, when the load is at \( F \), \( p = 0 \).

The influence line for \( p \) is shown in Fig. 15'13(d).

It will be seen later that in the case of a two hinged stiffening girder, \( p \) is constant, and does not vary with the load position.

(6) INFLUENCE LINE FOR BENDING MOMENT

We have

\[ Mx = \mu_x + Hy \]

Thus, the I.L. for \( Mx \) can be obtained by superimposing I.L. for \( \mu_x \) on the I.L. for \( Hy \).

The I.L. for \( \mu_x \) is a triangle having a maximum ordinate of \( \frac{1}{L} \) under the section \( X \).

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The I.L. for \( H.y \) will be a triangle. It should be noted that for the given section \( X, y \) is a fixed quantity equal to \( \frac{4d}{L^3} x (L-x) \).

Thus, the I.L. for \( H.y \) has a maximum ordinate of \( \frac{L}{4d} \) y

\[ = \frac{L}{4d} \cdot \frac{4d}{L^3} x (L-x) = \frac{x}{L} (L-x) \]

under the central hinge. Fig. 15'13(e) shows the I.L. for \( Mx \) in which the maximum ordinates of positive and negative bending moments are equal.

(7) MAXIMUM BENDING MOMENT DIAGRAM DUE TO A SINGLE POINT LOAD \( W \)

(a) Maximum negative bending moment diagram

From the I.L. for \( Mx \) it is clear that the maximum negative bending moment at section \( X \) occurs when the point load is at \( x \). In order to find the position of the section where absolute maximum negative B.M. occurs, differentiate Eq. 15'25 with respect to \( x \) and equate to zero.

Thus

\[ (L-x)(L-2x) - x(L-x) - 2x(L-x) = 0 \]

or

\[ 6x^2 - 6Lx + L^2 = 0 \]

which gives \( x = 0.211L \) or \( x = 0.789L \) \( (15.26) \)

Substituting the value of \( x \) in Eq. 15'25, we get

\( M_{max} = \frac{-0.096WL}{2} \)

(b) Maximum positive bending moment diagram

From Fig. 15'13(e), it is clear that the maximum positive bending moment at section \( X \) occurs when the point load is over the central hinge, its value being


\[ M_x(\text{max. + ve}) = W \left[ \frac{\frac{1}{2}x(L-x)}{L} - \frac{x(L-x)}{L} \right] \]

which is the equation of parabola.

\[ \frac{dM_x}{dx} = 0 = -(L-2x) - 2x \]

or

\[ x = \frac{L}{4} \]

Substituting in Eq. 15.27

\[ (\pm) M_{x, \text{max}} = \pm \frac{wL}{16} = \pm 0.0625 \frac{wL}{L} \]

Fig. 15.13 (f) shows the maximum positive and maximum negative bending moment diagrams due to point load \( W \).

(8) **MAXIMUM BENDING MOMENT DIAGRAM DUE TO UNIFORMLY DISTRIBUTED LOAD**

From Fig. 15.12(c) it is evident that the area of the negative portion of the I.L. diagram is equal to the area of the positive portion of the I.L. diagram for \( M_x \). Thus, the net area of I.L. diagram for \( M_x \) is zero. Hence it is concluded that there will be no bending moment anywhere in the girder due to its self weight or due to uniformly distributed load occupying the whole span. The I.L. diagram for \( M_x \) has a zero ordinate at the point \( O \), distant \( m \) from the left support. Let us first find the value of \( m \).

From triangle \( dx_1, f \),

\[ oo_1 = \frac{x(L-x)}{L}, \frac{1}{2} \frac{L}{L-x}(L-m) \]

From triangle \( dx_1, f \),

\[ oo_2 = \frac{x(L-x)}{L}, \frac{2}{3} \frac{L}{m} \]

Equating the two we get

\[ \frac{L-m}{L-x} = \frac{2m}{L} \]

or

\[ m = \frac{L^2}{3L-2x} \]  

(15.28)

\[ \text{Ordinate } oo_2 = \frac{x(L-x)}{L} \frac{2}{3} \frac{L}{3L-2x} = \frac{2x(L-x)}{(3L-2x)} \]

For the maximum negative bending moment at the section, the portion \( dx \) should be loaded with the U.D.L. while portion of should be empty; Similarly, for maximum positive B.M. at \( X \), portion of should be loaded while \( dx \) should be empty.

\[ \text{Area of } -ve \text{ portion of I.L. diagram} = \triangle dx_1 f - \triangle dx_2 f \]

\[ = \frac{1}{2} \cdot \frac{L}{L} \cdot \frac{x(L-x)}{L} - \frac{1}{2} \cdot \frac{2x(L-x)}{3L-2x} \]

\[ = \frac{x(L-x)(L-2x)}{2(3L-2x)} \]

If \( w \) is the intensity of U.D.L., we get

\[ (\pm) M_{x, \text{max}} = \pm \frac{wx(L-x)(L-2x)}{2(3L-2x)} \]  

(15.29)

To find the value of \( x \) at which absolute maximum positive or negative B.M. occurs, differentiate Eq. 15.29 with respect to \( x \) and equate to zero. Putting \( x = nL \) in Eq. 15.29 we get

\[ M_{x, \text{max}} = wL^2 \frac{n(1-n)(1-2n)}{2(3-2n)} \]

or

\[ 8n^2 - 24n^2 + 18n - 3 = 0 \]

\[ n = 0.234 \text{ or } x = 0.234L \]

Substituting the value of \( x \) in Eq. 15.29, we get

\[ (\pm) M_{x, \text{max}} = \pm 0.01883 wL^2 \]  

(15.30)

The loaded length \( m = \frac{L^2}{3L-2(0.234L)} = 0.395L \)

Fig. 15.13 (g) shows the maximum positive and maximum negative bending moment diagrams due to U.D.L.

(9) **INFLUENCE LINE FOR SHEAR FORCE**

Consider the section \( X \) distant \( x \) from the left support [Fig. 15.14(a)]. When the unit load is so placed that \( aL < x \), we have

\[ F_x = a + p(L-x) - \frac{pL}{2} \]  

(15.31)

or

\[ F_x = a + p \left( \frac{L}{2} - x \right) \]

(15.31)

or

\[ F_x = f_x + p \left( \frac{L}{2} - x \right) \]

(15.31)

where \( a = f_x \) = S.F. at \( X \) by considering the girder to be simply supported.

Now the equation of the parabola is

\[ y = \frac{4d}{L^2} x(L-x) \]
\[ \tan \theta = \frac{dy}{dx} = \frac{8d}{L^2} \left( \frac{L}{2} - x \right) \]  
\hspace{1cm} (15.32)

or

\[ \theta = \text{inclination of the tangent at } X', \text{ to the horizontal.} \]

---

**Diagram**

(a) **THE SUSPENSION BRIDGE**

(b) **I.L. FOR S.F.** \( x < \frac{L}{4} \)

(c) **I.L. FOR S.F.** \( x = \frac{L}{4} \)

(d) **I.L. FOR S.F.** \( \frac{L}{4} < x < \frac{L}{2} \)

(e) **S.F.D.** \( x < \frac{L}{2} \)

---

**SHEAR FORCE DIAGRAM**

\[ F_x = -(1 - \alpha) + \frac{pL}{2} - px + 1 \]

where \( p = \frac{4\alpha}{L} \), from Eq. 15.24

\[ \therefore \quad F_x = -(1 - \alpha) + \frac{4\alpha}{L} \left( \frac{L}{2} - x \right) + 1 \]

When \( x = 0 \), \( F_x = -(1 - \alpha) + 2\alpha = (-1 + 3\alpha) \)

When \( x = \alpha L \), \( F_x = -1 + 3\alpha - 4\alpha^2 \)

or \( -3\alpha - 4\alpha^2 \)

When \( x = L \), \( F_x = -(1 - \alpha) - 2\alpha + 1 = -\alpha \)

Fig. 15.14(e) shows the S.F.D. for a typical case when \( \alpha \) is less than \( \frac{1}{2} \).
Example 15'8. Derive from first principles the bending moment diagram for symmetrical suspension bridge with three pinned stiffening girder of length L subjected to a point load W at a distance a from the central pin. Draw to scale the bending moment diagram for the value of a that gives the largest bending moment under the load.

**Solution.**

![Bending Moment Diagram](image)

Fig. 15'15.

Fig. 15'15 (a) shows the free body diagrams of the cable and the three hinged stiffening girder. For the girder, the simply supported reactions at the ends can be obtained by taking moments about D and F. Thus,

\[ V_D = \frac{W}{L} \left( \frac{L}{2} + a \right) \]

and

\[ V_F = \frac{W}{L} \left( \frac{L}{2} - a \right) \]  \hspace{1cm} (1)

Taking moments about the central hinge, we get

\[ M_E = 0 = \frac{W}{L} \left( \frac{L}{2} - a \right) \frac{L}{2} + p \frac{L}{2}, \quad \frac{L}{2} - p \left( \frac{L}{4} \right) \]

or

\[ p = \frac{4W}{L^2} \left( \frac{L}{2} - a \right) \]

The bending moment at any point is given by

\[ M_x = \mu_x + H_y \]

\[ \mu_x = \frac{W}{L} \left( \frac{L}{2} - a \right) \frac{L}{2} + p \frac{L}{2}, \quad \frac{L}{2} - p \left( \frac{L}{4} \right) \]

Since \( y \) varies parabolically with \( x \), the \( H_y \) diagram will be a parabola having a maximum ordinate of \( \frac{pL^2}{8d} \). \( d = \frac{pL^2}{8} \) under the central hinge.

The \( \mu_x \) diagram will be a triangle having a maximum ordinate of \( \frac{W}{L} \left( \frac{L}{2} - a \right) \left( \frac{L}{2} + a \right) = \frac{W}{L} \left( \frac{L^2}{4} - a^2 \right) \) under the load.

The bending moment diagram is shown in Fig. 15'15 (b). The net B.M. under the load is given by

\[ M = -\frac{W}{L} \left( \frac{L^2}{4} - a^2 \right) + 2W \left( \frac{L}{2} - a \right)^2 - 2W \left( \frac{L}{2} - a \right)^2 \]

Substituting the value of \( p \) from (2), we get

\[ M = -\frac{W}{L} \left( \frac{L^2}{4} - a^2 \right) + 2W \left( \frac{L}{2} - a \right)^2 - 2W \left( \frac{L}{2} - a \right)^2 \]

For maxima, differentiate \( M \) with respect to \( a \).

\[ \frac{dM}{da} = 0 = -2W + 4W \left( \frac{L}{2} - a \right) \left( \frac{L^2}{4} - a^2 \right) \]

or

\[ a - 2 \left( \frac{L}{2} - a \right) + 3 \left( \frac{L^2}{4} - a^2 \right) = 0 \]

or

\[ 3a^2 = \frac{L^2}{4} \]

\[ a = \pm 0.288 \, L \]

The bending moment diagram for this value of \( a \) is shown in Fig. 15'15 (c), in which the ordinate under load

\[ \frac{W}{L} \left( \frac{L^2}{4} - (0.288 \, L)^2 \right) = 0.167 \, WL \]

and the ordinate under the central hinge

\[ \frac{4W}{L^2} \left( \frac{L^2}{2} - (0.288 \, L) \right)^2 = 0.106 \, WL \]

Example 15'9. The three hinged stiffening girder of a suspension bridge of 100 m span subjected to two point loads of 10 kN each placed at 20 m and 40 m respectively from the left hand hinge. Determine the B.M. and S.F. in the girder at section 30 m from each end. Also, determine the maximum tension in the cable which has a central dip of 10 m.

**Solution.**

To find the simply supported reactions \( V_D \) and \( V_F \), take moments about hinges D and F. Consider the equilibrium of the girder alone.
Thus,

$$V_F = \frac{1}{100} \left( (10 \times 40) + (10 \times 20) \right) = 6 \text{ kN}$$

$$f_n = \frac{1}{100} \left( (10 \times 60) + (10 \times 80) \right) = 14 \text{ kN}$$

In order to find $H$, take moments about the central hinge $E$.

Thus

$$M_E = 0 = \mu_F + H d$$

where

$$\mu_F = (-6 \times 50), \text{ and } d = 10 \text{ m}$$

\therefore

$$M_E = (-6 \times 50) + 10H = 0$$

or

$$H = 30 \text{ kN}.$$

The equation of the parabola is

$$y = \frac{4d}{L^2} \cdot x(L-x) = \frac{4 \times 10}{100 \times 100} \cdot x(100-x)$$

$$y = \frac{4x}{1000} (100-x)$$

At $x = 30$

$$y = \frac{4 \times 30}{1000} (100-30) = 8.4 \text{ m}$$

Thus at points $G$ and $H$, distant 30 m from $D$ and $F$ respectively, $y = 8.4 \text{ m}$.

The B.M. at any point is given by

$$M = \mu_X + H y$$

\therefore

$$M_G = (-14 \times 30) + (10 \times 10) + (30 \times 8.4) = -68 \text{ kN-m}$$

and

$$M_H = (-6 \times 30) + (20 \times 8.4) = +72 \text{ kN-m}$$

The S.F. at any point is given by

$$F_X = f_X + H \tan \theta$$
The free body diagrams for the cable and the stiffening girder are shown in Fig. 15'17. Since there will be no B.M. and S.F. anywhere in the girder due to uniformly distributed dead load, only live load has been shown.

Live load per metre run by girder
\[ \frac{V_f}{V_D} = \frac{1}{250} \left( 4 \times 125 \times 62.5 \right) = 125 \text{ kN} \]

\[ V_D = (4 \times 125) - 125 = 375 \text{ kN} \]

\[ M_E = \mu E + H y \]

\[ (-125 \times 125) + 25 H = 0 \]

or

\[ H = \frac{125 \times 125}{25} = 625 \text{ kN} \]

The equation of the cable is

\[ y = \frac{4d}{L^2} x (L-x) = \frac{4 \times 25}{250 \times 250} x (250-x) \]

or

\[ x = \frac{x}{625} (250-x) \]

At \( x = 60 \text{ m} \)
\[ y = \frac{60}{625} (250-60) = 18.25 \text{ m} \]

At \( x = 200 \text{ m} \)
\[ y = \frac{200}{625} (250-200) = 16 \text{ m} \]

\[ M_{E0} = \mu + H y \]

\[ = \left( -375 \times 60 + \frac{4 \times 60 \times 60}{2} \right) + (625 \times 18.25) \]

\[ = -3894 \text{ kN-m} \]

and

\[ M_{E00} = (-125 \times 50) + (625 \times 16) = +3750 \text{ kN-m} \]

\[ F_{E0} = F_0 + H \tan \theta \]

\[ = (-375 + 60 \times 4) + (625 \times 0.208) = -5 \text{ kN} \]

\[ F_{E00} = (125) + (625)(-0.24) = -25 \text{ kN} \]

Again, the equivalent U.D.L. transferred to the cable due to the live load

\[ \frac{8d}{L^2} x = \frac{8 \times 25}{240 \times 250} \times 625 = 2 \text{ kN/m} \]

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Equivalent U.D.L. transferred to the cable due to dead load, per metre run

\[ = \left( \frac{4 \times 8 \times 1}{2} \right) = 2 \text{ kN/m} \]

Total \( p = 2 + 2 = 4 \text{ kN/m} \]

Maximum tension in the cable is given by

\[ T = \frac{pL}{2} \sqrt{\frac{1 + \frac{L^2}{16d^2}}{2}} \]

\[ = \frac{4 \times 250}{2} \sqrt{\frac{1 + \frac{250 \times 200}{16 \times 25 \times 25}}{2}} = 1345 \text{ kN} \]

Example 15'11. A suspension bridge cable of span 80 m and central dip 8 m is suspended from the same level at two towers. The bridge cable is stiffened by a three hinged stiffening girder which carries a single concentrated load of 10 kN at a point 20 m from one end. Sketch the S.F. diagram for the girder.

Solution

The simply supported reactions are given by

\[ V_D = \frac{1}{80} \left( 10 \times 60 \right) = 7.5 \text{ kN} \]

\[ V_F = \frac{1}{80} \left( 10 \times 20 \right) = 2.5 \text{ kN} \]

To find the value of \( p \) consider the equilibrium of the stiffening girder and find the bending moment about \( E \).

\[ M_E = 0 = \left( -2.5 \times 40 \right) - \left( p \times 40 \times 20 \right) + \left( \frac{p \times 80}{2} \times 40 \right) \]
At any section distant \( x \) from \( D \), the S.F. is given by
\[
F_x = -7.5 \times 5 - 0.125 \times 25 = 5 \text{kN}
\]
At \( x = 0 \), \( F_x = -7.5 + 5 = -2.5 \)
At \( x = 20 \), \( F_x = -7.5 + 5 - (0.125 \times 20) = +10 \)
At \( x = 80 \), \( F_x = -7.5 + 5 - 0.125 \times 80 + 10 = -2.5 \)

The complete S.F. diagram is shown in Fig. 15'18(b).

**Example 15'12. A suspension bridge cable hangs between two points \( A \) and \( B \) separated horizontally by 120 m and with \( B \) 20 m above \( A \). The lowest point in the cable is 4 below \( A \). The cable supports a stiffening girder weighing \( \frac{1}{2} \) kN/m run which is hinged vertically below \( A \), \( B \) and the lowest point of the cable. Calculate the maximum tension which occurs in the cable when a 10 kN load crosses the girder from \( A \) to \( B \).

**Solution**

\[
p = \frac{100}{800} = 0.125 \text{kN/m}
\]

\[
pL = 0.125 \times 80 = 5 \text{kN}
\]

At any section distant \( x \) from \( D \), the S.F. is given by
\[
F_x = -7.5 \times 5 - 0.125 \times 25 = 5 \text{kN}
\]
At \( x = 0 \), \( F_x = -7.5 + 5 = -2.5 \)
At \( x = 20 \), \( F_x = -7.5 + 5 - (0.125 \times 20) = +10 \)
At \( x = 80 \), \( F_x = -7.5 + 5 - 0.125 \times 80 + 10 = -2.5 \)

The complete S.F. diagram is shown in Fig. 15'18(b).

Given:
\[
\begin{align*}
L_1 = 4 \text{ m} & \quad d_1 = 20 + 4 = 24 \text{ m} \quad L = 120 \text{ m} \\
L_2 = \sqrt{\frac{d_1}{d_4}} & = \sqrt{\frac{4}{24}} = 0.408
\end{align*}
\]

\[L_1 + L_2 = 120 \text{ m}\]

From (i) and (ii)
\[
L_2 = 35 \text{ m} \quad L_2 = 85 \text{ m}
\]

Let the 10 kN load be at a distance of \( x \) from \( D \).

Then the simply supported reactions due to the live load of 10 kN are:
\[
\begin{align*}
V_D &= \frac{1}{120} (120 - x) 10 = \left( 10 - \frac{x}{12} \right) \\
V_F &= \frac{1}{120} (10 x) = \frac{x}{12}
\end{align*}
\]

In order to find the value of \( p \), take moments about \( E \) of all forces to the right of it. Thus
\[
M_E = 0 = \left( -\frac{x}{12} \times 85 \right) - \frac{p}{2} \left( \frac{85}{2} \right)^2 + \left( 60p \right) \left( 75 \right)
\]

or
\[
17 \cdot 5p - \frac{x}{12} = 0 \tag{iii}
\]

The above equation gives the relationship between the load position and the value of \( p \). The maximum tension in the cable depends upon the maximum value of \( p \). The maximum value of \( p \) will evidently occur when the load is on the central hinge, i.e., when \( x = 35 \text{ m} \). Thus, from (iii), we get

\[
p = \frac{35}{12 \times 17.5} = 0.167 \text{kN/m}
\]

Also, the U.D.L. transferred to the hangers due to dead load
\[
= \frac{1}{2} \text{kN/m} = 0.333 \text{kN/m}
\]

\[
\text{Total U.D.L.} = p' = 0.167 + 0.333 = 0.5 \text{kN/m}
\]

\[
\text{The maximum tension will occur at} \ B.
\]

To find \( V_B \), take moments about \( A \)
\[
V_B = \frac{1}{L} \left[ \frac{p'L^2}{2} + H(d_4 - d_4) \right]
\]

or
\[
V_B = \frac{1}{L} \left[ \frac{p'L^2}{2} + \frac{p'L^2}{2d_4} (d_4 - d_4) \right]
\]

From which \( V_B = p'L_2 \)

Similarly, \( V_A = p'L_1 \)

\[
T_B = \sqrt{H^2 + \left( \frac{p'L_2}{2} \right)^2}
\]

But \[
H = \frac{p'L_1^2}{2d_4} \quad \text{[Eq. 15'12](a)}
\]

\[
T_B = p'L_2 \sqrt{\frac{L_2^2}{4d_4^2} + 1} = (0.5 \times 85) \sqrt{\frac{85 \times 85}{4 \times 24 \times 24} + 1} = 86.5 \text{kN}
\]
Example 15.13. A suspension cable, stiffened with a three hinged girder, has 100 m span and 10 m dip. The girder carries a load of 0.4 kN/m. A live load of 10 kN rolls from left to right. Determine (i) the maximum B.M. anywhere in the girder, (ii) the maximum tension in the cable.

Solution

(a) Maximum B.M.

From Fig. 15.13, the absolute maximum negative B.M., due to point load, in the girder occurs at $x=a/2 = 0.211 \times 100 = 21.1$ m from either end, in value being:

$$M_{\text{max. neg.}} = 0.096 \times W \times L = 0.096 \times 10 \times 100 = 96 \text{ kN-m}$$

The absolute maximum positive B.M. occurs at $x=a/2 = 0.25 \times 100 = 25$ m from either end, in value being:

$$M_{\text{max. pos.}} = 0.0625 \times W \times L = 0.0625 \times 10 \times 100 = 62.5 \text{ kN-m}$$

The greatest B.M. is $-96$ kN-m at 21.1 m from either ends. It should be noted that there will be no B.M. anywhere in the girder due to U.D.L. covering the whole span.

(b) Maximum cable tension

From Fig. 15.13 (c), the maximum value of $H$, due to point load, occurs when the load is on the central hinge, in value being

$$H = W \times \frac{L}{4d} = \frac{10 \times 100}{4 \times 10} = 25 \text{ kN}$$

$p$ (due to U.D.D.L.) = 0.4 kN-m.

$H$ due to U.D.D.L. = $\frac{pL^2}{8d} = \frac{0.4 \times 10 \times 100}{8 \times 10} = 50 \text{ kN}$

Total $H = 25 + 50 = 75 \text{ kN}$

$$T_{\text{max}} = \sqrt{V^2 + H^2}$$

$$T_{\text{max}} = H \sqrt{1 + \frac{16d^2}{L^2}}$$

$$T_{\text{max}} = 75 \sqrt{1 + \frac{16 \times 100}{100 \times 100}} = 76 \text{ kN}.$$
For maxima, \( \frac{dM}{dx} = 0 \)
\[-WL + \frac{2WxL}{85} - \frac{3Wx^2L^2}{85} = 0 \]
Cancelling \( W \) and substituting the value of \( L \), we get
\[-1 + 2a + \frac{2 \times 120a^2}{85} - \frac{3 \times 120a^3}{85} = 0 \]
or
\[a^3 - 1'14a^2 + 0'36 = 0 \]
\[a = 0'18; \quad aL = 0'18 \times 120 = 21'6 \text{ m} \]
Substituting the value of \( W, a \) and \( L \) in (2), we get
\[M_{\text{max}} = -10 \times 0'18 \times 120(1 - 0'18) + 10 \times 0'18 \times 120 \frac{a}{85} \]
\[= -177 + 54'9 - 9'9 = -132 \text{ kN-m} \]
at 21'6 m from left support.

Example 15.15. A symmetrical three pinned stiffening girder of a suspension bridge is 600 ft. long. Find the magnitude of the largest bending moment that can be exerted by a moving load 20 tons uniformly distributed over a length of 30 ft. Indicate the position of the load for this condition.

Solution

Let the absolute maximum B.M. occur at a point \( X \) distant \( x \) from \( D \), when the head of the load is ahead of it by a distance \( k \),

Keeping the position of the section fixed, let us first find out the value of \( k \) to get maximum B.M. at \( X \).

The simply supported reactions are:
\[V_F = \frac{20}{600} (x + k - 15) \text{ and } V_D = 20 - V_F \]

To find \( p \), take moments about \( E \). Thus
\[M_E = 0 = \left[ -\frac{20}{600} (x + k - 15) \times 300 \right] + \left\{ p \times 300 \times 300 \right\} - \frac{P}{2} \left( 300 \right)^2 \]
\[0 = \frac{x + k - 15}{4500} \]

The bending moment at the fixed section is given by
\[M_x = -V_D \times \left\{ \frac{pL}{2} \right\} - x - \frac{px^2}{3} + \frac{20}{30} \left( 30 - k \right)^2 \]
\[= -\left\{ \frac{20x}{30} \left( x + k - 15 \right) \right\} + \left\{ \frac{x + k - 15}{4500} \times 300x \right\} \]
\[= -\left\{ \frac{x + k - 15}{4500} \times \frac{x^2}{2} \right\} + \left\{ \frac{(30 - k)^2}{3} \right\} \]

For maxima, \( \frac{dM_x}{dk} = 0 \)
\[0 = +\frac{x}{30} + \frac{x}{15} - \frac{x^2}{9000} + \frac{2(30 - k)(-1)}{3} \]
\[x = \frac{3}{20} \times 30 + 30 \]

Substituting this value of \( k \) in (2), we get
\[M_x = -20x + \frac{x}{30} \left\{ x - 15 + \left\{ \frac{-3}{20} x + \frac{x^2}{6000} + 30 \right\} \right\} \]
\[+ \frac{x}{15} \left\{ x - 15 + \left\{ \frac{-3}{20} x + \frac{x^2}{6000} + 30 \right\} \right\} \]
\[= \frac{x^2}{9000} \left\{ x - 15 + \left\{ \frac{-3}{20} x + \frac{x^2}{6000} + 30 \right\} \right\} \]
\[+ \frac{3}{1} \left\{ \frac{30 - \left( \frac{-3}{20} x + \frac{x^2}{6000} + 30 \right)}{600} \right\} \]

Simplifying and rearranging, we get
\[M_x = -1'8'5x + \frac{17x^2}{200} + \frac{7x^2}{1200} - \frac{17x^2}{180000} - \frac{x^4}{108'000,00} \]

For getting the absolute maximum B.M., treat \( x \) as variable.

Thus \( \frac{dM_x}{dx} = 0 \), for maxima
\[0 = -1'8'5 + \frac{17x}{100} + \frac{7x^2}{600} - \frac{17x^2}{60000} - \frac{x^4}{27'000,000} \]
or
\[0 = -1'8'5 + \frac{109}{600} x + \frac{17x^2}{60000} + \frac{x^4}{27'000,000} \]
Using Newton's method, try \( x = 150 \)

\[
f(x) = 18.5 - 27.25 + 6.374 + 12 = -2.26
\]

\[
f'(x) = -\frac{109}{600} + \frac{17x}{30000} + \frac{x^3}{90000000}
\]

\[
= -0.018 + 0.00572 + \ldots = 0.1
\]

Better value of \( x = 150 - \frac{-2.26}{0.1} = 150 - 22.6 = 127.4 \)

Try \( x = 127.4 \)

\[
f(x) = 18.5 - 109 \times 127.4 + 17(127.4)^2 + (127.4)^3
\]

\[
= 0.0309
\]

\[
f'(x) = -0.018 + 0.0072 + \ldots = 0.108
\]

Better value of \( x = 127.4 - \frac{0.0309}{0.108} = 127.7 \)

Use approximate value of \( x = 128 \) ft.

Thus the absolute maximum B.M. occurs at a section 128 ft. from the left support. To get the value of absolute maximum B.M. substitute \( x = 128 \) ft in Eq. 4. Thus.

\[
M_{\text{max}} = (-18.5 \times 128) + \frac{17}{200} (128)^3 + \frac{7}{1200} (128)^4
\]

\[
= -\frac{17}{18000000} (128)^3 - \frac{(128)^4}{108000000}
\]

\[
= -1081 \text{ ft}-\text{ft.}
\]

This occurs when \( k = \frac{3}{20} \times 128 - \frac{(128)^3}{6000} + 30 \)

\[
= 13.5 \text{ ft.}
\]

Hence absolute maximum bending moment occurs at a section 128 ft from left support, when the head of the load is at a distance \((128 + 13.5) = 141.5 \) ft from the support.

15.7. TWO HINGED STIFFENING GIRDER

A two hinged stiffening girder is a statically indeterminate structure, since there are three unknowns to be determined (i) the reaction \( V_D \) at the hinged support \( D \), (ii) the reaction \( V_F \) at the hinged support \( F \), and (iii) the pull \( p \) exerted by the cables. Only two equations of statical equilibrium are available. In the case of a three hinged girder, an additional equation \( M_E = 0 \) was available at the central hinge. The problem of two hinged girder can be solved approximately by the strain energy method. The solution by strain energy is not within the scope of the book. The problem will therefore, be solved approximately by assuming the girder to be infinitely rigid, so that the pull \( p = \frac{W}{L} \), where \( W \) is the point load placed anywhere on the girder. On the assumption, the pull in the cable is constant, irrespective of the load position, while in the case of a three hinged girder, \( p = \frac{4a}{L} \) and depends upon the load position.

(1) INFLUENCE LINE FOR \( H \)

For the cable, \( H = \frac{pL^2}{8d} \)

Consider a unit load placed at distance \( aL \) from the left support. Then

\[
p = \frac{W}{L} = \frac{1}{L}
\]

\[
H = \frac{L}{L.8d} = \frac{L}{8d}
\]

Thus, the horizontal pull at the abutments of the cable is constant, and does not vary with the load position. The influence lines for \( p \) and \( H \) will thus be rectangles, as shown in Fig. 15.22 (b) and (c) respectively.

(2) BENDING MOMENT DIAGRAM

The bending moment at any section \( X \), distant \( a \) from \( D \) is given by

\[
M_x = M_x + H.y
\]

The \( M_x \)-diagram is a triangle having a maximum ordinate of \( \frac{aL(L-aL)}{L} = aL(1-a) \) under the unit load.

The \( H.y \)-diagram is a parabola since \( y \) varies parabolically with \( x \), having a maximum value of the ordinate equal to \( \frac{L}{8d} \) at the middle of the span.

At the point \( X \), the ordinate of the \( H.y \)-diagram will be

\[
\frac{L}{8d} = \frac{L}{8d} \cdot \frac{4d}{L^2} \cdot x(L-x) = \frac{x}{2L}(L-x)
\]

The B.M.D. is shown in Fig 15.22(d).
(3) INFLUENCE LINE FOR BENDING MOMENT

The B.M. at section $X$ distant $x$ from $D$ is given by

$$M_x = \mu_x + H_y$$

Thus the I.L. for B.M. is obtained by superimposing I.L. for $\mu_x$ on the I.L. for $H_y$.

The I.L. for $\mu_x$ is a triangle having a maximum ordinate of $\frac{x(L-x)}{L}$ under the section.

The value of $y$ is fixed for a given section $X$. Also the value of $H$ is fixed, and does not vary with the load position. Hence the quantity $H_y$ is fixed, its value being

$$H_y = \frac{L-x}{L} \frac{4d}{E} \frac{x(L-x)}{L} = \frac{1}{2} (L-x)$$

Thus the I.L. for $H_y$ is a triangle having an ordinate of $\frac{x(L-x)}{2L}$.

Fig. 15'22(e) shows the I.L. for bending moment at $X$.

By the inspection of the Fig. 15'22(e), we have

$$\begin{align*}
\frac{d_x}{d} &= \frac{1}{2} x \frac{x_x}{x} \\
\frac{d_g}{d} &= x_g + x_h = h_f \\
\frac{d}{x} &= x_g + x_h = h_f \\
But \quad (dx_g + dx_h) + (dg_g + dh_f) &= L \\
\frac{d_g}{x} &= x_g = h_f \\
\end{align*}$$

$$\text{And area of } \triangle dx_g h = \text{area of } \triangle dd_g + \triangle dh_f$$

Thus, in the case of two hinged girder also, there will be no bending moment anywhere in the girder due to uniformly distributed dead load or live load covering the whole span.

(4) MAXIMUM BENDING MOMENT DIAGRAM DUE TO SINGLE POINT LOAD $W$

When a point load $W$ rolls over the girder, the maximum +ve B.M. at $X$ will occur [Fig. 15'22(e)] when the load is either on the left or right hinge, and its value is equal to $\frac{Wx}{2L} (L-x)$.

The maximum negative B.M. at $X$ will occur when the load is on the section itself, its value being equal to $-\frac{Wx}{2L} (L-x)$.

$$\therefore \quad (\pm) M_{max} = \frac{Wx}{2L} (L-x)$$
The variation is thus parabolic. Fig. 15'22 (f) shows the max B.M.D.

For absolute max. B.M.,
\[
\frac{dM_{max}}{dx} = \frac{W}{2} - \frac{Wx}{L} = 0
\]
\[x = \frac{L}{2}\]
\[\pm M_{max} = \frac{W}{2} \cdot \frac{L}{2} = \frac{WL}{2}\]  \(15.40\)

(5) **MAXIMUM BENDING MOMENT DIAGRAM DUE TO UNIFORMLY DISTRIBUTED LOAD**

By inspection of Fig. 15'22(e), the maximum negative B.M. at X will occur when only portion gh is loaded, while the maximum positive B.M. at X will occur when the portion dg and hf are loaded keeping portion gh empty. In either case, the maximum B.M. is given by:
\[
(\pm)M_{max} = \frac{w}{2} \cdot \frac{L}{2} \cdot \frac{x}{2L} \cdot (L-x) - \frac{wx}{8}(L-x)
\]  \(15.41\)

The variation is parabolic. Fig. 15'22(g) shows the maximum B.M.D. The absolute maximum B.M. will evidently occur at:
\[x = \frac{L}{2}\]
\[\pm M_{max} = \frac{w}{8} \cdot \frac{L}{2} = \frac{WL^3}{32}
\]  \(15.42\)

(6) **INFLUENCE LINE FOR SHEAR FORCE**

The S.F. at any section X, distant x from the left hinge is given by:
\[F_s = f_s + H \tan \theta
\]
where \(f_s\) = shear force at X due to simply supported actions.

The I.L. for \(f_s\) will have zero ordinates at the ends, and ordinates of \(\frac{x}{L}\) and \(\frac{L-x}{L}\) and the section.

The I.L. for \(H \tan \theta\), in which both \(H\) and \(\tan \theta\) are constant, will be rectangle having the ordinate:
\[H \tan \theta = \frac{L}{8d} \cdot \frac{8d}{L^2} \left( \frac{L}{2} - x \right) = \left( \frac{1}{2} - \frac{x}{L} \right)
\]  \(15.43\)

Thus, the I.L. for \(F_s\) will be given by superimposing the I.D. for \(F_s\) over the I.L. for \(H \tan \theta\) as shown in Fig. 15'22 (i). From Fig. 15'22(i), we have:
\[
dd_1 = f_1 = \left( \frac{1}{2} - \frac{x}{L} \right) ; x_1 = \frac{x}{L}
\]
\[x_2 = \left( \frac{1}{2} - \frac{x}{L} \right) ; x_2 = \left( \frac{1}{2} - \frac{x}{L} \right)
\]
\[x_3 = \left( \frac{1}{2} - \frac{x}{L} \right) ; x_3 = \left( \frac{1}{2} - \frac{x}{L} \right)
\]
\[x_4 = \left( \frac{1}{2} - \frac{x}{L} \right) ; x_4 = \left( \frac{1}{2} - \frac{x}{L} \right)
\]

Now
\[
\frac{\Delta e}{e_f} = \frac{x_{x_s}}{f_1} = \frac{1}{2 - \frac{x}{L}} = \frac{1}{2 \left( \frac{1}{2} - \frac{x}{L} \right)}
\]
\[x_{1} + e_f = \left( \frac{1}{2} - \frac{x}{L} \right)
\]
\[x_{2} = \left( \frac{1}{2} - \frac{x}{L} \right)
\]
\[\text{Area } x_{x_s} e = \frac{L}{2} \cdot \frac{L}{2} = \frac{L}{8}
\]

Also, \[\text{Area } (dd_1 x, x) + \text{area } (e_f) = \frac{1}{2} \left[ \left( \frac{1}{2} - \frac{x}{L} + \frac{1}{2} \right) \right] + \left[ \frac{1}{2} \left( \frac{L}{2} - x \right) \left( \frac{1}{2} - \frac{x}{L} \right) \right]
\]
\[= \frac{L}{8}
\]

Hence the area of positive S.F. is equal to the area of negative shear force. It can, therefore, be concluded that there will be no S.F. anywhere in the girder due to uniformly distributed dead and/or live load covering the whole span.

15'8. **TEMPERATURE STRESS IN TWO HINGED GIRDER**

In the case of a cable subjected to a change of temperature \(t\), the change in the dip \(d\) is given by Eq. 15'18.
\[3d = \frac{3}{16} \cdot \frac{L^2}{d} at
\]  \(1\)

The horizontal pull \(H\) and the U.D.L. \(p\) carried by the suspenders depend directly or indirectly on the value of the dip \(d\).

Let \[\delta p = \text{change in pull } p, \text{ due to change in } 3d.
\]
Due to this change in $\delta p$, the change in the maximum deflection at the centre of the girder is given by

$$\delta(\Delta) = \frac{5}{384} \frac{\delta p L^4}{EI}$$

where

$\Delta$ = deflection at the centre of the girder.

$EI$ = flexural rigidity of the girder.

Now, when dip $d$ increases (i.e. when the cable sags), the suspenders become loose. The suspenders can remain taut only if the girder sags by an equal amount at every point. Thus, we have the compatibility equation.

$$\delta \Delta = \delta d$$

or

$$\delta p = \frac{72}{5} \frac{E I a t}{L d}$$

(15.43)

Increase in B.M. at the girder = $\frac{\delta p L^3}{8}$

Increase in the stress in the girder = $\frac{\delta p L^3}{8} \cdot \frac{D}{2I}$

$$= \frac{72}{5} \frac{E I a t}{L d} \cdot \frac{L D}{16I}$$

$$= \frac{9}{10} \frac{D}{d} \cdot \frac{E a t}{16I}$$

(15.45)

where

$D$ = height of the girder.

Thus, the change in the stress is independent of the span and the moment of inertia, and depends on $\frac{D}{d}$ ratio.

Example 15.16. A suspension bridge with two hinged stiffening girder has a span of 100 m and the cables have a central dip of 10 m. The stiffening girders are 4 m deep, and have moment of inertia equal to $1.64 \times 10^{10}$ mm$^4$. If the temperature falls through 22 Kelvin, calculate (i) the flange stress, (ii) increase in the tension in the cable. Take $E=2 \times 10^8$ N/mm$^2$ and $a=11 \times 10^{-4}$ per 1 K.

Solution

From Eq. 15.45. stress in the flanges of the stiffening girder due to fall of temperature

$$= \frac{9}{10} \frac{D}{d} \cdot \frac{E a t}{16I}$$

CABLES AND SUSPENSION BRIDGES

$$= \frac{9}{10} \times \frac{4}{10} \times 2 \times 10^8 (11 \times 10^{-4}) \times 22$$

$$= 17.42 \text{ N-mm}^2 = 17.42 \times 10^3 \text{ kN/mm}^2$$

The increase in load $\delta p$ on the girder is given by

$$\delta p = \frac{72}{5} \frac{E I a t}{L d}$$

where $EI=(2 \times 10^6)(1.64 \times 10^{10})=3.28 \times 10^{18} \text{ kN-mm}^2$

$$= 3.28 \times 10^4 \text{ kN-m}^2$$

$$\therefore \delta p = \frac{72}{5} \frac{(3.28 \times 10^4) \times 11 \times 10^{-4} \times 22}{100 \times 100 \times 10} = 0.1143 \text{ kN-m}$$

$$= \frac{9}{10} \frac{L^3}{8d} = \frac{0.1143 (100)^3}{8 \times 10} = 14.29 \text{ kN}$$

$$\therefore \text{ Change in the cable tension is}$$

$$\delta T = \delta H \sqrt{1 + \left(\frac{4d}{L}\right)^2}$$

$$= 14.29 \sqrt{1 + \left(\frac{4 \times 10}{100}\right)^2} = 15.39 \text{ kN}$$

Example 15.17: A suspension cable has a span of 160 m and a central dip of 16 m, and is suspended from the same level at both towers. The bridge is stiffened by a stiffening girder hinged at the end supports. The girder carries a single concentrated load of 8 kN at a point 40 m from left end. Assuming equal tensions in the suspension hangers, calculate (i) the horizontal tension in the cable and (ii) the maximum positive and negative bending moments.

If the 8 kN load rolls from left to right, what will be the value of absolute maximum B.M. and S.F. and where do they occur?

Solution. (Fig. 15.22)

(i) Load per metre run in the hangers $= \frac{W}{L}$

$$= \frac{8}{160} = 0.05 \text{ kN/m}$$

Horizontal tension in the cable $= \frac{PL^3}{8d}$

$$= \frac{0.05 \times 160 \times 160}{8 \times 16} = 10 \text{ kN}$$

(ii) The simply supported reactions are

$$V_F = \frac{8 \times 40}{160} = 2 \text{ kN} \ ; \ V_D = 8 - 2 = 6 \text{ kN}$$
The maximum negative B.M. will occur under the load,

\[-(-)M_{max} = -(-Vb \times 40) + \left(\frac{Pl}{2} \times 40\right) - \frac{P}{2} = -240 - \frac{0.05 \times 160 \times 40}{2} - \frac{0.05}{2} \times 1600 = -120 \text{ kN} \cdot \text{m}\]

The maximum positive B.M. will occur in the portion to the right of the load. Measuring \( x \) from \( F \), we have

\[M = -V_F x + \frac{pL}{2} x^2 - \frac{p}{2} x^2\]

\[= 2x - \frac{0.05 \times 160}{2} x^2 - \frac{0.05}{2} x^2 = 2x - \frac{0.05}{2} x^2\]

For maxima, \( \frac{dM}{dx} = 0 = -0.05 x \)

\[x = \frac{2}{0.05} = 40 \text{ m}\]

\[(-)M_{max} = 2x - \frac{0.05}{2} x^2 = (2 \times 40) - \frac{0.05}{2} (40)^2 = +40 \text{ kN} \cdot \text{m}\]

(iii) Absolute maximum (±) B.M. occurs at the mid span.

\[\text{(±) } M_{\text{max, mid}} = \frac{WL}{8} = \frac{8(160)}{8} = 160 \text{ kN} \cdot \text{m}\]

(iv) From the I.I. for S.F. (Fig. 15.22), it is clear that the absolute maximum S.F. occurs under the load, its value being equal half the load = \( \frac{1}{2} \times 8 = 4 \text{ kN} \), irrespective of the position of the load.

**PROBLEMS**

1. A cable is suspended between two points which are at the same level 120 m apart horizontally. The cable carries uniform load of 15 N per horizontal metre, and two concentrated loads, one of 900 N at 40 m horizontally from one end and the other of 300 N at 40 m horizontally from the other end. Determine the horizontal distance from one end to the lowest point and maximum tension in the cable.

2. The cables of a suspension bridge of 100 m span are suspended from piers which are 12 metres and 6 metres respectively above the lowest point of the cable. The load carried by each cable is 1 kN/m of span. Find (i) the length of the cable between the piers, (ii) the horizontal pull in the cable.

3. A suspension cable has a span of 400 ft measured horizontally and the level of the left-hand end \( A \) is 8 ft below the level of the right-hand end \( B \). A load of 12 tons is carried by the cable at a point \( C \) which is 150 ft horizontally from \( B \) and 21 ft below the level of \( A \). The weight of cable may be taken as 3 lb/ft of horizontal distance. Determine the horizontal and vertical reactions at the ends and the maximum tension in the cable.

4. A suspension cable of 160 m span and 16 m central dip carries a load of \( \frac{1}{2} \) kN per linear horizontal metre, calculate the maximum and minimum tension in the cable. Find the horizontal and vertical forces in each pier under the following alternative conditions:

   (i) if the cable passes over frictionless rollers on the top of the piers.

   (ii) if the cable is firmly clamped to saddles carried on frictionless rollers on the tops of the piers. In each case the backstay is inclined at 30° with the horizontal.

5. An unstiffened suspension cable carries a total load of 40 kN uniformly distributed over a span of 160 m. The suspension and anchor cables are attached to saddles free to move horizontally on the piers, one saddle being 12 m and the other 20 m above the lowest point on the cable. The anchor cables are inclined at 45° to the vertical and their weight may be neglected. Determine the greatest and the least tension in the suspension cables, the greatest thrust on a pier and the tension in an anchor cable.

6. A suspension cable of span \( L \), having in the shape of a symmetrical parabola is strengthened by a stiffening girder pinned at the abutments and at the centre. Show that for the passage across the bridge of a uniformly distributed load longer than the span the maximum ± shearing force occurs at the abutments. Find the magnitude of these forces and of the maximum ± shearing force at the centre of the span and state the corresponding loading conditions in all cases. (U.L.)

7. The roadway for a suspension bridge of span 2L is stiffened by longitudinal girders of length \( L \), pin-joined together at the centre of the span and hinged at their outer ends the abutments.

   Their girders are supported by a large number of vertical tie rods attached to suspending chains, the lengths of tie rods being such that each chain is the form of a parabola with the axis vertical.

   Show that the greatest hogging and sagging bending moments set up in the bridge by concentrated load \( W \) advancing across the bridge are \( \frac{W L}{8} \) and \( \frac{W L}{3\sqrt{3}} \). (Cambridge)
8. The cable of a suspension bridge have a span of 160 m and a central dip of 20 m. Each cable is stiffened by girder hinged at the ends at midspan to constrain the cable to retain its parabolic shape. There is a uniform dead load of 1/4 kN per horizontal metre of span over the whole of the girder and in addition a load of 3/4 kN per horizontal metre and 40 m long.

Determine the maximum cable tension when the live load is situated on the left hand of the stiffening girder with its right end over the central hinge. Sketch the S.F. and B.M. diagrams for the girder showing on them the maximum positive and negative values.

9. The towers of a 500 ft span suspension bridge are of unequal height, one is 60 ft and the other 20 ft above the lowest point of the cable, which is immediately above the inner pin of a three-pinned stiffening girder hinged at the towers. Find the maximum tension in the cable due to a point load of W crossing the bridge. (U.L.)

10. A suspension bridge cable hangs between two points A and B separated horizontally by 300 ft and with B 50 ft above A. The lowest point in the cable is 10 ft below A. The cable supports a stiffening girder weighing 1/4 t/ft run which is hinged vertically below A, B and the lowest point of the cable. Calculate the maximum temperature which occurs in the cable when a 20 t load crosses the girder from A to B. (St. Andrews)

11. The towers of a suspension bridge with a three pinned stiffening girder are 15 m and 10 m high respectively and are 50 m apart. The cable dips 4 m below the top of the 10 m tower and its lowest point is immediately above the pin in the stiffening girder. Find the position and magnitude of the largest bending moment which a point load of 4 kN can induce in the girder together with position of the load.

12. A steel cable 2 cm diameter is stretched across two poles 100 metres apart. If the central dip is 2 m at a temperature of 58°F, calculate the stress intensity in the cable. Calculate the fall of temperature necessary to raise the stress to 550 kg/cm². Weight of steel = 7.8 g/cm³ and α = 6.2 x 10⁻⁶ per °F.

13. A suspension bridge with two hinged stiffening girder has a span of 130 metres, the cables having a central dip of 15 metres at 65°F. If the stiffening girder is 5 m deep, calculate the flange stress due to a fall of 25°F in temperature at the central section, given I for the section = 4 x 10⁶ cm⁴. Find also the increase in the horizontal tension in the cable. Take E = 2.10 x 10⁸ kg/cm² and α = 6.2 x 10⁻⁶ per °F.

14. The two parts of the three pinned stiffening girder of a suspension bridge are 400 ft and 300 ft long respectively. Find the position and magnitude of the maximum bending moment due to a uniformly distributed load w per foot run for 100 ft on both sides of the inner pin. (U.L.)

Answers

1. 46.67 m; 6750 kN.
2. (i) 102.22 m, (ii) 143.1 kN, (iii) 155.5 kN, 146 kN.
   (iv) 167.63 kN, 115.77 kN.
3. V_A = 6.81 tons, V_B = 5.725 tons, H = 47.9 tons, T_A = 48.4 tons.
4. Max. tension = 107.7 kN, Minimum tension = H = 100 kN
   (i) 6.7 kN horizontally, 93.8 kN vertically.
   (ii) 97.7 kN vertically, No force horizontally.
5. (i) Greatest tension = 56.4 kN,
   (ii) Least tension = H = 51.6 kN,
   (iii) Greatest thrust on the pier = 74.18 kN,
   (iv) Tension in anchor cables = 73 kN.
6. T_A = 95 kN
   M_A = 135 kN-m at 56 m from left support.
   M_A = 125 kN-m at 120 m from left support.
7. H = 3.35 W, T_A = 3.564 W.
8. H = 166 tons.
9. M_A = 21.91 kN-m at 12.15 m from left support.
10. 489 kg/cm²; 48°F.
11. 97.6 kg/cm², 828 kg.
12. 7410 w at 174 ft from the remote end of the 400 ft length.
16
Arches

141. INTRODUCTION

An arch may be looked upon as a curved girder, either a solid rib or braced, supported at its ends and carrying transverse loads which are frequently vertical. Since the transverse loading at any section normal to the axis of the girder is at an angle to the normal face, an arch is subjected to three restraining forces: (i) thrust, (ii) shear force and (iii) bending moment. Depending upon the number of hinges, arches may be divided into for classes (Fig. 16'1):

1. Three hinged arch
2. Two hinged arch
3. Single hinged arch
4. Fixed arch (hingeless arch).

(a) THREE HINGED ARCH  (b) TWO HINGED ARCH

(c) SINGLE HINGED ARCH  (d) FIXED ARCH

Fig. 16'1.
Type of arches

A three hinged arch is a statically determinate structure while the rest three arches are statically indeterminate. In bridge construction, especially in railroad bridges, the more frequently used arches are the two-hinged and the fixed end ones.

16.2. LINEAR ARCH (THEORETICAL ARCH)

Consider a system of jointed linkwork inverted about AB, with loads as shown in Fig. 16'2 (a). Under a given system of loading, every link will be in a state of compression. The magnitudes of pushes (or thrusts) T1, T2, T3, etc., can be known by the rays Od, Oe, etc., in the force polygon. For any arch under a given system of loading, the final lines of actions of thrust T1, T2, T3, etc., in the respective segments can be plotted by the usual graphical methods if the horizontal reactions at A and B are known. The line of thrust (i.e., the actual lines of action of thrusts T1, T2, T3, etc.) is known as the theoretical arch or linear arch.

![Fig. 16'2. Theoretical arch.](image)

It is, however, not possible to construct the actual arch of the shape of theoretical arch. The moving loads will change the shape of the theoretical arch, and it cannot be made to change its shape to
suit the varying load positions. In actual practice, therefore, an arch
is made parabolic, circular or elliptic in shape.

Consider a cross-section PQ of the arch [Fig. 16'2 (c)]. Let T
be the resultant thrust acting through D along the linear arch. The
thrust T is neither normal to the cross-section nor does it act through
centre C of the cross-section.

The resultant thrust T can be resolved normal and tangential
to the section PQ. Let N be the normal component and F be the
tangential component. Evidently, the tangential component F will
cause shear force at the section PQ. The normal component N acts
eccentrically, the eccentricity being equal to CD. Thus, the action of
N acting at D is twofold: (i) a normal thrust N at C, and (ii) a
bending moment \( M = N \cdot e \) at C. Hence, unlike beams, a section of
arch is subjected to three straining actions: (i) Shear force \( F \),
(ii) Bending moment \( M \), and (iii) Normal thrust \( N \). The shear force
F is also sometimes known as the radial shear.

163. EDDY'S THEOREM

Consider a section at \( P \) distant \( x \) from \( A \), of an arch, shown
in Fig. 16'3. Let the other co-ordinate of \( P \) be \( y \). For the given

![Diagram](image_url)

Fig. 16'3.

system of loads, the linear arch can be constructed (if \( H \) is known).
Since funicular polygon represents the bending moment diagram to
one scale, the vertical intercept \( P_1 P_2 \) at the section \( P \) will give the
bending moment due to external load system. If the arch is drawn
to a scale of 1 cm = \( p \) m, load diagram is plotted to a scale 1 cm = \( q \)

N and if the distance of pole \( O \) from the load line is \( r \), the scale of
bending moment diagram will be 1 cm = \( p \cdot q \cdot r \) N-m.

Now, theoretically, the B.M. at \( P \) is given by

\[
M_P = -V_1 x + W_1 (x - a) + H y
\]

where \( \mu_x = -V_1 x + W_1 (x - a) \)

= Usual bending moment at a section due to load
system on a simply supported beam.

From Fig. 16'3, we have,

\[
\mu_x = (P_1 P_2) \times \text{scale of B.M. diagram}
\]

\[
= -P_1 P_2 (p \cdot q \cdot r)
\]

and \( Hy = (P_2 P_3) \times \text{scale of B.M. diagram} \)

\[
= P_2 P_3 (p \cdot q \cdot r)
\]

Hence \( M_P = \mu_x + Hy = -P_1 P_2 (p \cdot q \cdot r) + PP_2 (p \cdot q \cdot r) \)

\[
= (P_2 P_3 (p \cdot q \cdot r)
\]

Hence the ordinate between the linear arch and the actual
arch gives the bending moment. This is known as Eddy's theorem
and may be stated as below:

"The bending moment at any section of an arch is equal to the
vertical intercept between the linear arch and the centre line of the
actual arch".

164. THREE HINGED ARCH

A three hinged arch is a statically determinate structure, having
a hinge at each abutment or springing, and also at the crown.
There are in all four reaction components (two at each hinge, i.e.,
\( H \) and \( V \)). Three equations are available from the static equilibrium
and one additional equation is available from the fact that the B.M.
at the hinge at the crown is zero. Thus, the value of \( H \) can be easily
calculated for any given load system.

![Diagram](image_url)

Fig. 16'4.
The arch shown in Fig. 16'4 (a), is subjected to a number of loads \( W_1, W_2, \) etc. Let the reactions at \( A \) and \( B \) be \( (H, V_A) \) and \( (H, V_B) \) respectively. Since the B.M. at \( C \) is zero, we have,

\[
M_C = \mu_C + HY = 0
\]

\[
H = -\frac{\mu_C}{Y} \quad \text{...(16'1)}
\]

The value of \( H \) is thus known. The value of \( V_A \) can be known by taking moments, of all forces, about \( B \). Similarly, \( V_B \) can be known. After having known the reaction components, the value of radial shear \( (F) \) and normal thrust \( (N) \) at any section \( P \) can be easily calculated, with reference to Fig. 16'4 (b) where equilibrium of left portion \( PA \) has been shown. The vertical and horizontal sections on the section \( P \) are : 

\[
V = V_A - W_1 \quad \text{and} \quad H = H.
\]

Now, resolving along the section at \( P \), we get

\[
F = H \sin \theta - V \cos \theta \tag{16'2}
\]

Similarly, resolving normal to the section, we get

\[
N = H \cos \theta + V \sin \theta \quad \text{[(16'2 a)]}
\]

**THREE HINGED PARABOLIC ARCH**

The equation of a parabola, with origin at the left hand hinge \( A \) [Fig. 16'4 (a)] can be written as

\[
y = kx(L - x) \quad \text{(i)} \]

where \( k \) is a constant

At 

\[
x = \frac{L}{2}, \text{let } y = r = \text{central rise}.
\]

Substituting in (i), we get

\[
r = k \left( \frac{L}{2} \right) \left( \frac{L - \frac{L}{2}}{2} \right) = k \frac{L^3}{4}
\]

\[
k = \frac{4r}{L^3}
\]

\[
y = \frac{4r}{L^3} x (L - x) \quad \text{...(16'3)}
\]

This is the equation of a parabolic arch.

According to Eddy's theorem, the vertical intercept between the linear arch and the centre line of the actual arch gives the B.M. at a section. Due to uniformly distributed load, the linear arch will be a parabola. It will pass through the hinge at the crown. The centre line of the actual arch is also parabolic, passing through the central hinge. These two parabolas pass through three common

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points and hence they overlap each other. Therefore a parabolic arch will not have B.M. due to U.D.L. It will be subjected to pure compression.

**THREE HINGED CIRCULAR ARCH**

Let us now consider the centre line of the arch to be segment of a circle of radius \( R \), subtending an angle of 20 at the centre. It is always convenient to have the origin at \( D \), the middle of the span. Let \((x, y)\) be the co-ordinates of the point \( P \). Draw line \( PC_1 \) parallel to \( AB \).

Then

\[
OP^2 = OC_1^2 + PC_1^2
\]

or

\[
R^2 = (y + (r - r))^2 + x^2 \quad \text{...(16'4)}
\]

Equation (16'4) connects \( y \) with \( x \).

Also,

\[
r(2R - r) = \frac{L}{2} \cdot \frac{L}{2} = \frac{L^3}{4} \quad \text{...(16'5)}
\]

From equation (16'5), the value of the radius can be calculated for the known values of the span and the rise.

The co-ordinates of \( P \) (i.e. \( x \) and \( y \)) can also be expressed as trigonometric functions. Thus, if \( OP \) makes an angle \( \beta \) with \( OC \),

\[
x = OP \sin \beta = R \sin \beta
\]

and

\[
y = OC_1 - OD = R \cos \beta - R \cos \theta = R (\cos \beta - \cos \theta)
\]

**Example 16'1.** A parabolic arch hinged at the springings and crown has a span of 20 m. The central rise of the arch is 4 m. It is loaded with a uniformly distributed load of intensity 2 kN/m on the left 8 m length. Calculate (a) the direction and magnitude of reaction at the hinges, (b) the bending moment, normal thrust and shear at 4 m
and 15 m from the left end, and (c) maximum positive and negative bending moments.

Solution

Fig. 16.6.

(a) For vertical reaction at A, take moments about B. Thus

\[ V_A \times 20 = 2 \times 8(20 - 4) \]

\[ V_A = 12.8 \text{ kN} \]

Hence \[ V_B = 8 \times 2 - 12.8 = 3.2 \text{ kN} \]

Since the bending moment at the hinge C is zero, we have

\[ M_C = (3.2 \times 10) + H \times 4 = 0 \]

\[ H = \frac{320}{4} = 8 \text{ kN} \]

\[ \therefore \text{Reaction at } A = R_A = \sqrt{V_A^2 + H^2} = \sqrt{(12.8)^2 + 8^2} = 15.09 \text{ kN} \]

Its inclination with the horizontal is given by

\[ \tan \theta_A = \frac{V_A}{H} = \frac{12.8}{8} = 1.6; \quad \therefore \theta_A = 58^\circ \]

(b) Reaction at B = \( R_B = \sqrt{V_B^2 + H^2} = \sqrt{(3.2)^2 + 8^2} = 8.62 \text{ kN} \)

Its inclination with the horizontal is given by,

\[ \tan \theta_B = \frac{V_B}{H} = \frac{3.2}{8} = 0.4; \quad \therefore \theta_B = 21^\circ 48' \]

The magnitude and direction of the reaction at the crown will be the same as that of the reaction and the hinge B, i.e., 8.62 kN at 21° 48' with the horizontal. This is due to the fact that there is no

**ARCHES**

Loading between B and C. The reaction at B passes through C since \( MC = 0 \).

(b) The bending moment diagram is shown in Fig. 16.6(b).

The equation of the parabola is

\[ y = \frac{4r}{L} x(20 - x) = \frac{4 \times 4}{400} x(20 - x) = \frac{x}{25}(20 - x) \]

Thus

\[ \frac{dy}{dx} = \frac{20 - 2x}{25} \]

At \( x = 4 \) m, \( y = \frac{4}{25}(20 - 4) = 2.56 \text{ m} \)

\[ \tan \theta = \frac{dy}{dx} = \frac{20 - 2 \times 4}{25} = 0.48 \]

\[ \therefore \theta = 25^\circ 38' \]

\[ \sin \theta = 0.433 \text{ and } \cos \theta = 0.901 \]

\[ M_4 = -(12.8 \times 4) + (8 \times 2.56) + (4 \times 2 \times 2) = -14.72 \text{ kN-m} \]

Vertical shear at the section,

\[ V = 12.8 - 2 \times 4 = 4.8 \text{ kN} \]

From Fig. 16.7(a),

\[ F = H \sin \theta - V \cos \theta (\uparrow) \]

\[ = 8 \times 0.433 - 4.8 \times 0.901 = -0.861 \text{ kN} \]

Hence \( F = 0.861 \text{ kN} \uparrow \)

\[ N = H \cos \theta + V \sin \theta \]

\[ = 8 \times 0.901 + 4.8 \times 0.433 = 9.286 \text{ kN} \]

At \( x = 15 \), \( y = \frac{15}{25}(20 - 15) = 3.0 \text{ m} \)

\[ \frac{dy}{dx} = \tan \theta = \frac{20 - 2 \times 15}{15} = 0.4 \]

**Fig. 16.7.**
STRENGTH OF MATERIALS OF THEORY AND STRUCTURES

\[ \theta = 21^\circ 48' \text{ (inclination with } BA) \]
\[ \sin \theta = 0.3714 \text{ and } \cos \theta = 0.9285 \]
\[ M_{15} = (-32 \times 5) + 8 \times 30 = +8.0 \text{ kN.m} \]

From Fig. 16.7(6),
\[ F = H \sin \theta - V \cos \theta \]
\[ = 8 \times 0.3714 - 32 \times 0.9285 = 2.97 - 2.97 = 0 \]
and
\[ N = H \cos \theta + V \sin \theta \]
\[ = 8 \times 0.9285 + 32 \times 0.3714 = 8.616 \text{ kN.} \]

(c) Maximum positive and negative B.M.

Maximum negative B.M. will occur somewhere under the U.D.L. Let it occur at \( x \) from the left hinge.
\[ M_x = (-128 \times x) + \frac{25}{2} \]
\[ = -128x + \frac{8x}{25} (20-x) \]
\[ \frac{dM_x}{dx} = 0 = -128 + 2x + \frac{32}{5} - 16x = 0 \]
from which \( x = 4.7 \) m

\[ M_{max} (\text{negative}) = -128 \times 4.7 + 4.7 \times \frac{8}{25} (4.7)(20-4.7) \]
\[ = 15 \text{ kN.m.} \]

The maximum positive B.M. will evidently occur somewhere in the portion BC for which the equation of B.M. is given by,
\[ M_x = -32x + 8y, \text{ } y \text{ being measured from } B \]
\[ = -32x + \frac{8x}{25} (20-x) \]
\[ \frac{dM_x}{dx} = 0 = -32 + \frac{32}{5} - 16x = 0 \]
from which \( x = 5 \).

Hence maximum positive B.M. occurs where the radial shear is zero.

\[ M_{max} (\text{positive}) = -32 \times 5 + \frac{8 \times 5}{25} (20-5) \]
\[ = +8 \text{ kN.m.} \]

Example 16.2. A symmetrical parabolic arch with a central hinge, of rise \( r \) and span \( L \), is supported at its ends on pins at the same level. What is the value of the horizontal thrust when a load \( W \) which is uniformly distributed horizontally covers the whole span?

Example 16.3. An arch in the form of a parabola with axis vertical has hinges at the abutments and the vertex. The abutments are at different levels, the horizontal span being \( L \) and the heights of vertex above the abutments being \( h_1 \) and \( h_2 \).

Show that the horizontal thrust due to a load w/unit length uniformly distributed across the span is
\[ \frac{wL^2}{2(\sqrt{h_1} + \sqrt{h_2})^2} \] (Cambridge)
Let $L_1$ be the distance of the vertex from the left hand abutment. With $C$ as origin, the equation of parabola is $y = kx^2$.

For $CA$, therefore, $h_1 = kL_1^2$

or

$h = h_1$

For $CB$, we have $h_2 = k(L - L_1)^2$

or

$k = \frac{h_2}{(L - L_1)^2}$

Equating the two, we get

$h_1 = h_2$

$L_1^2 = (L - L_1)^2$

or

$L_1(L - L_1) = \sqrt{h_1h_2}$

Taking moments about $B$,

$Hh_1 = V_A \frac{L\sqrt{h_1}}{\sqrt{h_1} + \sqrt{h_2}} - \frac{wL^2}{2} \frac{h_1L^2}{(\sqrt{h_1} + \sqrt{h_2})^2}$ (i)

By taking moments about $B$,

$H(h_1 - h_2) + \frac{wL^2}{2} = V_A L$

$V_A = H(h_1 - h_2) + \frac{wL}{2}$

Substituting the value of $V_A$ in (i) we get

$Hh_1 = \left[ H(h_1 - h_2) + \frac{wL}{2} \right] \frac{L\sqrt{h_1}}{\sqrt{h_1} + \sqrt{h_2}} - \frac{wL^2}{2} \frac{h_1L^2}{(\sqrt{h_1} + \sqrt{h_2})^2}$

or

$H\left[ h_1 - \frac{(h_1 - h_2)\sqrt{h_1}}{\sqrt{h_1} + \sqrt{h_2}} \right] = \frac{wL^2}{2(\sqrt{h_1} + \sqrt{h_2})} - \frac{wL^2}{2(\sqrt{h_1} + \sqrt{h_2})^2}$

Example 16.4. A three hinged parabolic arch of 20 m clear span and 4 m central rise carries a point load of 4 kN at 4 m horizontally from the left hand hinge. Calculate the normal thrust and shear force at the section under the load. Also, calculate the maximum B.M. positive and negative.

Solution.

Taking $A$ as origin, we have the equation of the parabola

$y = \frac{4r}{L^2} x(L-x) = \frac{4 \times 4}{20 \times 20} x(20-x)$

$= \frac{x}{25}(20-x)$

Taking moments about $B$, we get

$20V_A + 4 \times 16 = 0$

or

$V_A = \frac{64}{20} = 3.2$ kN

Taking moments about $C$, we get

$Mc = 4H - (0.8 \times 10) = 0$
or \[ H = \frac{8}{4} = 2 \text{kN} \]

Now, bending moment at any section is
\[ M_x = \mu_x + Hy \]

The \( \mu_x \)-diagram is a triangle having maximum ordinate \( = 3.2 \times 4 = 12.8 \) kN-m under the point load. The \( Hy \)-diagram is a parabola having a maximum ordinate \( = 2 \times 4 = 8 \) kN-m under the central hinge.

At \( x = 4, y = \frac{4}{25} (20 - 4) = 2.56 \) kN-m

Under the point load,
\[ M_p = (-3.2 \times 4) + (2 \times 2.56) = -7.68 \text{kN-m} \]

This is evidently the maximum negative B.M.

The maximum positive B.M. will occur somewhere in the portion \( BC \). Measuring \( x \) from \( B \), the equation of B.M. for the portion \( BC \) is,
\[ M_x = -0.8x + 2 \frac{x}{25} (20 - x) \]
\[ \frac{dM_x}{dx} = -0.8 + \frac{40}{25} - \frac{4}{25}x = 0 \]
which gives \( x = 5 \) m

\[ M_{max.} (= +ve) = (-0.8 \times 5) + \frac{2}{25} \times 5(20 - 5) = +2 \text{kN-m} \]

The equation of the parabola is \( y = \frac{x}{25} (20 - x) \)
\[ \tan \theta = \frac{dy}{dx} = \frac{20}{25} - \frac{2x}{25} \]
\[ \tan \theta \text{ (at } x = 4 \text{ m)} = 0.8 - 0.32 = 0.48 \]
\[ \theta = 25.38^\circ ; \sin \theta = 0.433; \cos \theta = 0.901 \]

Considering the point load slightly to the right of \( P \), we get
\[ F = H \sin \theta - V_a \cos \theta \]
\[ = (2 \times 0.433) - (3.2 \times 0.901) = -2.017 = 2.017 \uparrow \downarrow \]
and \[ N = H \cos \theta + V_a \sin \theta \]
\[ = (2 \times 0.901) + (3.2 \times 0.433) = 3.188 \text{kN} \]

Example 16.5. A symmetrical three hinged circular arch has a span of 16 m and a rise to the central hinge of 4 m. It carries a verti-

The diagram is a triangle having maximum ordinate \( = 3.2 \times 4 = 12.8 \) kN-m under the point load; the \( Hy \)-diagram is a parabola having a maximum ordinate \( = 2 \times 4 = 8 \) kN-m under the central hinge.

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\[ M_x = -0.8x + 2 \frac{x}{25} (20 - x) \]
\[ \frac{dM_x}{dx} = -0.8 + \frac{40}{25} - \frac{4}{25}x = 0 \]
which gives \( x = 5 \) m

\[ M_{max.} (= +ve) = (-0.8 \times 5) + \frac{2}{25} \times 5(20 - 5) = +2 \text{kN-m} \]

The equation of the parabola is \( y = \frac{x}{25} (20 - x) \)
\[ \tan \theta = \frac{dy}{dx} = \frac{20}{25} - \frac{2x}{25} \]
\[ \tan \theta \text{ (at } x = 4 \text{ m)} = 0.8 - 0.32 = 0.48 \]
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and \[ N = H \cos \theta + V_a \sin \theta \]
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\[ M_x = -0.8x + 2 \frac{x}{25} (20 - x) \]
\[ \frac{dM_x}{dx} = -0.8 + \frac{40}{25} - \frac{4}{25}x = 0 \]
which gives \( x = 5 \) m

\[ M_{max.} (= +ve) = (-0.8 \times 5) + \frac{2}{25} \times 5(20 - 5) = +2 \text{kN-m} \]

The equation of the parabola is \( y = \frac{x}{25} (20 - x) \)
\[ \tan \theta = \frac{dy}{dx} = \frac{20}{25} - \frac{2x}{25} \]
\[ \tan \theta \text{ (at } x = 4 \text{ m)} = 0.8 - 0.32 = 0.48 \]
\[ \theta = 25.38^\circ ; \sin \theta = 0.433; \cos \theta = 0.901 \]

Considering the point load slightly to the right of \( P \), we get
\[ F = H \sin \theta - V_a \cos \theta \]
\[ = (2 \times 0.433) - (3.2 \times 0.901) = -2.017 = 2.017 \uparrow \downarrow \]
and \[ N = H \cos \theta + V_a \sin \theta \]
\[ = (2 \times 0.901) + (3.2 \times 0.433) = 3.188 \text{kN} \]

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The diagram is a triangle having maximum ordinate \( = 3.2 \times 4 = 12.8 \) kN-m under the point load; the \( Hy \)-diagram is a parabola having a maximum ordinate \( = 2 \times 4 = 8 \) kN-m under the central hinge.

At \( x = 4, y = \frac{4}{25} (20 - 4) = 2.56 \) kN-m

Under the point load,
\[ M_p = (-3.2 \times 4) + (2 \times 2.56) = -7.68 \text{kN-m} \]

This is evidently the maximum negative B.M.

The maximum positive B.M. will occur somewhere in the portion \( BC \). Measuring \( x \) from \( B \), the equation of B.M. for the portion \( BC \) is,
\[ M_x = -0.8x + 2 \frac{x}{25} (20 - x) \]
\[ \frac{dM_x}{dx} = -0.8 + \frac{40}{25} - \frac{4}{25}x = 0 \]
which gives \( x = 5 \) m

\[ M_{max.} (= +ve) = (-0.8 \times 5) + \frac{2}{25} \times 5(20 - 5) = +2 \text{kN-m} \]

The equation of the parabola is \( y = \frac{x}{25} (20 - x) \)
\[ \tan \theta = \frac{dy}{dx} = \frac{20}{25} - \frac{2x}{25} \]
\[ \tan \theta \text{ (at } x = 4 \text{ m)} = 0.8 - 0.32 = 0.48 \]
\[ \theta = 25.38^\circ ; \sin \theta = 0.433; \cos \theta = 0.901 \]

Considering the point load slightly to the right of \( P \), we get
\[ F = H \sin \theta - V_a \cos \theta \]
\[ = (2 \times 0.433) - (3.2 \times 0.901) = -2.017 = 2.017 \uparrow \downarrow \]
and \[ N = H \cos \theta + V_a \sin \theta \]
\[ = (2 \times 0.901) + (3.2 \times 0.433) = 3.188 \text{kN} \]

Example 16.5. A symmetrical three hinged circular arch has a span of 16 m and a rise to the central hinge of 4 m. It carries a verti-
Its inclination with the horizontal is given by
\[ \tan \theta = \frac{12}{8} = 1.5, \quad \therefore \theta = 56° 18'. \]

Reaction at \( B = \sqrt{1.5^2 + 6^2} = \sqrt{16 + 64} = 8.94 \text{ kN} \)

Its inclination with the horizontal is given by
\[ \tan \theta = \frac{4}{8} = 0.5, \quad \therefore \theta = 26° 34'. \]

(c) At 6 m from left hand hinge, \( x = (8 - 6) = 2 \text{ m} \)
\[ y = (100 - 2^2)^{1/2} - 6 = 9.8 - 6 = 3.8 \text{ m}. \]
\[ M_x = (-12 \times 6) + (8 \times 3.8) - 16 \times 2 \]
\[ = -9.6 \text{ kN-m} \]

Example 16.6. A three hinged circular arch consists of a portion \( AC \) of radius 3 m and rise of the hinge \( C \) with respect to the left abutment is 3 m. The right hand portion \( CB \) is of radius 8 m and the horizontal distance \( BC \) is 7 m. If a concentrated load of 10 kN acts at 6 m from the left hand end, determine the reactions at the hinges and maximum bending moment on the arch.

Solution.
(Fig. 16'12)

The rise of the crown above the hinge \( B \)
\[ r = \sqrt{8^2 - 7^2} = 4.13 \text{ m} \]

Taking moments about \( B, \)
\[ H(4.13 - 3) + V_A \times 10 = 10 \times 4 \]

Taking moments about \( C, \)
\[ H \times 3 = V_A \times 3 \]

or
\[ H = V_A \]

Substituting in equation (1)
\[ 11.13V_A = 40 \]

or
\[ V_A = 3.59 \text{ kN} = H \]

and
\[ V_B = 10 - 3.59 = 6.41 \text{ kN} \]

Reaction at \( A = \sqrt{(3.59)^2 + (3.59)^2} = 5.08 \text{ kN} \]
\[
\frac{dM}{d\theta} = 10.77 \cos 0 - \sin 0 = 0.
\]
\[\tan \theta = 1 \quad \text{or} \quad \theta = 45^\circ \]
\[M_{\text{max.}}(+ve) = 10.77 (\sin 45^\circ - 1 + \cos 45^\circ) = 4.45 \text{ kN-m} \]

The maximum negative B.M. in portion BC will evidently occur just below the load.

Height of point of application of load above \(O = \sqrt{8^2 - 3^2} = 7.42 \text{ m.} \]

Height of the hinge \(B\) above \(O = \sqrt{8^2 - 7^2} = 3.87 \text{ m.} \]

Rise of the point of application of the load above the hinge \(B\)
\[= 7.42 - 3.87 = 3.55 \text{ m.} \]

\[M_{\text{max.}}(-ve) = -6.41 \times 4 + 3.59 \times 3.55 = -12.90 \text{ kN-m} \]

Hence, the maximum B.M. over the span = 12.90 kN-m

Example 16.7. A frame shown in Fig. 16.13 is hinged to the ground at \(A\) and \(E\) and hinged also at \(C\) and has rigid corners at \(B\) and \(D\). Find the reactions at \(A\) and \(E\) when a uniformly distributed load of 40 kN per metre run covers \(BD\) and draw the bending moment diagram, figuring significant values on the diagram.

Solution

\[
M_B = \frac{1}{2} \cdot 3 \cdot 3 \cdot 40 = 180 \text{ kN-m}.
\]

\[
M_{E_1} = \frac{1}{2} \cdot 3 \cdot 3 \cdot 1.5 = 7.5 \text{ kN-m}.
\]

\[
M_{E_2} = \frac{1}{2} \cdot 5 \cdot 5 \cdot 1.5 = 31.25 \text{ kN-m}.
\]

The I.L. diagram for \(H\) will therefore be a triangle having maximum ordinate of \(\frac{L}{4r}\) under the central hinge, as shown in Fig. 16.14 (b).

\[
(12H + 360) = 6V_e
\]

Equating (1) and (2), we get
\[H = 32.727 \text{ kN} \]

Substituting in (1), we get
\[32.727 + 720 = 6V_e \]

or
\[V_e = 125.45 \text{ kN} \]

Bending moment at the head of the column \(BA\)
\[= H \times 6 = 32.727 \times 6 = 196.36 \text{ kN-m} \]

Bending moment at the head of column \(DE\)
\[= 32.727 \times 5 + 163.64 \text{ kN-m} \]

On \(BD\) the B.M. at any distance \(x\) from \(B\) is given by
\[M_y = 32.727 - 125.45 x + 40x^2 \]

which is evidently zero at the central hinge \(C\) where \(x = 3 \text{ m.} \) The bending moment diagram is shown in Fig. 15.13 (b).

16.4. MOVING LOADS ON THREE HINGED ARCHES

(1) INFLUENCE LINE FOR \(H\)

Let us consider a unit load rolling from \(A\) to \(B\). At any instant let the load be at a distance \(\alpha L\) from \(A\). The vertical reactions at \(A\) and \(B\) will be \((1-\alpha)\) and \(\alpha\) respectively.

For \(H\), equate \(M_c\) to zero.

Thus \(M_c = 0 = (H \times x) - \alpha \frac{L}{2} \)

or
\[H = \frac{\alpha L}{2r} \]

Thus, \(H\) varies linearly with \(\alpha\).

At \(A\), \(\alpha L = 0\) and hence \(H = 0\).

At, \(\alpha L = \frac{L}{2}\), and hence \(H = \frac{L}{4r}\)

The I.L. diagram for \(H\) will therefore be a triangle having maximum ordinate of \(\frac{L}{4r}\) under the central hinge, as shown in Fig. 16.14 (b).

(2) INFLUENCE LINE FOR B.M. AT \(P\)

Let us now draw the I.L. from B.M. at \(P\) distance \(x\) from \(A\)
\[M = \mu + H \times y \]

Thus, the influence line for \(M\) consists of \((i)\) I.L. for \(\nu\), and \((ii)\) I.L. for \(H\). The I.L. for \(\mu\) will be a triangle having a maximum
ordinate of \( \frac{x(L-x)}{L} \) under the section. The I.L. for \( H_y \) will also be a triangle having a maximum ordinate

\[
y = \frac{4}{4r} x(L-x) = \frac{x}{L} (L-x)
\]

Using the influence lines, let us now plot the maximum positive and negative bending moment diagram due to (i) single point load \( W \) and (ii) U.D.L.

(3) MAXIMUM BENDING MOMENT DIAGRAM DUE TO SINGLE POINT LOAD \( W \)

By the inspection of the I.L. for \( M_p \), it is clear that maximum negative B.M. will occur when the point load is at the section \( P \).

Thus, \( M_{\text{max.}} (- \text{ve}) = \frac{-W x(L-x)}{L} + \frac{W x(L-x)}{L} = -\frac{W x(L-x)}{L} - \frac{W x(L-x)}{L} = -\frac{W x(L-x)}{L} \)  

To get the absolute maximum negative B.M., differentiate (i) with respect to \( x \) and equate it to zero.

Thus \( \frac{dM_{\text{max.}}}{dx} = 0 = \frac{Lx^2 - 2WxL + W^2}{L^3} \)  

Solving gives \( x = (0.5 \pm 0.289) \)  

Substituting the value of \( x \) in (i), we get

\[
M_{\text{max.}} (- \text{ve}) = -\frac{WL}{6\sqrt{3}} = -0.096WL
\]

The maximum negative bending moment diagram will be a third degree polynomial, as shown in Fig. 16.14(d) below the base.

The maximum positive bending moment at \( P \) will occur when the load \( W \) is at the central hinge, as is clear from the I.L. diagram for \( M_p \).

Thus \( M_{\text{max.}} (+ \text{ve}) = \frac{W x(L-x)}{L} - \frac{W x(L-x)}{L} = \frac{W x(L-x)}{L} - \frac{W x(L)}{2} \)

To get the absolute maximum positive B.M. differentiate (ii) with respect to \( x \) and equate it to zero.

Thus \( \frac{dM_{\text{max.}}}{dx} = 0 = L - 4x \)

\[
x = \frac{L}{4} = 0.25L
\]

Fig. 16.14

It must be noted here that both \( x \) and \( y \) are fixed quantities for \( P \). The I.L. for \( M_p \) will be obtained by superimposing the two \( I.L. \) as in Fig. 16.14 (c).
Substituting the value of \( x \) in (ii), we get

\[
M_{\text{max}. \text{max.}} = \frac{WL}{16} = +0.0625 \text{ W}L.
\]

The maximum positive bending moment diagram will be a second degree polynomial (i.e., parabola), as shown in Fig 16.14(d) above the base.

(4) MAXIMUM BENDING MOMENT DIAGRAM DUE TO U.D.L.

Maximum negative B.M. at \( P \) will occur when the \( AO \) is loaded, while the maximum positive B.M. at \( P \) will occur when the span \( BO \) is loaded. Since the ordinates \( P_1P_2 \) and \( C_1C_2 \) are equal the area of \( \Delta s \) \( AP_1B \) and \( AC_1B \) are equal and hence area of \( \Delta s \) \( AP_1O_1 \) and \( BC_1O_1 \) will be equal. Therefore, the maximum negative B.M. at \( P \) will be equal to the maximum positive B.M. at \( P \).

Thus \( M_{\text{max.}} = \pm \frac{wx}{\text{area of triangle } AP_1O_1} \) (1)

Let us, therefore, locate the point \( O \) first. Let the distance \( AO=a \), and \( OB=(L-a) \).

Now \( OO_1=C_1C_2 \times \frac{2}{L} \times a = \frac{2a}{L} C_1C_2 \)

Also, \( OO_1=P_1P_2 \times \frac{1}{(L-a)} \times (L-a) = \frac{L-a}{L} \times P_1P_2 \)

Equating the two, we get

\[
\frac{2a}{L} C_1C_2 = \frac{L-a}{L} \times P_1P_2 \]

or

\[
\frac{2a}{L} = \frac{L-a}{L-x} \text{ since } C_1C_2 = P_1P_2 \]

\[
\therefore a = AO = \frac{L^3}{3L-2x}
\]

Hence, the ordinate \( OO_1 = \frac{2}{L} \times \frac{L^3}{(3L-2x)} \times \frac{x(L-x)}{L} = \frac{2x(L-x)}{(3L-2x)} \)

Now area \( AP_1O_1 \) = area \( AP_1B \) = area \( AO_1B \)

\[
= \frac{1}{2} L \times \frac{x(L-x)}{L} - \frac{1}{2} L \times \frac{2x(L-x)}{(3L-2x)}
\]

\[
= \frac{x(L-x)(L-2x)}{2(3L-2x)}
\]

Substituting the value in (1), we get

\[
M_{\text{max.}} = \pm \frac{wx(L-x)(L-2x)}{2(3L-2x)} \quad (4)
\]

To obtain absolute maximum (±) B.M., differentiate (4) with respect to \( x \) and equate it to zero. Putting \( x=nL \) in (4) we get

\[
M_{\text{max.}} = \pm \frac{wxL}{(6-4n)}
\]

\[
\frac{dM_{\text{max.}}}{dx} = 0
\]

\[
=(6-4n)(1-n)(1-2n)-n(1-2n)-2n(1-2n)+4(n-1)(1-2n)
\]

or

\[
8n^3-24n^2+18n-3=0
\]

From which, \( n=0.234 \)

or

\[
x=0.234L
\]

Substituting the value of \( x \) in (4), we get

\[
M_{\text{max.}}(\pm) = \pm 0.01883 \text{ w}L^3
\]

The loaded length \( AO = \frac{L^3}{3L-2(0.234L)} = 0.395L \) (5)

The maximum negative and positive diagrams are shown in Fig. 16.14(c).

(5) INFLUENCE LINES FOR RADIAL SHEAR (F) AND NORMAL THRUST (N)
Let us now draw the influence line diagrams for radial shear (F) and normal thrust (N) at a section P distant x from A. When the load is in AP, consider the equilibrium of the portion PB (Fig. 16'16 (b)) and when the load is in PB, consider the equilibrium of the portion AP (Fig. 16'16(a)). When the unit load is at a distance aL from A, the reactions at A and B will be (1 - a) and a respectively while the horizontal thrust H will be equal to \( \frac{L}{2r} \) as proved earlier.

![Fig. 16'16](image)

When the load is between A and P, consider Fig. 16'16 (b) from which

\[
F_p = V_A \cos \theta + H \sin \theta (\downarrow) \tag{I}
\]

and

\[
N_p = H \cos \theta - V_A \sin \theta (\rightarrow) \tag{II}
\]

Again, when the load is between P and B, we get from Fig. 16'16(a),

\[
F_p = H \sin \theta - V_A \cos \theta (\uparrow) \tag{III}
\]

and

\[
N_p = H \cos \theta + V_A \sin \theta (\leftarrow) \tag{IV}
\]

By the inspection of equation (I) and (III) it is clear that the influence line for \( F_p \) can be obtained by superimposing \( V \cos \theta \) diagram (i.e. I.L. for S.F. for simple beam, every ordinate of which is multiplied by cos \( \theta \)) on \( H \sin \theta \) diagram, keeping in view the fact that both the quantities are additive when the load is in \( AP \) and are subtractive when the load is in \( BP \). Fig. 16'15 (b) shows the I.L. diagram for \( F_p \).

Similarly, I.L. for \( N_p \) can be obtained by superimposing \( V \sin \theta \) diagram on \( H \cos \theta \) diagram keeping in view the fact that both the quantities are subtractive when the load is in \( AP \) and are additive when the load is in \( BP \). Fig. 16'15(c) shows the I.L. diagram for \( N_p \).

**Example 16'8.** A three hinged parabolic arch has a span of 40 m and a central rise of 8 m. Five wheel loads of 4, 4, 6, 6 and 5 tonnes spaced 2, 3, 2 and 3 m in order, cross the arch from left to right with the 4 kN load leading. When the leading load is 25 m from the left hand hinge, calculate the horizontal thrust in the arch. Also, calculate the bending moment, normal thrust and shear force at the section under the tail load.

**Solution**

![Fig. 16'17](image)

Fig. 16'17 shows the I.L. diagram for \( H \) having a maximum ordinate \( \frac{L}{4r} = \frac{40}{4 \times 8} = 1.25 \) at the central hinge. The loads have been shown in the required position. The influence line ordinates under the various loads are:

- Under first 4 kN load, ordinate \( \frac{1.25}{20} \times 15 = 0.938 \)
- Under next 4 kN load, ordinate \( \frac{1.25}{29} \times 17 = 1.062 \)
- Under 6 kN load, ordinate \( \frac{1.25}{20} \times 18 = 1.125 \)
- Under next 6 kN load, ordinate \( \frac{1.25}{20} \times 18 = 1.125 \)
- Under last 5 kN load, ordinate \( \frac{1.25}{20} \times 15 = 0.938 \)

\( H = (4 \times 0.938) + (4 \times 1.062) + (6 \times 1.25) + (6 \times 1.25) + (5 \times 0.938) = 3.75 + 4.25 + 7.50 + 6.75 + 4.69 = 26.94 \text{ kN} \)

At the section under the tail load:

Considering the tail load slightly to the right side of the section, \( V_A = \frac{1}{40} [(4 \times 15) + (4 \times 17) + (6 \times 20) + (6 \times 22) + (5 \times 25)] \)

\[= 12 \text{ kN} \]

The equation of the parabola is

\[ \frac{4r}{L^2} \times (L-x) = \frac{4 \times 8}{1600} x (40-x) = \frac{x}{50} (40-x) \]
\[ \frac{dy}{dx} = \tan \theta = \frac{4 - x}{5} \]

At \( x = 15 \), \( y = \frac{15}{25} (40 - 15) = 7.5 \)
\[ \tan \theta = \frac{4 - 15}{5} = 0.2 \]
\[ \theta = 11^\circ 18' \quad \sin \theta = 0.196 \quad \cos \theta = 0.981 \]
\[ M = -(12.625 \times 15) + (26.94 \times 7.5) = 12.68 \text{ kN} \cdot \text{m} \]
\[ F = H \sin \theta - V_A \cos \theta = 26.94 \times 0.196 - 12.625 \times 0.981 = 7.12 \text{ kN} \]

or
\[ F = 7.12 \text{ kN} \]
\[ N = H \cos \theta + V_A \sin \theta = 26.94 \times 0.981 + 12.625 \times 0.196 = 28.89 \text{ kN} \]

**Example 16\(^{1}\)** A three hinged parabolic arch has a horizontal span of 30 m with central rise of 5 m. A point load of 10 kN moves cross from left to right. Calculate the maximum positive and negative moment at the section 8 m from the left hinge.

Also, calculate the position and amount of the absolute maximum B.M. that may occur in the arch.

**Solution. (Fig. 16\(^{1}\)4)**

The equation to the parabola is

\[ y = \frac{4r}{L^2} x(L - x) = \frac{4 \times 5}{30 \times 30} x(30 - x) = \frac{x}{45} (30 - x) \]

At \( x = 8 \), \( y = \frac{8}{45} (30 - 8) = 3.91 \text{ m} \).

By inspection of I.L. for B.M. at any section, it is clear that maximum negative B.M. occurs when the load is on the section while maximum positive B.M. at the section occurs when the load is on the central hinge.

**When the load is on the section:**

\[ V_B = \frac{10 \times 8}{30} = \frac{8}{3} \text{ kN} \]

For \( H \),
\[ M_c = - \frac{8}{3} \times 15 + H \times 5 \]

or
\[ H = \frac{40}{5} = 8 \text{ kN} \]
\[ M_{mae} (\text{-ve}) = - \frac{8}{3} \times 22 + 8 \times 3.91 = -27.39 \text{ kN} \cdot \text{m} \]

**When the load is on the central hinge:**

\[ V_A = \frac{10}{2} = 5 \text{ kN} \]

For \( H \),
\[ M_c = 0 = -5 \times 15 + H \times 5 \]
\[ H = 15 \text{ kN} \]
\[ M_{mae} (\text{+ve}) = -5 \times 8 + 15 \times 3.91 = +18.65 \]

The absolute maximum bending moments are +0.0625 \( \text{WL} \) and -0.096 \( \text{WL} \). Out of these two, the negative bending moment is greater.

Hence \( M_{mae, max} = -0.096 \text{ WL} \)
\[ = -0.096 \times 10 \times 30 \]
\[ = -28.8 \text{ kN} \cdot \text{m} \]

This occurs at 0.211 \( L = 6.33 \text{ m} \) from either end-hinge.

**16\(^{1}\). TWO HINGED ARCH**

A two hinged arch is statically indeterminate to single degree, since there are four reaction components to be determined while the number of equations available from statical equilibrium is only three. Considering \( H \) to be the redundant reaction, it can be found out by the use of Castigliano’s theorem of least work.

Thus, assuming the horizontal span remaining unchanged, we have,

\[ \frac{\partial U}{\partial H} = 0 \]

where \( U \) is the total strain energy stored in the arch. Here also, the strain energy stored due to thrust and shear will be considered negligible in comparison to that due to bending.

\[ U = \int \frac{M^2 \delta s}{2EI} \]

\[ \frac{\partial U}{\partial H} = \int \frac{2M \cdot \delta M}{2EI} \cdot \frac{\delta H}{ds} = \frac{M}{EI} \cdot \frac{\partial M}{\partial H} \cdot ds \]

Now
\[ M = \mu + Hy; \quad \frac{\partial M}{\partial H} = y \]

\[ \frac{\partial U}{\partial H} = 0 = \int \left( \mu + Hy \right) \cdot \frac{\delta y ds}{EI} \]

\[ \frac{\partial U}{\partial H} = 0 = \int \frac{\mu y ds}{EI} \]
or

\[ H = - \frac{\mu y ds}{EI} \left( \frac{y^2 ds}{EI} \right) \]

Taking \( dx = ds \cos \theta \), and \( I = I_0 \sec \theta \) where \( I_0 \) is the moment of inertia at the crown, we get

\[ H = - \frac{\mu y dx}{I^2} \]

If the two hinges are forced a distance \( \lambda \) apart by the thrust, \( \lambda \) must be added to the right hand side of equation (1). Thus,

\[ \left( \frac{\mu + H y}{EI} \right) y ds = \lambda \]

or

\[ H = \frac{\lambda EI}{\mu y ds} \]

\[ \frac{I^2}{\mu y ds} \] (16.7)

Alternative method:

Equation 16.7 can also be obtained by the consideration of the flexural deformation of a curved rib.

Let \( ACB \) represent the centre line of a curved rib subjected to variable B.M. Let us find the horizontal and vertical displacements of end \( B \) with reference to \( A \). Consider the effect of B.M. on an element of length \( ds \). Let this element turn through an angle \( \delta i \), the part \( AC \) of the rib being unchanged. The chord \( CB \) will therefore turn to a position \( CB_1 \) through an angle \( \delta i \). \( B_1B_2 \), thus, gives the horizontal displacement while \( BB_2 \) gives the vertical displacement of \( B \).

Now \( B_1B_2 = BB_1 \cos BB_2 \)

\[ = (CB, di) \cos BCD \]

\[ = di, (CB) \cos BCD = di(CD) \]

\[ = ydi \] (1)

But

\[ \frac{M}{I} = \frac{E}{R} = E, \frac{di}{ds} \]

\[ \therefore \]

\[ \frac{di}{ds} = \frac{M ds}{EI} \]

Substituting in (1), we get

\[ B_1B_2 = y \cdot \frac{M ds}{EI} = \frac{My ds}{EI} \]

The total horizontal displacement of \( B = \int \frac{My ds}{EI} \)

It can be proved, similarly, that the total vertical displacement of \( B = \int \frac{My ds}{EI} \).

Now, for two hinged arch having no yielding of the supports, we have

\[ \int \frac{My ds}{EI} = 0 \]

or

\[ \int \frac{(\mu + H y) ds}{EI} = 0 \]

From which,

\[ H = - \frac{\mu y ds}{I^2} \] (16.7)

It is to be noted that in the above equation, \( \frac{I^2}{\mu y ds} \) is the property of an arch while \( \frac{My ds}{EI} \) depends both on the property of the arch as well as on the loading.

16.7. TWO HINGED PARABOLIC ARCH: EXPRESSION FOR \( H \).

Consider a two hinged parabolic arch of horizontal span \( L \) and central rise \( r \), subjected to a point load \( W \) at a distance \( x \) from the left support.
The equation of arch is \( y = \frac{4r}{L^2} \alpha(L-x) \)

Now \( H = \frac{\int_0^L \mu y \, dx}{\int_0^L y^2 \, dx} \) \( \tag{1} \)

The numerator is \( \int_0^L \mu y \, dx = \int_0^L \mu y \, dx + \int_0^L \mu |dx = a + b \) \( \tag{2} \)

The quantity \( a = \int_0^L W(L-x) \frac{4r}{L^2} \alpha(L-x) \, dx \)

\( = \frac{(1-a)4rLW}{L^2} \left( \frac{L^4}{3} - \frac{L^4}{4} \right) = \frac{(1-a)4rLW}{L^2} \left( \frac{-L^4}{3} + \frac{L^4}{4} \right) \) \( \tag{3} \)

The quantity \( b = \int_0^L W a(L-x) \frac{4r}{L^2} \alpha(L-x) \, dx \)

\( = \frac{4raW}{L^2} \int_0^L (L-x)^3 \, dx \)

\( = \frac{4raW}{L^2} \int_0^L (L^3 x^3 - 2Lx^2) \, dx \)

\( = \frac{4raW}{12L^2} \left( 1 - 6a^2 - 3a^4 + 8a^4 \right) \) \( \tag{4} \)

Substituting the values of \( a \) and \( b \) in (2), we get

The numerator is \( \left[ \frac{(1-a)4rLW}{L^2} \left( \frac{L^4}{3} - \frac{L^4}{4} \right) \right] \)

\( + \frac{4raW^2}{12} \left( 1 - 6a^2 - 3a^4 + 8a^4 \right) \) \( \tag{5} \)

Again, the denominator is \( \int_0^L y^2 \, dx = \frac{16r^2}{L^4} \int_0^L (x^2L^4 + x^2 - 2Lx^2) \, dx \)

\( = \frac{16r^2}{L^4} \left( \frac{L^5}{3} + \frac{L^5}{5} - \frac{L^5}{2} \right) \)

\( = \frac{16r^2L}{30} \left( 10 + 6 - 15 \right) \)

\( = \frac{8}{15} r^2L \) \( \tag{16.9} \)

\[ H = \frac{(1-a)4rLW}{L^2} \left( \frac{L^4}{3} - \frac{L^4}{4} \right) + \frac{4raW^2}{12} \left( 1 - 6a^2 - 3a^4 + 8a^4 \right) \]

which reduces to

\( H = \frac{5}{8} W \frac{L}{r} a(1-a)(1+a-a^2) \) \( \tag{16.10} \)

16.8. TWO HINGED CIRCULAR ARCH : EXPRESSION FOR \( H \)

Let \( \theta \) be the half angle subtended by the arch at the centre. Let the load \( W \) be acting at a section which makes an angle \( \phi \) with the centre line.

Consider any point \( P \) subtending an angle \( \beta \) with the centre line.

The co-ordinates of \( P \) are given by

\( x = R\sin \theta - \sin \beta \) \( \tag{i} \)

and

\( y = R(\cos \beta - \cos \theta) \) \( \tag{ii} \)

Also, \( ds = Rd\beta \)

Now, \[ \int_0^B y^2 \, ds = 2 \int_0^\phi \int_0^B \cos \beta - \cos \theta \, Rd\beta \]

\( = 2R \int_0^\phi \cos \beta - 2 \cos \beta \cos \theta + \cos^2 \theta \, d\beta. \]

which on simplification gives

\[
\int_0^\phi \cos \beta d\beta = 2 \cos \theta \int_0^\phi \cos \beta d\beta + \cos^2 \theta \int_0^\phi d\beta
\]
\[
\int_A \mu y ds = R^2 \left( 40 \cos^2 \theta - 26 - 3 \sin 2 \theta \right) \quad (16.11)
\]
\[
= R^2 (20 + 0 \cos 2 \theta - 1.5 \sin 2 \theta) \quad [16.11 (a)]
\]

To find \( \mu y ds \), assume an equal load \( W \) placed symmetrically on the other side so that the integrations may be simplified. In that case \( V_A = V_B = W \).

\[
\int_A \mu y ds = 2 \left\{ \int_0^\phi \mu y ds + \int_0^\phi \mu y ds \right\} = 2(a + b) \quad (1)\]

The integral \( a \) is:

\[
= - \int_0^\phi R^2 \left[ (\sin \theta - \sin \phi) \cos \beta - \cos \theta \right] d\beta
- \int_0^\phi R^2 \left[ (\cos \theta - \cos \phi) \sin \beta - \sin \phi \right] d\beta
- \int_0^\phi R^2 \left[ (\sin \beta - \sin \phi) \cos \beta - \cos \phi \right] d\beta
- \int_0^\phi R^2 \left[ (\cos \beta - \cos \phi) \sin \beta - \sin \phi \right] d\beta
\]

The second integral \( b \) is:

\[
= - W R^2 \left[ \int_0^\phi \left( \sin \theta (\cos \phi - \cos \theta) - \sin \beta \cos \beta + \cos \theta \sin \beta \right) d\beta \right]
\]

which on simplification gives

\[
(b) = - W R^2 \left[ \sin^2 \theta - \frac{\sin 2 \theta}{2} \cos \theta + \frac{\cos 2 \theta}{4} \cos \theta - \sin 2 \theta \sin \theta \sin \phi \right.
\]
\[
+ \frac{\phi \sin 2 \theta}{2} - \frac{\cos 2 \phi}{4} + \cos \theta \cos \phi \left. \right] \quad (2)
\]

Adding (2) and (3) and simplifying, we get

\[
\int_A \mu y ds = - W R^2 \left[ \sin^2 \theta - \sin^2 \phi - 2 \cos \theta (\cos \theta - \cos \phi + \sin \theta)
\right.
\]
\[
- \phi \sin \phi \left. \right] \quad (4)
\]

Hence,

\[
H = \frac{1}{2} \int_A y^2 ds = \frac{2 W R^2 [\sin^2 \theta - \sin^2 \phi - 2 \cos \theta (\cos \theta - \cos \phi + \sin \theta - \phi \sin \phi)]}{R^2 (40 \cos^2 \theta + 26 - 3 \sin 2 \theta)}
\]

Hence \( H \) for a single isolated \( W \) is given by

\[
H = \frac{W [\sin^2 \theta - \sin^2 \phi - 2 \cos \theta (\cos \theta - \cos \phi + \sin \theta - \phi \sin \phi)]}{(40 \cos^2 \theta + 26 - 3 \sin 2 \theta)} \quad (16.12)
\]

---

**Example 16.10.** A parabolic arch, hinged at the ends has a span 30 m and rise 5 m. A concentrated load of 12 kN acts at 10 m from the left hinge. The second moment of area varies as the secant of the slope of the rib axis. Calculate the horizontal thrust and the reactions at the hinges. Also, calculate the maximum bending moment anywhere on the arch.

**Solution.** Fig. (16.19)

\[
V_A = \frac{12 \times 20}{30} = 8 \text{ kN}
\]

\[
V_B = 12 - 8 = 4 \text{ kN}
\]

\[
H = \frac{\int_0^\phi \mu y dx}{\int_0^\phi \phi \phi dx}
\]

The equation of the parabola with \( A \) as origin, is

\[
y = \frac{4r}{L^2} x(L - x) = \frac{4 x}{900} x(30 - x) = \frac{x}{45} (30 - x)
\]

For \( AC \), \( \mu = -8x \)

For \( CB \), \( \mu = -4(30 - x) \)

\[
\int_0^3 8x^2 dx = \int_0^3 8x^2 x(30 - x) + \int_0^3 30 (30 - x) dx
\]

\[
= \frac{30}{45} \int_0^3 8x^2 (30 - x) dx + \int_0^3 30 4x (30 - x)^2 dx
\]

\[
= \frac{8}{45} \int_0^3 450x^2 + \frac{x^4}{4} - 20x^2 dx
\]

\[
= \frac{44,000}{9}
\]

\[
\int_0^3 \frac{x^2}{45^2} dx = \int_0^3 \frac{x^2}{45^2} (30 - x) dx
\]

\[
= \int_0^3 \frac{1}{45^2} \int_0^3 (900x^2 + 20x^3 - 15x^4 - 15x^4) dx
\]

\[
= \frac{400}{15}
\]
578

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\[ H = \frac{44000}{9 \times 500} = 12.22 \text{ kN} \]

Reaction at \( A = R_A = \sqrt{8^2 + 12.22^2} = 14.61 \text{ kN} \)

Its inclination with the horizontal is given by

\[ \tan \theta_A = \frac{8}{12.22} = 0.655 \]

or \( \theta_A = 33^\circ 14' \)

Reaction at \( B, R_B = \sqrt{4^2 + 12.22^2} = 12.85 \text{ kN} \)

Its inclination with the horizontal is given by

\[ \tan \theta_B = \frac{4}{12.22} = 0.327 \]

or \( \theta_B = 18^\circ 6' \)

Maximum negative B.M. will occur in \( AC \), just below the load.

Rise of the arch at the point of application of the load is given by

\[ y = \frac{x}{45} (30-x) = \frac{10}{45} (30-10) = 40 \text{ m} \]

\[ M_{max} (-ve) = 12.22 x \frac{40}{9} \times 8 \times 10 = -25.69 \text{ kN-m} \]

Let the maximum positive moment occur at distance \( x \) from \( B \)

\[ M_x = -4x + 12.22 \times \left( \frac{30-x}{45} \right) \]

\[ \frac{dM_x}{dx} = 0 = -4 + \frac{12.22 \times 30}{45} - \frac{12.22 \times 2x}{45} \times 2x \]

From which \( x = 7.65 \text{ m} \)

\[ M_{max} (+ve) = -4 \times 7.65 + 12.22 \times \frac{7.65 \times (30-7.65)}{45} \]

\[ = -30.60 + 46.40 = 15.80 \text{ kN-m} \]

Maximum bending moment is \(- 25.69 \text{ kN-m} \) which occurs below the load.

**Example 16.11.** A parabolic two hinged arch has a span of 32 metres and a rise of 8 m. A uniformly distributed load of 1 kN/m covers 8 m horizontal length of the left side of the arch. If \( l = l_0 \) sec \( \theta \)

where \( \theta \) is the inclination of the arch of the section to the horizontal, and \( l_0 \) is the moment of inertia of the section at the crown, find out the horizontal thrust at hinges and bending moment at 8 m from the left hinge. Also find out normal thrust and radial shear at this section.

**Solution.** (Fig. 16.19)

Taking moments about \( B \) for vertical reaction at \( A, \)

\[ V_A \times 32 = 8 \times 1 \times 28 \]

or \( V_A = 7 \text{ kN} \); and \( V_B = 8 - 7 = 1 \text{ kN} \)

The rise of the arch at any section distant \( x \) from \( A \) is given by

\[ y = \frac{4 \times 8}{32^2} x (32-x) = \frac{x}{32} (32-x) \]

\[ - \int_0^L \mu y dx = \int_0^8 \left( 7x - \frac{x^2}{2} \right) \frac{x}{32} (32-x) dx + \int_0^{32} \frac{1}{32} (32-x) \frac{x}{32} dx \]

\[ = \frac{8}{3} \left[ 7x + \frac{x^2}{2} \right] - \frac{32}{32} + \frac{4}{4} \left[ x^4 + x^2 \right] - \frac{2x^2 \times 32}{3} \]

\[ = 8 \left[ \frac{7}{3} - 1 \times 8 + \frac{8^2}{320} \right] - \left[ \left( 16 \times 32 + \frac{32^2}{128} - \frac{2 \times 32^2}{3} \right) \right] \]

\[ = 2477.07 \]

\[ \int_0^L y dx = \int_0^{32} \frac{x^2 (32-x)^2}{32^2} dx \]

\[ \int_0^L y dx = \frac{1}{32^2} \int_0^{32} \left[ 1024 x^2 + x^4 - 64 x^2 \right] dx \]

\[ = \frac{1}{32^2} \left[ \frac{1024 x^3}{3} + \frac{x^5}{5} - \frac{64 x^4}{4} \right]_0 \]

\[ = 1024 \times 32 + \frac{32^3}{3} - \frac{64 \times 32^2}{3} = 1092.16 \]

\[ H = - \int_0^L \mu y dx = 2477.07 \]

\[ \int_0^L y dx = 1092.16 = 2.27 \text{ kN} \]

Now, \( y = \frac{x}{32} (32-x) \)

\[ \frac{dy}{dx} = -\frac{32-2x}{32} = -\frac{1}{16} \]

At \( x = 8, y = \frac{8}{32} (32-8) = 6 \text{ m} \)

and \( \tan \theta = 1 - \frac{8}{16} = 0.5 \)

\[ \theta = 26^\circ 34' ; \sin \theta = 0.447 ; \cos \theta = 0.894 \]
B.M. \( (x^2+xy) = (-1 \times 24) + 2.27 \times 6 = -24 + 13.62 \\
= -10.38 \text{ kN-m} \)

Vertical shear \( \nu = V = 1 \text{ kN} \)

Normal thrust \( N = H \cos \theta - V \sin \theta = 2.27 \times 0.894 - 1 \times 0.447 \\
= 1.583 \text{ kN-m} \)

Radial shear \( F = H \sin \theta + V \cos \theta = 2.27 \times 0.447 + 1 \times 0.894 \\
= 1.909 \text{ kN} \)

169. MOVING LOADS ON TWO HINGED ARCHES

(1) INFLUENCE LINE FOR \( H \)

Let us consider a unit point load at a distance \( aL \) from \( A \). The vertical reaction at \( A \) will be \((1-a) \) while the vertical reaction at \( B \) will be equal to \( x \). For this load position the horizontal thrust, as proved earlier, is given by

\[ H = \frac{5}{8} \cdot \frac{L}{r} \cdot (1+\alpha)(1-\alpha) \]  \( (16.10) \)

Since \( \alpha \) is the variable, it is clear that equation 16.10 is a fourth degree polynomial. The I.L. for \( H \) can very easily be plotted by giving \( x \) different values.

At \( A, \ alpha = 0 \); \quad H = 0 \)
At \( B, \ alpha = 1 \); \quad H = 0 \)
At \( C, \ alpha = \frac{1}{2} \); \quad H = \frac{5}{8} \cdot \frac{L}{r} \cdot \frac{1}{2} \cdot \left( 1 + \frac{1}{2} - \frac{1}{4} \right) = \frac{25}{128} \frac{L}{r} \]  \( (16.15) \)

The I.L. diagram is shown in Fig. 16.21 (b).

(2) INFLUENCE LINE FOR B.M.

The B.M. at any point \( P \) distance \( x \) from \( A \) is given by

\[ M = Hx = \mu + Hy \]

Thus, the I.L. for \( M_F \) can be obtained by superimposing the I.L. for \( \mu \) on the I.L. for \( H \). The I.L. for \( \mu \) will have maximum ordinate of \( \frac{x(L-x)}{L} \) under \( P \). The I.L. for \( H \) will be similar to I.L. for \( H \), with maximum ordinate of \( \frac{25}{128} \frac{L}{r} y \) under the crown. The I.L. for \( M_F \) is shown in Fig. 16.21 (c).

(3) INFLUENCE LINE FOR \( N_F \) AND \( F_F \)

Influence lines of normal thrust and radial shear at \( P \) can be obtained exactly in the same manner as that for a three hinged arch except for the difference that the \( H \sin \theta \) and \( H \cos \theta \) diagram will

Example 16.12. A semi-circular arch of constant section and span 2R is pinned at both supports. Find what part of the span must be covered by a uniformly distributed load \( w \) unit length so as to produce maximum sagging bending moment at the mid-span.

Solution

Let a unit point load be placed at a point \( D \) subtending an angle \( \phi \) with the central line. The co-ordinates of the point \( D \) are given by

\[ x = R(1 - \sin \phi) \]
\[ y = R \cos \phi \]
It has been proved in § 16’7 that the horizontal thrust for a two-hinged semicircular arch is given by

\[ H = \frac{W \cos^2 \phi}{\pi} \cos^2 \frac{\phi}{\pi} \], when \( W = 1 \).

\[ H = \frac{W \cos^2 \phi}{\pi} \cos^2 \frac{\phi}{\pi} \], when \( W = 1 \).

Let us now draw the influence line for B.M. at B, the crown.

The bending moment is given by

\[ M_c = \mu + H_y = \mu + H \]

The influence line for \( \mu \) will be at triangle having a maximum

\[ \text{ordinate} = \frac{R \times R}{2R} = \frac{R}{2} \] under \( C \). The influence line for \( H_R \) will be a

cosine curve having a maximum ordinate \( \frac{\cos^2 \theta}{\pi} R = \frac{R}{\pi} \) under \( C \).

The influence line for \( M_c \) is thus the result of superimposition of the two, as shown in Fig. 16’22.

The influence line for \( M_c \) will have zero ordinates at points \( d \) and \( e \). To find the position of those points, write the equation of B.M. at \( C \) and equate it to zero.

The vertical reaction at \( B \) is given by

\[ V_B = \frac{1 \times R (1 - \sin \phi)}{2R} = \frac{1 - \sin \phi}{2} \]

\[ M_c = -V_b R + H \]

\[ = -\left( \frac{1 - \sin \phi}{2} \right) R + \frac{\cos \phi}{\pi} R \]

Equating this to zero, we get

\[ 2 \cos^2 \phi + \pi \sin \phi - \pi = 0 \]

or

\[ \sin \phi = 0.571 \]

\[ \phi = 34^\circ 50' \]

or

\[ \phi = 34^\circ 50' \]

Maximum negative B.M. at \( P \) will evidently occur when the load is on the section \( P \). In that case \( \alpha L = 10 \) m.

\[ \alpha = \frac{10}{30} = \frac{1}{3} \]
Also, \( V_A = W(1-a) = 10 \times \left(1 - \frac{1}{3}\right) = \frac{20}{3} \)

\[ M_{max}(+ve) = -\frac{20}{3} \times 10 + 10 \times 19 \times 4.44 = 2143 \text{ kN-m.} \]

Maximum positive B.M. will occur when the load is some where in \( CB \). Let the load be at a distance \( x \) from \( A \). Then,

\[ H = \frac{5}{8} \times 10 \times \frac{30}{3} x(1-a)(1+a-a^2) \]

\[ = 37.5 \left( a-2a^2+a^3 \right) \]

\[ V_A = W(1-a) = 10(1-a) \]

Now, \( M_f = -10(1-a) 10 + 37.5(0.71-2 \times 0.358 + 0.225) \)

\[ = -10(1-a) 10 + 37.5(0.71-2 \times 0.358 + 0.225) \]

\[ \frac{dM_f}{dx} = 0 = -100 + 37.5 \times 4.44(1-6a^2+4a^3) \]

or

\[ (1-6a^2+4a^3) + 0.6 = 0 \]

or

\[ 4a^3 - 6a^2 + 1.6 = 0 \]

Solving this by trial and error, we get

\[ a = 0.71 \]

\[ \therefore \text{ Distance of the load from } A = 0.71 \times 30 = 21.3 \text{ m.} \]

Distance of the load from \( B = 8.7 \text{ m.} \)

\[ H \text{ for this load position} = 37.5(0.71-2 \times 0.358 + 0.225) \]

\[ = 9.35 \text{ kN} \]

\[ V = 10(1-0.71) = 2.9 \text{ kN} \]

\[ \therefore M_{max}(+ve) \text{ at } P = -2.9 \times 10 + 9.35 \times 4.44 = +12.5 \text{ kN-m.} \]

16.10. TEMPERATURE EFFECTS

(1) Three Hinged Arch

Due to increase in temperature, the length of the arch \( ACB \) will increase. Since the two end hinges are rigidly fixed, the crown \( C \) will rise from \( C \) to \( C_1 \). Thus \( ACB \) will be new centre line of the arch. No temperature stresses will, therefore, be induced. Let us find out the value of \( CC_1 \) for a given increase (or decrease) in the temperature.

\[ AC_1 \text{ is the position of the chord. Make } AC_2 = AC. \text{ Then increase in the length of the chord} \]

\[ = AC_1 - AC_2 = CC_2 = AC \text{ at} \]

\[ CC_1 = CC_2 \sec CC_1 \]

\[ = CC_2 \sec DC \text{A, since } CAC_1 \text{ is small} \]

\[ = (AC \times t) \frac{AC}{CD} = \frac{AC^2}{CD} \times t \]

\[ = \frac{(AD^2+DC^2)}{CD} t = \frac{4t^2}{4r} \times t \]

(16.16)

(2) Two Hinged Arch

Since there is no central hinge in the case of two hinged arch the end hinges will exert a horizontal thrust on the arch to prevent the ends from moving out when the temperature of the arch increases. Due to this horizontal thrust, there will be bending moment at all the sections.

Let \( H_t \) be the horizontal thrust induced due to a rise in temperature by \( t \). The increase in horizontal span of arch \( = L \times t \) where \( \alpha = \text{co-efficient of thermal expansion} \). The bending moment on any element at a height \( y \) is \( M = H_y y \). We have already seen that total increase in span due to bending of curved bar = \( \int_0^L \frac{My}{E} ds \).

Evidently,

\[ \int_0^L \frac{M y^2}{EI} ds = L \alpha t \]

or

\[ \int_0^L \frac{H_y y^2}{EI} ds = L \alpha t \]

or

\[ H_t = \int_0^L \frac{y^2 ds}{EI} = \int_0^L \frac{y^2 ds}{EI} \]

(16.17)

Example 16.14. A two hinged parabolic arch of span 40 m and rise 8 m is subjected to a temperature rise of 22 K. Calculate the maximum bending stress at the crown due to the temperature rise if
The equation of the parabola is
\[ y = \frac{4r}{L^2} x(L-x) = \frac{4 \times 8}{40 \times 40} x(40-x) \]
\[ = \frac{x}{50} (40-x) \]
\[ E=2.1 \times 10^8 \text{ N/mm}^2 \]
\[ El. L = t = (2.1 \times 10^8) I (40 \times 11 \times 10^{-6} \times 22) \]
\[ = 2.0328 \times 10^6 \text{ J/mm}^3 \]
(where \( I \) is in m^4 units)
\[ A \]
\[ B \]
\[ C \]
\[ 30\mathrm{m} \]
\[ \theta \]
\[ \phi \]
\[ \sin \theta \]
\[ \cos \theta \]
\[ \text{Maximum B.M. at crown} = 1495 I \times 8 \]
\[ = 11960 \text{ J/mm} \]
\[ \text{Maximum bending stress at crown} \]
\[ f = \frac{M}{Z} = \frac{11960 I}{I \times 0.5} \]
\[ = 11960 \times 0.5 = 5980 \text{ kN/m}^2 \]
\[ = 5.98 \text{ N/mm}^2. \]

\textbf{Example 16.15.} A steel two hinged circular arch rib has a span of 30 m and a rise of 3 m. The rib section is uniform throughout with an overall depth of 0.7 m. Neglecting all effect except those due to bending, find, from first principles, the bending stress at the crown due
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\[ M = W_0 = W R \sin \theta \]

\[ F = \frac{M}{Z} = \frac{38829 I}{10035} \]
\[ = 13590 \text{ kN/m}^3 \]
\[ = 13.59 \text{ N/mm}^3 \]

**Example 16'16.** A semicircular ring of radius \( R \) and uniform flexural stiffness, loaded by equal and opposite forces \( R \) at the points \( A \) and \( B \), is shown in Fig. 14'96. Show that the separation of points \( A \) and \( B \) is given by

\[ \delta = \pi WR^2 \left( \frac{R^3}{2EI + \frac{R}{AE}} \right) = \frac{\pi WR^2}{2EI} \left( 1 + \frac{I}{AR^2} \right) \]
Solution

Fig. 16-27.

Let the tension in the tie rod be \( H \).

If \( U \) is the total strain energy due to bending, we have

\[
\frac{\partial U}{\partial H} = \text{shortening of tie rod} = \frac{HL}{AE} = \frac{H\sqrt{3}R^2}{50EI} = \frac{-\sqrt{3}HR^3}{50EI} \tag{1}
\]

Consider an element of length \( ds = R\,d\theta \). Let \( x \) and \( y \) be the co-ordinates of the element the origin being taken at \( A \). We have thus,

\[
x = R \left( \sin \frac{\pi}{3} - \sin \theta \right) = R \left( \frac{\sqrt{3}}{2} - \sin \theta \right)
\]

\[
y = R \left( \cos \theta - \cos \frac{\pi}{3} \right) = R \left( \cos \theta - \frac{1}{2} \right)
\]

\[
M_P = \frac{W}{2} \left[ \frac{R}{2} \sin \theta \right] + HR \left( \cos \theta - \frac{1}{2} \right)
\]

\[
\frac{\partial M_P}{\partial H} = R \left( \cos \theta - \frac{1}{2} \right)
\]

\[
\frac{\partial U}{\partial H} = 2 \int_0^{\pi/3} M_P \cdot \frac{1}{EI} \cdot ds = 2 \int_0^{\pi/3} \left\{ \frac{W}{2} R \left( \frac{\sqrt{3}}{2} - \sin \theta \right) + HR \left( \cos \theta - \frac{1}{2} \right) \right\} \times R \left( \cos \theta - \frac{1}{2} \right) R\,d\theta
\]

\[
= 2R^2 \int_0^{\pi/3} \left[ -\frac{W\sqrt{3}}{4} \cos \theta + \frac{W \sin \theta \cos \theta}{2} + H \cos \theta - \frac{H \cos \theta}{2} \right] d\theta
\]

\[
= -\frac{2R^2}{EI} \left[ \frac{W\sqrt{3}}{4} \sin \theta + \frac{W \sin \theta \cos \theta}{2} + H \cos \theta - \frac{H \cos \theta}{2} \right]_0 \to \frac{1}{2}
\]

16.11. Reaction Locus for Two Hinged Arch

The reaction locus is a line which gives the point of intersection of the two reactions for any position of an isolated load.

Parabolic Arch

Fig. 16-28.

In Fig. 16-28, \( AM \) shows the reaction locus due to a single point load rolling from \( A \) to \( B \). The vertical line through the point \( P \) (where the unit load is acting at any instant) intersects the reactions locus at the point \( M \). Thus, \( MA \) and \( MB \) gives the directions of reaction at \( A \) and \( B \). Thus, we get the line of pressure \( AMB \) (or the linear arch) at once. The B.M. at any point is then equal to the horizontal thrust multiplied by the vertical distance between the line of pressure and the centre line of the arch at the point.

Let \( h_y = MN \) ordinate of reaction locus at any point \( P \).

Then, by triangle of force \( AMN \), we have

\[
\frac{AN}{H} = \frac{MN}{V_A}
\]

or

\[
\frac{aL}{H} = \frac{h_y}{1-a}
\]

or

\[
h_y = \frac{aL(1-a)}{H}
\]
Substituting in (1), we get

\[ h_f = MN = \frac{AN}{H} V_A = \frac{R(1 - \sin \phi)}{\pi} \left( \frac{1 + \sin \phi}{2} \right) \]

or

\[ h_f = \frac{R}{2} \]

Thus, the ordinate \( h_f \) is constant and does not depend on the load position. The reaction locus is thus a straight line parallel to \( AK \), and having ordinate to \( \frac{\pi R}{2} \).

16.12 FIXED ARCH

In the case of a fixed arch, there are six reaction components (i.e. \( V, H \) and \( M_r \) at either end) to be determined. Since only three equations are available from static equilibrium, fixed arch is statically indeterminate to third degree. We must have, therefore, three equations from the consideration of elastic deformations. Since the ends \( A \) and \( B \) are position-fixed and direction-fixed we have the following conditions to satisfy:

(a) Horizontal movement of \( B \) with respect to \( A = 0 \).

\[ \int_A^B \frac{M y}{E I} \, ds = 0 \quad (1) \]

or for finite elements, \[ \sum_A^B \frac{M y}{E I} \, ds = 0 \]

(b) Vertical movement of \( B \) with respect to \( A = 0 \).

\[ \int_A^B \frac{M x}{E I} \, ds = 0 \quad (2) \]

or \[ \sum_A^B \frac{M x}{E I} \, ds = 0 \]

(c) Change of slope of the end \( B \) with respect to \( A = 0 \).

Now \[ M = \frac{E}{I} \frac{d^2 y}{d s^2} \]

\[ \oint_A^B \frac{M x}{E I} ds = 0 \quad (3) \]

or \[ \sum_A^B \frac{M x}{E I} ds = 0 \]
Thus we have got three additional equations, the simultaneous solution of which gives the required results.

Fig. 16'30 shows a fixed parabolie arch subjected to a single point load \( W \) at a distance \( \alpha L \) from \( A \). The equation for B.M. at any point \( p \) distant \( x \) from \( A \) may be written as

\[
M = \mu + M_A + (M_B - M_A) \frac{x}{L} + Hy
\]

where \( M_A \) and \( M_B \) are the fixing moments at ends \( A \) and \( B \).

Substituting the value of \( M \) in equations (1), (2) and (3) we get

\[
\frac{M}{EI} \int ds = \frac{\mu}{EI} \int ds + M_A \int \frac{yds}{EI} + M_B - M_A \int \frac{x^2ds}{EI} + H \int \frac{yds}{EI} = 0
\]  
(16'20)

\[
\frac{M}{EI} \int ds = \frac{\mu}{EI} \int ds + M_A \frac{y}{EI} \int ds + M_B - M_A \frac{x^2}{EI} + H \frac{y}{EI} \int ds = 0
\]  
(16'21)

and

\[
\frac{M}{EI} \int ds = \frac{\mu}{EI} \int ds + M_A \frac{y}{EI} \int ds + M_B - M_A \frac{x^2}{EI} + H \frac{y}{EI} \int ds = 0
\]  
(16'22)

From the above three equations, \( M_A \), \( M_B \) and \( H \) can be determined.

Example 16'18. A circular arched rib of uniform cross-section is fixed at \( A \) and \( B \) and is subjected to vertical loads as shown in Fig. 16'31 (a). Find the magnitudes of the vertical reactions at \( A \) and \( B \).

Solution

Due to the skew-symmetrical loading, there will be no deflection and bending moment at the midspan. There is no resultant load on the span, and hence there will be no horizontal reaction at the supports. The arch can, therefore, be split into two halves, as shown in Fig. 16'31 (b). Let the S.F. at the midspan be \( F \). Take \( F \) as the redundant.

Consider any point \( P \) subtending an angle \( \theta \) with the central line. We have

\[
M_P = -FR \sin \theta; \quad +10 \ R \ (\sin \theta - \sin 30^\circ)
\]

Also,

\[
\frac{\partial M_P}{\partial F} = -R \sin \theta.
\]

Now,

\[
\frac{2U}{dF} = \int_0^{30^\circ} M_P \frac{\partial M_P}{\partial F} \frac{dF}{EI} ds = 0
\]

or

\[
\int_0^{30^\circ} R \frac{\partial U}{\partial F} = FR \int_0^{30^\circ} \sin \theta d\theta = 10R \int_0^{30^\circ} (\sin \theta - \sin 30^\circ \sin \theta) d\theta
\]

\[
= FR \left[ \frac{1}{4} \sin \frac{30^\circ}{2} \right] - 10R \left[ \frac{1}{4} \sin \frac{20^\circ}{2} + \sin 30^\circ \cos \frac{30^\circ}{2} \right]
\]

\[
= FR \left[ 30^\circ - \frac{\sqrt{3}}{8} \right] - 10R \left[ \left( \frac{30^\circ - \sqrt{3}}{8} + \frac{1}{2} \right) \right]
\]

\[
= FR \left( 0.524 - 0.216 \right) - 10R \left[ \left( 0.524 - 0.216 \right) + 0.25 - 0.262 \right]
\]

\[
= 0.308 FR - 0.8 R
\]

Equating this to zero, we get

\[
F = \frac{0.8}{0.308} = 2.6 \text{ kN}
\]
Example 16.19. A thin circular proving ring of radius 10 cm and uniform flexural stiffness $EI$ carries concentrated load 10 kN applied at the ends of a diameters. Find the maximum bending moment in the ring, and estimate the separation of the loaded points. The thickness of the ring in its own plane is 20 mm and the breadth is 40 mm. Take $E=2\times10^5 \text{N/mm}^2$.

Solution

Let the load be $P$ and radius be $R$.

Due to the symmetrical loading, it is evident that there will be no axial load in the ring at the loaded points $A$ and $B$. Cut the ring in two parts at the level of $AB$ and fix the point $A$ and $B$. The upper half ring [Fig. 16.32(a)] carries load of $\frac{1}{2}P$ and moments $M_0$ at $A$ and $B$. The only unknown is $M_0$.

At any point $P$, the B.M. is given by

$$M_P = M_0 - \frac{1}{2}PR\sin \theta$$

(1)

Also

$$ds = R \, d\theta$$

$$\frac{dU}{dM_0} = \int_0^\pi M_P \, \frac{dM_P}{dM_0} \, ds = 0$$

or

$$\int_0^\pi \left( M_0 - \frac{1}{2}PR\sin \theta \right) R \, d\theta = 0$$

(2)

which gives $\theta = 32.5^\circ$

The B.M. diagram is shown in Fig. 16.32(b).

The deflection of loaded points is given by

$$\delta = \frac{\partial U}{\partial P}$$

$$= \int_0^\pi \left( M_0 - \frac{1}{2}PR\sin \theta \right) \frac{1}{2} \frac{R \sin \theta}{EI} \, d\theta$$

$$= \int_0^\pi \left( M_0 - \frac{1}{2}PR\sin \theta \right) \frac{1}{2} \frac{R \sin \theta}{EI} \, d\theta$$

(3)

where

$$I = \frac{12}{16} \times 40 \times 20^3 = 2.67 \times 10^4 \text{mm}^4$$

$$\delta = \frac{1.870 \times 10000(100)^3}{4\pi \times 2.1 \times 10^5 \times 2.67 \times 10^4}$$

$$= 0.265 \text{mm}.$$
(a) I.L. for H

The I.L. for H can be obtained exactly in the same manner as that for a three hinged arch. Thus, the I.L. for H will be a triangle having a central ordinate of \( \frac{L}{4r} = \frac{60}{4 \times 12.5} = 1.2 \), as shown in Fig. 16.33 (b).

(b) I.L. for \( P_{DE} \)

```
Fig. 16.33.
```
When the unit load is at $F$, $V_A = 0.5$ and $H = 1.2$.

\[ P_{IH} = \frac{1}{7.42} \left[ \left( 1.2 \times 15 \right) - \left( 0.5 \times 10 \right) \right] = 1.75 \text{ (comp.)} \]

The I.L. for $P_{IH}$ is shown in Fig. 16:30(d).

(d) I.L. for $P_{DH}$

Consider the equilibrium of the forces of the left of the section $aa$ and take moments about the point $E$ where the members $DE$ and $IH$ meet. Thus

\[ P_{DH} = \frac{M_F}{z} \]

where $z$ is perpendicular distance of $DH$ from $F$.

The inclination $\phi$ of $DH$ is given by

\[ \tan \phi = \frac{4}{10} = 0.4 \quad \Rightarrow \quad \phi = 21.8^\circ \text{ and } \sin \phi = 0.371 \]

\[ z = DF \sin \phi = 20 \times 0.371 = 7.42 \text{ m} \]

\[ P_{DH} = \frac{M_F}{7.42} \]

When the unit load is at $D$, $V_A = \frac{5}{6}$ and $H = 0.4$

\[ P_{DH} = \frac{1}{7.42} \left[ \left( 0.4 \times 15 \right) - \left( \frac{5}{6} \times 30 \right) + \left( 1 \times 20 \right) \right] = 0.134 \text{ (comp.)} \]

When the unit load is at $E$, $V_A = \frac{2}{3}$ and $H = 0.8$

\[ P_{DH} = \frac{1}{7.42} \left[ \left( \frac{2}{3} \times 30 \right) - \left( 0.8 \times 15 \right) \right] = 1.077 \text{ (tension)} \]

When the unit load is at $F$, $V_A = 0.5$ and $H = 1.2$

\[ P_{DH} = \frac{1}{7.42} \left[ \left( 1.2 \times 15 \right) - \left( 0.5 \times 30 \right) \right] = 0.403 \text{ (comp.)} \]

The I.L. for $P_{DH}$ is shown in Fig. 16:33(e).

(e) I.L. for $P_{DI}$

Pass a section $bb$ and consider the equilibrium of all the forces to the left of it.

Thus,

\[ P_{DI} = \frac{M_F}{DF} = \frac{M_F}{20} \]

When the unit load is at $D$, $V_A = \frac{5}{6}$ and $H = 0.4$

\[ P_{DI} = \frac{1}{20} \left[ \left( \frac{5}{6} \times 30 \right) - \left( 0.4 \times 15 \right) \right] = 0.95 \text{ (comp.)} \]

When the unit load is at $F$, $V_A = 0.5$ and $H = 1.2$

\[ P_{DI} = \frac{1}{20} \left[ \left( 1.2 \times 15 \right) - \left( 0.5 \times 30 \right) \right] = 0.15 \text{ (tension)} \]

The I.L. for $P_{DI}$ is shown in Fig. 11:33(f).
Find (a) the horizontal distance of the hinge $B$ from one end, (b) horizontal and vertical reaction at the abutments $A$ and $C$, and (c) the bending moment at the point of application of the 70 kN load.

8. An arch in the form of a parabola with axial vertical has hinges at the abutments and the vertex. The abutments are at different levels, the horizontal span being 80 m and the heights of the vertex above the abutments being 16 m. Calculate:

(a) The horizontal thrust, and (b) maximum negative B.M. due to a U.D.L. of 25 kN/m run over the whole span.

9. A symmetrical 3-pinned parabolic arch has a span of 50 m and a rise of 24 m. Find the maximum bending moment at the quarter point of the arch caused by a uniformly distributed load of 10 kN per m run which can occupy any portion of the span. Indicate the position of the load for this condition.

$U.L.$

10. A three-hinged parabolic arch has horizontal span of 240 ft. and a rise of 24 ft. Derive from first principles the influence line for the horizontal thrust at the abutments and bending moment at the quarter span. Plot these influence lines on squared paper.

$U.E.$

11. A three-hinged parabolic arch has a horizontal span of 40 m with a central rise of 5 m. A point load of 8 kN moves across from left to right. Calculate the maximum positive and negative B.M. at the 1 m distance from the left hand hinge. Also, calculate the position and amount of the absolute maximum B.M. that may occur in the arch.

12. A three hinged parabolic arch has a span of 60 m and a central rise of 10 m. Calculate the maximum moments at quarter-span produced by a uniformly distributed load of 1 ton per horizontal foot, 60 ft. long. In each case, state the position of load for which maximum value occurs.

13. A three hinged parabolic arch has a horizontal span of 40 m with a central rise of 5 m. A point load of 8 kN moves across from left to right. Calculate the maximum positive and negative B.M. at the 10 m distance from the left hand hinge. Also, calculate the position and amount of the absolute maximum B.M. that may occur in the arch.

$U.L.$

14. A parabolic arch, hinged at the ends has a span of 60 m and a rise of 12 m. A concentrated load of 8 kN acts at 15 m from the left hinge. The second moment of area varies as the secant of the slope of the rib axis. Calculate the horizontal thrust and the reactions at the hinge. Also, calculate the maximum bending moments anywhere on the arch.

15. A parabolic two hinged arch has a span of 80 metres and a rise of 10 m. A uniformly distributed load of 2.5 kN/m covers half of the span. If $I=I_{C}$, sec 6, find out the horizontal thrust at the hinges and radiate shear at this section.

16. A parabolic two hinged arch has a span $L$ and central rise $r$. Calculate the horizontal thrust at the hinges due to (a) U.D.L. w over the whole span, and (b) U.D.L. w over half the span.

$U.E.$

17. A two hinged semicircular arch of radius 10 m is subjected to a load of 10 kN acting on the section subtending an angle of 45° with the central line of the arch at its centre. Working from first principles, calculate (a) the horizontal thrust at the hinges, (b) the vertical reactions at the hinges, (c) maximum positive and negative bending moments.

18. A two hinged parabolic arch has a span of 40 m and a central rise of 8 m. Calculate the maximum positive and negative B.M. at a section distant 12 m from the left hinge, due to a single point load of 6 kN rolling from left to right.

19. A circular arch rib, of uniform section, hinged at the springings, has a span of 40 m and a central rise of 9 m. If the rib section is symmetrical and has a depth of 80 cm, calculate the maximum bending stress due to a rise of temperature of $60^\circ$C; $\alpha=0.000062$ per $^\circ$C and $E=2.1 \times 10^8$ kN/m².

20. A uniform arch rib covers a span of 40 m, the centre line of the rib being the segment of a circle subtending an angle of 120° at the centre. The arch is pinned at the two supports and carries a vertical load of 8 kN at the crown of the arch.

Calculate the reactions at the supports and construct the bending moment diagram for the arch.

21. A two hinged semicircular arch, of uniform flexural stiffness $EI$ carries central vertical load $W$. Show that the horizontal thrust at support is $W\\cos\theta$.

22. A steel bar of constant $EI$ is hinged into the form of a semicircle of large radius $r$, and is attached by end hinges to two rigid anchorages. Find the bending moment developed at its centre by a rise in temperature $t$ if the coefficient of linear expansion is $\alpha$. Find also the value of the centrally applied point load which would cause the same central bending moment.

Answers

1. $H=12.5$ kN; $M_{\max}=9.38$ kN-m; $M_{\text{max}}=28.13$ kN-m
2. $M_{\text{max}}(\text{-})=-11.25$ kN-m at 9 m from either hinge
3. $M_{\text{max}}(\text{+})=20$ kN-m at 4 m from either hinge
4. $2.72$ kN $\uparrow$; 3.28 kN; 6 kN; 16.44 kN-m
4. \(-245 \text{ kN-m at first quarter} ; +98 \text{ kN-m at third quarter}

5. (a) \(wx - \frac{wx^2}{2L} - \frac{wx^3}{2L}

\quad \text{or}

(b) \(\frac{wx^2}{4c}

(c) \(-\frac{w}{32}(8Lx-10x^2-L^2) ; \quad +\frac{wx^3}{16}

(d) \(-\frac{3wL^2}{160} \quad \text{when} \quad x = \frac{2}{3}L

5. (a) 19.25 ; 13.05 kN

(b) 13.05 kN

(c) \(-36.27 \text{ kN-m}

(d) \(+13.73 \text{ kN-m}

7. (a) 5.76 m from \(A

(b) \(H=120 \text{ kN} ; V_A=78.5 \text{ kN} ; V_C=81.5 \text{ kN}

(c) \(-35.9 \text{ kN-m}

8. (a) \(H=163.3 \text{ kN} ; M_{ms} (-ve)=1364 \text{ kN-m}

9. \(M_{ms} = +1875 \text{ kN-m ; load length ; Left 57.5 m or right 40 m}

10. \(H=131.25 \text{ t for the load placed centrally on the arch}

\quad \text{M}_{ms} = -928 \text{ ton-ft for the extending from} \quad 22.5 \text{ ft.} \quad \text{to} \quad 82.5 \text{ ft. from nearer springing}

11. 20 kN-m ; 30 kN-m ;\( -30.72 \text{ kN-m} ; 8.44 \text{ m from either ends}

12. 29.30 kN ; \(-32.32 \text{ kN-m} ; 31.26 \text{ kN} ; 5.60 \text{ kN} \uparrow \quad \downarrow

13. (a) 36.25 tons

(b) \(M_D = +135 \text{ t-ft} ; M_E = +315 \text{ t-ft}

(c) \(M_{ms} = M_E = +314 \text{ t-ft}

14. \(H=5.56 \text{ kN} ; V_A=6 \text{ kN} ; V_B=2 \text{ kN} ; M_{ms} = -39.96 \text{ kN-m}

15. 100 kN ; \(-290 \text{ kN-m} ; 103.1 \text{ kN} ; 0

16. (a) \(\frac{wx^2}{8r} \quad \text{or} \quad \frac{wx^3}{16r}

17. (a) 1.592 kN (b) \(V_A=8.54 \text{ kN} ; V_B=1.46 \text{ kN}

(c) \(M_{ms} (-ve)=13.47 \text{ kN-m} ; M_{ms} (+ve)=2.03 \text{ kN-m}

18. 10.4 kN-m ; 18.4 kN-m

19. 81.4 \text{ kg/cm}^2

20. \(H=5.11 \text{ kN} ; M_{ms} = -21.07 \text{ kN-m} (-ve)

22. (a) \(M = \frac{4E \ell a t}{\pi r} \text{ (hoggling)}

(b) \(W = \frac{8E \ell a t}{11416r^2}

SECTION 3
ADVANCED STRENGTH OF MATERIALS

17. BENDING OF CURVED BARS
18. STRESSES DUE TO ROTATION
19. VIBRATIONS AND CRITICAL SPEEDS
20. FLAT CIRCULAR PLATES
21. UNSYMMETRICAL BENDING
22. ELEMENTARY THEORY OF ELASTICITY
Bending of Curved Bars

17'1. INTRODUCTION : BARS WITH SMALL INITIAL CURVATURE

In 'simple bending' the relations between the straining actions and the stresses and strains were established for a 'straight beam'. The well known formula:

\[ \frac{M}{I} = \frac{f}{y} = \frac{F}{R} \]

is sometimes called the 'straight-beam formula'. The results of simple bending can be applied, with sufficient accuracy, to the beams or bars having small initial curvature.

It is common practice to distinguish rods or bars of small and large initial curvature. The chief characteristic of such a division is the ratio of the depth of the section \( h \) in the plane of curvature to the radius of curvature \( R_0 \) of the rod axis. If this ratio \( \frac{h}{R_0} \) is 0.2 and less, it is taken that the rod has a small curvature. For a rod of large curvature, the ratio \( \frac{h}{R_0} \) is comparable with unity. This division is purely conventional.

Let us now take the case of a beam or bar with small initial curvature \( R_0 \) about the neutral surface \( EF \) (Fig. 17'1). Let the beam \( ABCD \), under the action of pure bending, be bent to \( A'B'C'D' \). The radius of curvature decreases from \( R_0 \) to \( R \), and the central angle increases from \( \theta \) to \( (\theta + \delta \theta) \).

Consider any surface \( GH \), distant \( y \) from the neutral surface \( EF \).

The treatment that follows is based on the same assumptions as those of simple bending of straight bars.
Strain in \( GH = \frac{G'H' - GH}{GH} \)
\[ = \frac{(R+y)(\theta + \theta') - (R_0 + y)\theta}{(R_0 + y)\theta} \]
\[ = \frac{R(\theta + \theta') + y\theta + y\theta' - R_0 \theta - y\theta}{(R_0 + y)\theta} \]
\[ = \frac{R(\theta + \theta') + y\theta - R_0 \theta}{(R_0 + y)\theta} \]
But
\[ E\theta = E'\theta' = R_0 \theta = R(\theta + \theta') \]
\[ \therefore \frac{R_0 - R}{R} = \frac{y\theta}{\theta} \]

Stress \( f = E\varepsilon = Ey \left( \frac{1}{R} - \frac{1}{R_0} \right) \)
or
\[ \frac{f}{y} = E \left( \frac{1}{R} - \frac{1}{R_0} \right) \]

But
\[ \frac{f}{y} = \frac{M}{I} \] (from simple theory of bending).

17.2 BARS WITH LARGE INITIAL CURVATURE

In the above analysis, it is assumed that \( y \) is negligible in comparison to the initial radius of curvature \( R_0 \). However, there are practical cases of bars, such as hooks, links and rings, etc., which have large initial curvature (or small radius of curvature). In such a case, the dimensions of the cross-section are not very small in comparison with either the radius of curvature or with the length of the bar. The treatment that follows is based on the following assumptions by Winkler.

1. Transverse sections which are plane before bending remain plane after bending.
2. Longitudinal fibres of the bar, parallel to the central axis, exert no pressure on each other.
3. The working stresses are below the limit of proportionality.
4. The line joining the centroids of the cross-sections of the bar, called the centre line, is the plane curve and that the cross-sections have an axis of symmetry in this plane.

The bar is subjected to the action of forces lying in the plane of symmetry so that bending takes place in this plane.

Consider a curved beam of constant cross-section, subjected to pure bending produced by couples \( M \) applied at the ends. Let \( AB \) and \( CD \) be two adjacent cross-sections of the beam subtending a small angle \( \theta \) at the centre of curvature, before bending. Let the bending moment \( M \) cause the plane \( CD \) to rotate through \( \Delta \theta \) (Fig. 17.2), changing the centre of curvature from \( O \) to \( O' \). Let the distance of the centroidal axis from the centre of curvature be changed from initial value of \( R \) to \( \rho \). Consider any fibre \( PQ \) distant \( y \) from
the centroidal axis. The section CD rotates about the point H; hence the layer GH is the neutral layer. It should be noted that the quantities \( R, p \) and \( y \) are measured from the centroidal axis; and not from the neutral axis.

Let

\[ \varepsilon_0 = \text{strain of the centroidal fibre} \]

\[ \varepsilon = \text{strain of any other fibre } PQ \text{ distant } y \text{ from the centroidal layer.} \]

\[ \varepsilon_0 = \frac{FF'}{EE} \text{ or } FF' = \varepsilon_0 (FE) = \varepsilon_0 R \, d\theta \]  \hspace{1cm} \text{(1)}

\[ \varepsilon = \frac{Q'Q}{PQ} = \frac{Q'O' + Q'Q''}{PQ} = \frac{FF' + Q'Q''}{PQ} \]

or

\[ \varepsilon = \varepsilon_0 \frac{R \, d\theta + y \, \Delta \theta}{(R+y) \, d\theta} = \frac{\varepsilon_0 \, R + y \, \frac{\Delta \theta}{d\theta}}{R+y} \]  \hspace{1cm} \text{(2)}

Let

\[ \omega = \text{angular strain} = \frac{\Delta \theta}{d\theta} \]

Substituting this and adding and subtracting \( \varepsilon_0 \) to the numerator of (2), we get

\[ \varepsilon = \varepsilon_0 \frac{R+y}{R+y} \left( \varepsilon + \varepsilon_0 \right) \]

\[ \varepsilon = \varepsilon_0 + \frac{(\omega - \varepsilon_0)}{R+y} \]  \hspace{1cm} \text{(17')}

Within the elatic property of the material,

\[ f = E \, \varepsilon = E \left[ \varepsilon_0 + \frac{(\omega - \varepsilon_0)}{R+y} \right] \]  \hspace{1cm} \text{(176)}

where \( f = \text{bending stress or normal stress} \) (also known as circumferential stress).

It will be assumed, for simplicity, that the section of the beam is symmetrical about the plane of curvature. The y-axis is then the axis of symmetry of the section [Fig. 17.2 (b)] and the moment of elementary forces \( f \cdot dA \) with respect to this axis is zero.

There are two unknowns in Eq. 17.6: \( \varepsilon_0 \) and \( \omega \). To determine these quantities we use the two equations of statics which state that the sum of the normal forces distributed over a cross-section is equal to zero and the moment of these forces is equal to the external moment \( M \). These equations are:

\[ \int_A f \cdot dA = \int_A E \left[ \varepsilon_0 + \frac{(\omega - \varepsilon_0)}{R+y} \right] dA = 0 \]

and

\[ \int_A f \cdot ydA = \int_A E \left[ \varepsilon_0 + \frac{(\omega - \varepsilon_0)}{R+y} \right] y \, dA = M \]  \hspace{1cm} \text{(11)}

The above equations may be simplified as follows:

From (1),

\[ \int_A f \, dA = \int_A E \varepsilon_0 dA + \int_A E \omega \, dA \]

or

\[ \varepsilon_0 = \frac{\omega \varepsilon_0}{R+y} \]  \hspace{1cm} \text{(17')} \]

Also, from (11),

\[ M = E \varepsilon_0 \int_A y \, dA + \int_A \frac{y^2}{R+y} \, dA \]  \hspace{1cm} \text{(IV)}

But

\[ \int_A \frac{y}{R+y} \, dA = 0 \]

and let

\[ \int_A \frac{y}{R+y} \, dA = -mA \text{ or } m = \frac{1}{A} \int_A \frac{y}{R+y} \, dA \]  \hspace{1cm} \text{(177)}
Substituting these in (I.II) and (IV), we get

\[ \varepsilon_0 = (\omega - \varepsilon_0)^{\frac{m}{AR}} \]

and

\[ M = E(\omega - \varepsilon_0)^{\frac{m}{AR}} \]

Solving these two, we get

\[ \omega = \frac{1}{E} \left( \frac{M}{R} + \frac{M}{mR} \right) \]

Substituting these values in Eq. 17.6, we get

\[ f = \frac{M}{AR} \left( 1 + \frac{1}{m} \frac{y_{0}}{R+y} \right) \]

The above expression for \( f \) is generally known as the Winkler-Bach formula. The distribution of \( f \), given by Eq. 17.12 is hyperbolic (and not linear as in the case of straight beams) and is shown in Fig. 17.2 (c).

In the above expression the quantity \( m \) is a pure number, and is the property of each particular shape of the cross-section, defined by Eq. 17.7. Its value can be determined by performing the integration or by graphical method. The quantity \( mA \) is called the modified area of the cross-section.

Reduction of the above formula for the case of the straight beam

Eq. 17.12 can be reduced for the case of a straight beam. Rewriting it,

\[ f = \frac{M}{AR} + \frac{M}{mAR} \cdot \frac{y}{R+y} \]

\[ = \frac{M}{AR} \left[ \frac{y}{R+y} \right] + \frac{My}{R+y} \]

\[ = \frac{M}{AR} \left( 1 + \frac{y}{R+y} \right) \int \frac{y^{2}dA}{1+y/R} \]

...\( (17.13) \)

For a straight beam, \( R \) is infinitely large. Hence Eq. 17.13 reduces to

\[ f = 0 + \frac{My}{\int y^{2}dA} = \frac{My}{I} \]

...\( (17.14) \)

where \( \int y^{2}dA = I \)

Eq. 17.14 is the same as our usual bending stress formula.

Location of neutral axis

The neutral axis can be located by equating Eq. 17.12 to zero, since the bending stress at the neutral axis is zero.

\[ 1 + \frac{1}{m} \frac{y_{0}}{R+y_{0}} = 0 \]

or

\[ y_{0} = -\frac{mR}{1+m} \]

...\( (17.15) \)

where \( y_{0} \) = distance of neutral axis from the centroidal layer.

The negative sign suggests that the neutral axis is towards the centre of curvature (see sign conventions below).

Sign Convention

The following sign convention will be followed:

1. A bending moment \( M \) will be taken as positive if it decreases the radius of curvature, and negative if it increases the radius of curvature.

2. \( y \) is positive when measured towards the convex side of beam, and negative when measured towards the concave side (or towards the centre of curvature).

3. With the above sign convention, if \( f \) comes out to be positive it will denote tensile stress while negative sign will mean compressive stress.

17.3. ALTERNATIVE EXPRESSION FOR \( f \)

In the previous treatment, the integrals

\[ \int \frac{y}{AR+y} dA \text{ and } \int \frac{y^{2}}{AR+y} dA \]

were expressed as \(-mA\) and \(mAR\) respectively, and substituted in Eqs. III and IV to get the final expression for \( f \). We shall now denote the above integrals in terms of a new parameter \( A' \) defined below:

\[ \int \frac{y}{AR+y} dA = \int \left( 1 - \frac{R}{ya+R} \right) dA = \int dA - \int \frac{R}{ya+R} dA \]
SRENGTH OF MATERIALS AND THEORY OF STRUCTURES

But
\[
\int dA = A
\]
and let
\[
\int_A \frac{R}{y+R} dA = A'
\]...(17'16)
Hence
\[
\int \frac{y}{R+y} dA = A - A'
\]...(17'17)
Also
\[
\int (y-R) \frac{R}{y+R} dA + \int \frac{R^2}{y+R} dA
\]
\[= \left[ y.dA \right]_R - R dA + \left[ \frac{R^2}{y+R} dA \right]
\]
\[= 0 - RA + RA' = R(A' - A) \]...(17'18)

Substituting these values in Eqs. III and IV of the previous article, and simplifying as before, we get the following alternative expression for \( f \):
\[
f = \frac{M}{R(A' - A)} \left( \frac{A'}{A} - \frac{R}{y+R} \right) \]...(17'19)
Comparing Eqs. 17'7 and 17'17, we have
\[
\int \frac{y}{R+y} dA = -mA = A - A'
\]
\[m = \frac{A' - A}{A} = \left( \frac{A'}{A} - 1 \right) \]...(17'20)
\[A' - A = mA \]...(17'20 (a))

Substituting these values of \( (A' - A) \) and \( \frac{A'}{A} \) in Eq. 17'19, we get
\[
f = \frac{M}{RA} \left( m + 1 - \frac{R}{y+R} \right) = \frac{M}{R} \left( m + \frac{y}{y+R} \right)
\]
or
\[
f = \frac{M}{RA} \left( 1 + \frac{m}{R} \frac{R}{y+R} \right)
\]
which is the same as Eq. 17'12.

17'4. DETERMINATION OF FACTOR \( m \) FOR VARIOUS SECTIONS

We shall now determine the values of factor \( m \) for various shapes of cross-section by evaluating the integral defined by Eq. 17'7.

1. RECTANGULAR CROSS-SECTION

Consider a rectangular normal section of a curved beam. The width parallel to the axis of curvature is \( b \) while the dimension measured along the direction of radius of curvature is \( d \). Let \( R \) be the distance of the centroidal axis from the axis of curvature.

From Eq. 17'7,
\[m = \frac{1}{A} \int_A \frac{y}{R+y} dA = - \frac{1}{bd} \int \frac{1}{y+R} y \cdot bdy
\]
\[= - \frac{1}{d} \left[ y \frac{y}{R+y} + \frac{1}{2} \left( \frac{R}{y+R} \right)^2 \right]_{dy}
\]
\[= - \frac{1}{d} \left[ y-R \log_e (R+y) \right]_{y-R}
\]
or
\[m = \frac{R}{d} \log_e \left( \frac{R+d/2}{R-d/2} \right) = 1 - \frac{R}{d} \log_e \left( \frac{R_s}{R_1} \right) \]...(17'21)

Also \[\log_e \left( \frac{R+d/2}{R-d/2} \right) = \frac{d}{R} \left[ 1 + \frac{1}{3} \left( \frac{d}{2R} \right)^2 + \frac{1}{5} \left( \frac{d}{2R} \right)^4 + ... \right]
\]
\[m = \frac{1}{3} \left( \frac{d}{2R} \right)^2 + \frac{1}{5} \left( \frac{d}{2R} \right)^4 \]...(17'22)

2. TRAPEZOIDAL CROSS-SECTION

Consider an elementary strip of width \( b \), distant \( y \) from the centroidal axis. Let \( r \) be its distance from the axis of rotation.
Now
\[ r = (R + y) \quad \ldots (i) \]
\[ b = b_2 + \frac{b_1 - b_2}{b} \left( \frac{R_2 - r}{R} \right) \quad \ldots (ii) \]
\[ dA = b \, dr = \left\{ b_2 + \frac{b_1 - b_2}{b} \left( \frac{R_2 - r}{R} \right) \right\} \, dr \quad \ldots (iii) \]

For rectangular section, \( b_1 = b_2 = b \). Hence the above expression reduces to
\[ m = -1 + \frac{Rb}{A} \log \frac{R_2}{R_1} = \frac{R}{d} \log \left( \frac{R_2}{R_1} \right) - 1 \]
which is the same as Eq. 17'21.
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Fig. 17-6. Circular section.

Now

\[ m = -\frac{1}{A} \int_{-\pi/2}^{+\pi/2} \frac{y}{R+y} \, dA = -\frac{1}{\pi R^2} \int_{-\pi/2}^{+\pi/2} \frac{r \sin \theta}{R+r \sin \theta} \cdot 2 \pi \cos^2 \theta \, d\theta = -\frac{2 \pi}{R} \int_{-\pi/2}^{+\pi/2} \frac{\sin \theta \cos^2 \theta}{R+r \sin \theta} \, d\theta \]

Putting \( \frac{R}{r} = k \), we get

\[ m = -2 \frac{\pi}{R} \int_{-\pi/2}^{+\pi/2} \frac{\sin \theta - \sin^3 \theta}{\sin \theta + k} \, d\theta = \frac{2}{\pi} \int_{-\pi/2}^{+\pi/2} \left[ \frac{\sin^2 \theta - k \sin \theta + (k^2 - 1) \sin \theta + k}{\sin \theta + k} \right] \, d\theta = \frac{2}{\pi} \left[ \frac{\pi}{2} + (k^2 - 1) \pi \right] - 2k \sqrt{k^2 - 1} = 1 + 2(k^2 - 1) - 2k \sqrt{k^2 - 1} = -1 + 2k^2 - 2k \sqrt{k^2 - 1} = 1 + 2\left(\frac{R}{r}\right)^2 - 2\left(\frac{R}{r}\right)\sqrt{\left(\frac{R}{r}\right)^2 - 1} \quad [17'25 (a)] \]

Alternatively, if \( \frac{1}{R+r \sin \theta} \) is expanded in the converging series, and each term then under the integral (1) is integrated separately the final result is

\[ m = \frac{1}{4} \left( \frac{r}{R} \right)^2 + \frac{1}{8} \left( \frac{r}{R} \right)^4 + \frac{5}{64} \left( \frac{r}{R} \right)^6 + \ldots \quad [17'25 (b)] \]

Hollow circular section

For a hollow circular section of inner radius \( r_1 \) and outer radius \( r_2 \), \( m \) is given by

\[ m = -1 + \frac{2R}{r_2^3 - r_1^3} \left[ \sqrt{R^2 - r_1^2} - \sqrt{R^2 - r_2^2} \right] \quad [17'25 (c)] \]

(5) OTHER SECTIONS

For the T-section [Fig. 17-7 (a)]

\[ m = \frac{R}{A} \left( b_1 \log_s \frac{R_s}{R_1} + b_2 \log_s \frac{R_s}{R_2} \right) - 1 \quad (17'26) \]

For the I-section [17'7 (b)]

\[ m = \frac{R}{A} \left( b_1 \log_s \frac{R_s}{R_1} + b_3 \log_s \frac{R_s}{R_3} + b_4 \log_s \frac{R_s}{R_4} \right) - 1 \quad (17'27) \]

Example 17'1. A curved beam, rectangular in cross-section is subjected to pure bending with couple of +400 N.m. The beam has width of 20 mm, and depth of 40 mm and is curved in a plane parallel...
to the depth. The mean radius of curvature is 5 mm. Find the position of the neutral axis, and the ratio of the maximum to the minimum stress. Also, plot the variation of the bending stress across the section.

Solution.

\[ f = \frac{EM}{AR} \left( 1 + \frac{1}{m} \frac{y}{R+y} \right) \]

\[ = \frac{400000}{40 \times 20 \times 50} \left( 1 + \frac{1}{0.0591} \frac{y}{50+y} \right) \]

\[ = 10 \left( 1 + 16.92 \frac{y}{50+y} \right) \quad \ldots (1) \]

The stress is given by Eq. 17.12

\[ f = \frac{EM}{AR} \left( 1 + \frac{1}{m} \frac{y}{R+y} \right) \]

\[ = \frac{400000}{40 \times 20 \times 50} \left( 1 + \frac{1}{0.0591} \frac{y}{50+y} \right) \]

\[ = 10 \left( 1 + 16.92 \frac{y}{50+y} \right) \quad \ldots (1) \]

The stress distribution can be plotted by substituting various values of \( y \) in (1). Fig. 17.8 shows the stress distribution.

**Example 17.2.** Solve example 17.1 assuming the section to be circular, of diameter 40 mm. The mean radius of curvature of the beam is 50 mm.

Solution.

\[ m = \frac{R}{d} \log_e \left( \frac{R}{R_1} \right) - 1 \]

Here, \( R = 50, R_1 = 50 - 20 = 30 \) and \( R_2 = 50 + 20 = 70 \) mm.

\[ m = \frac{50}{40} \log_e \left( \frac{70}{30} \right) - 1 = 0.0585 \]

\[ m = \frac{50}{40} \log_e \left( \frac{70}{30} \right) - 1 = 0.0585 \]

Alternatively, from Eq. 17.22, considering first two terms,

\[ m = \frac{d}{2} \left( \frac{d}{2R} \right)^2 + \frac{d}{8} \left( \frac{d}{2R} \right)^4 \ldots \]

\[ m = \frac{1}{4} \left( \frac{40}{100} \right)^2 + \frac{1}{8} \left( \frac{40}{100} \right)^4 \ldots = 0.0432 \]
Stress $f$ is given by Eq. 17.12

$$f = \frac{M}{AR} \left( 1 + \frac{1}{m} \frac{y}{R+y} \right)$$

where

$$A = \frac{\pi}{4} (40)^2 = 1257 \text{ mm}^2$$

$$f = \frac{400 \times 10^3}{1257 \times 50} \left[ 1 + \frac{1}{0.0432} \frac{y}{50+y} \right]$$

$$= 6'364 \left[ 1 + 23'15 \frac{y}{50+y} \right] \ldots (2)$$

The position of neutral axis is given by

$$f=0 = 6'364 \left[ 1 + 23'15 \frac{y_0}{50+y_0} \right]$$

or

$$23'15 y_0 = 50 - y_0$$

\[ \therefore \quad y_0 = -\frac{50}{24'15} = -2'07 \text{ mm}. \]

The stress distribution across the section is shown in Fig. 17.9.

Example 17.3. Solve example 17.1 if the beam section is trapezoidal with dimensions shown in Fig. 17.10.

Solution.

The distance $d_1$ of the centroidal axis from the side $b_1$ of a trapezium is given by

$$d_1 = \frac{b_1 + 2b_2}{b_1 + b_2} \times \frac{d}{3} \quad \ldots (17.28)$$

\[ = \frac{30 + 40}{50} \times \frac{40}{3} = 18'6 \text{ mm} \]

\[ \therefore \quad d_1 = 40 - 18'6 = 21'4 \text{ mm} \]

$$R_s = R + d_1 = 50 + 21'4 = 71'4 \text{ mm}$$

$$R_s = R + d_1 = 30 + 18'6 = 48'6 \text{ mm}$$

$$A = (b_1 b_2) \frac{d}{2} = (30 + 20) \frac{40}{2} = 1000 \text{ mm}^2$$

From Eq. 17.23,

$$n = -1 + \frac{R_s}{A} \left\{ \left[ b_2 + (b_1 - b_2) \frac{R_s}{d} \right] \log_e \frac{R_s}{R_s} - (b_1 - b_2) \right\}$$

$$= -1 + \frac{50}{1000} \left\{ \left[ 20(30 - 20) \frac{71'4}{40} \right] \log_e \frac{71'4}{31'4} (30 - 20) \right\}$$

$$= 0'0547$$
The stress is given by Eq. 17.12,

\[ f = \frac{M}{AR} \left(1 + \frac{1}{m} \frac{y}{R+y}\right) \]

\[ = \frac{400 \times 10^6}{1000 \times 50} \left[1 + \frac{1}{0.0547} \frac{y}{50+y}\right] \]

\[ = 8 \left[1 + 18'28 \frac{y}{50+y}\right] \]

\( f_{\text{max}} = 8 \left[1 + 18'28 \frac{-18'6}{50-18'6}\right] = -78'6 \text{ N/mm}^2 \]

\( = 78'6 \text{ N/mm}^2 \) (compressive)

\( f_{\text{min}} = 8 \left[1 + 18'28 \frac{21'4}{50+21'4}\right] = 51'8 \text{ N/mm}^2 \) (tensile)

Ratio \( \frac{f_{\text{max}}}{f_{\text{min}}} = \frac{78'6}{51'8} = 1'52 \)

The position of N.A. is given by

\[ f = 0 = 8 \left[1 + 18'28 \frac{-y_0}{50+y_0}\right] \]

or \( 18'28 y_0 = -50 - y_0 \)

\[ \therefore y_0 = -\frac{50}{19'28} = -2'59 \text{ mm.} \]

The pressure distribution across the section is shown in Fig. 17.10 (b).

**17.5. BENDING OF CURVED BAR BY FORCES ACTING IN THE PLANE OF SYMMETRY**

Upto this stage, we have discussed the case of 'pure bending', i.e., bending of a curved bar by the action of pure couples. However, a curved beam may be subjected to a system of forces of \( P_1, P_2, \ldots P_n \) keeping it in equilibrium. It is assumed that these forces act in the plane of the centre line, which is the plane of symmetry of the bar. If we consider any normal section of the beam, the resultant force to the right of it may not be along the section, but may be inclined to the section. This inclined force can be resolved along the section and normal to the section. Therefore, as in the case of an arch, the section of the curved beam may be subjected to three straining actions:

(i) A normal force \( N \), acting normal to the section;

(ii) A shearing force \( F \), acting along or tangential to the section, and

(iii) A bending moment \( M \) acting in the plane of symmetry (i.e., symmetrical bending of a curved bar).

The stress at a point in a section will therefore be equal to the algebraic sum of the direct stress \( f_0 \) due to the normal force \( N \) and the bending stress \( f_b \). It is assumed that the normal force \( N \) acts through the centroid of the section and that the stress caused due to this is equal to \( \frac{N}{A} \) at each point on the area. Hence the final stress at any point is given by

\[ f = f_0 + f_b = \frac{N}{A} + \frac{M}{AR} \left(1 + \frac{1}{m} \frac{y}{R+y}\right) \quad \ldots (17.29) \]

\( N \) is assumed to be plus if it produces tensile stress, and minus if it produces compressive stress.

The shearing force \( F \) produces shearing stresses, the distribution of which is assumed to be the same as for a straight bar.

**17.9. STRESSES IN HOOKS**

The results of the previous article can now be applied to find the stresses in the horizontal section through the centre of curvature of a hook carrying a vertical load \( P \).
The horizontal section AC, passing through the centre of curvature is the most highly stressed section. BB is the centroidal axis of the horizontal section which may be of trapezoidal or any other shape. The load P which the hook supports acts eccentricaly with respect to the centroid of the section. Hence this force causes (i) bending moment \( M = P \times l \) (being negative since it increases the radius of curvature) and (ii) a tensile force \( P \) acting through the centroid of the section. Hence the stress at point on the horizontal section is given by Eq. 17'29.

\[
\sigma = \frac{P}{Am} \left( 1 + \frac{y}{m} \frac{R+y}{R+y} \right)
\]

(Since \( M = -P l \)).

Designating \( y \) as positive when measured towards the convex side and negative when measured towards concave side, stress \( f_1 \) at CC where \( y = -d_1 \) is given by

\[
f_1 = \frac{P}{A} - \frac{P l}{AR} \left( 1 - \frac{1}{m} \frac{d_1}{R-d_1} \right) \quad \ldots (17'30)
\]

Similarly, the stress \( f_2 \) at AA, where \( y = +d_2 \) is given by

\[
f_2 = \frac{P}{A} - \frac{P l}{AR} \left( 1 + \frac{1}{m} \frac{d_2}{R+d_2} \right) \quad \ldots (17'31)
\]

The bending action alone causes compressive stress at AA and tensile stress at CC, while the normal force \( P \) causes uniform tensile stress over the whole section. In a well-designed hook, both the stresses \( f_1 \) and \( f_2 \) are not very different.

If \( l = R \) (i.e. centre line of the load passing through the centre of the curvature of the hook) the above two equations reduce to the following simplified form:

\[
f_1 = \frac{P}{A} - \frac{P R}{AR} \left( 1 - \frac{1}{m} \frac{d_1}{R-d_1} \right) = \frac{P}{Am} \frac{d_1}{R-d_1} \quad \ldots (17'33)
\]

and

\[
f_2 = \frac{P}{A} - \frac{P R}{AR} \left( 1 + \frac{1}{m} \frac{d_2}{R+d_2} \right) = \frac{P}{Am} \frac{d_2}{R+d_2} \quad \ldots (17'34)
\]

Eqs. 17'33 and 17'34 clearly suggest that the final stress of CC (intrados) is tensile (positive) while the stress at AA (extrados) is compressive (negative). If, however, \( l \) is more than \( R \), \( f_1 \) is slightly reduced and \( f_2 \) slightly increased from the corresponding values given by Eqs. 17'33 and 17'34. In a well-designed hook, the centre of load passes through the centre of curvature.

Example 17'4. A central horizontal section of a hook is a symmetrical trapezium 50 mm deep, the inner width being 60 mm and the outer width being 30 mm. Estimate the extreme intensities of stress when the hook carries a load of 27 kN, the load line passing 40 mm from the inside edge of the section and the centre of curvature being in the load line. Also, plot the stress distribution across the section.

Solution.

\[
dl = \frac{b_1 + 2b_2}{b_1 + b_2} \times \frac{d}{3} = \frac{60 + 60}{60 + 30} \times 60 = 26'7 \text{ mm}
\]

\[
d_1 = d - d_1 = 60 - 26'7 = 33'3 \text{ mm}
\]

\[
R = R_1 + d = 40 + 26'7 = 66'7 \text{ mm}
\]

\[
R_2 = R_1 + d = 40 + 60 = 100 \text{ mm}
\]

\[
A = (b_1 + b_2) \frac{d}{2} = (60 + 30) \frac{60}{2} = 2700 \text{ mm}^2.
\]
From Eq. 17.13,
\[ m = -1 + \frac{R}{A} \left[ \left\{ b_2 + \left( b_1 - b_2 \right) \frac{R_2}{d} \right\} \log_a \frac{R_1}{R_2} \left( b_1 - b_2 \right) \right] \]
\[ = -1 + \frac{66.7}{2700} \left[ \left\{ 30 + (60-30) \frac{100}{70} \right\} \log_a \frac{100}{40} \left( 60-30 \right) \right] \]
\[ = 0.06975. \]
The bending stress at any point is given by Eq. 17.12
\[ f_b = \frac{M}{AR} \left( 1 + \frac{1}{m} \frac{y}{R+y} \right) \]
But \[ M = -P \times l \] (here \( l = R = 66.7 \text{ mm} \)
\[ = -2700 \times 66.7 \text{ N-mm} \]
\[ f_b = -\frac{27000 \times 66.7}{2700 \times 66.7} \left[ 1 + \frac{1}{0.06975} \times \frac{y}{66.7+y} \right] \]
or
\[ f_b = -10 \left( 1 + 14.34 \frac{y}{66.7+y} \right) \]

Also, direct stress
\[ f_d = \frac{P}{A} = \frac{27000}{2700} = 10 \text{ N/mm}^2. \]

(i) At the section CC,
\[ f_b = -10 \left( 1 + 14.34 \frac{26.7}{66.7-26.7} \right) \]
\[ = 85.9 \text{ N/mm}^2 \text{ (tensile)} \]
\[ f_d = 10 \text{ N/mm}^2 \text{ (tensile)} \]
\[ \therefore \text{ Total stress } f = f_b + f_d = 85.9 + 10 = 95.9 \text{ N/mm} \text{ (tensile).} \]

(ii) At the section AA,
\[ f_b = -10 \left( 1 + 14.34 \frac{33.3}{66.7+33.3} \right) = -57.8 \text{ N/mm}^2 \]
\[ f_d = +10 \text{ N/mm}^2 \]
\[ \therefore f = f_b + f_d = -57.8 + 10 = -47.8 \text{ N/mm}^2 \text{ (compressive).} \]

At any point distant \( y \) from the centroidal axis, the stress is given by
\[ f = f_b + f_d = -10 \left( 1 + 14.34 \frac{y}{66.7+y} \right) + 10 \]
\[ \text{At } y = -10, f = -10 \left( 1 + 14.34 \times 10 \right) + 10 \]
\[ = -10(1+1) + 10 = 0 \]
\[ \text{At } y = +10, f = -10 \left( 1 + 14.34 \times 10 \right) + 10 \]
\[ = -10(1+1) + 10 = 0 \]
\[ \therefore \text{ The stress distribution is shown in Fig. 17.12.} \]

Example 17.5. A central horizontal section of a hook is an \( I \) section with dimensions shown in Fig. 17.13. The hook carries a load \( P \), the load line passing 40 mm from the inside edge of the section, and the centre of curvature being in the load line. Determine the magnitude of the load \( P \) if the maximum stress in the hook is not to exceed the permissible stress of 120 N/mm². What will be the maximum compressive stress in the hook for that value of the load?

Solution.

At \( y = -20 \),
\[ f = -10 \left( 1 + 14.34 \times 20 \right) + 10 \]
\[ = 51.4 + 10 = 61.4 \text{ N/mm}^2 \]

At \( y = 0 \),
\[ f = -10(1+0) + 10 = 0 \]

At \( y = +20 \),
\[ f = -10 \left( 1 + 14.34 \times 20 \right) + 10 \]
\[ = -28.7 + 10 = -18.7 \text{ N/mm}^2 \]

At \( y = +30 \),
\[ f = -10 \left( 1 + 14.34 \times 30 \right) + 10 \]
\[ = -54.5 + 10 = -44.5 \text{ N/mm}^2 \]

The stress distribution is shown in Fig. 17.12.
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\[ A = (40 \times 20) + (20 \times 40) + (60 \times 30) = 3400 \text{ mm}^2 \]

To find the position of the centroidal axis take moments of individual areas, about \( CC \). Thus,

\[ 3400 d_1 = (60 \times 30 \times 15) + (40 \times 20 \times 50) + (40 \times 20 \times 80) \]
\[ = d_1 = 38.5 \text{ mm}; d_2 = 20 + 40 + 30 = 90 \text{ mm} \]
\[ = R_1 = 60 \text{ mm}; R_2 = R_1 + d_1 = 60 + 38.5 = 98.5 \text{ mm} \]
\[ R_3 = R_1 + d_2 = 60 + 90 = 150 \text{ mm}; R_4 = R_1 + 30 = 90 \text{ mm} \]
\[ R_5 = R_1 + 30 + 40 = 60 + 30 + 40 = 130 \text{ mm} \]

From Eq. 17.27,

\[ m = \frac{R}{A} \left( b_1 \log \frac{R_2}{R_1} + b_2 \log \frac{R_3}{R_2} + b_3 \log \frac{R_4}{R_3} \right) - 1 \]
\[ = \frac{98.5}{3430} \left[ 60 \log \frac{90}{60} + 20 \log \frac{130}{90} + 40 \log \frac{150}{130} \right] - 1 \]
\[ = 0.0837. \]

The bending stress at any point is given by Eq. 17.27,

\[ f_b = \frac{M}{AR} \left[ 1 + \frac{1}{m} \frac{y}{R+y} \right] \]

But \( M = -P \times l \) (here \( l = R = 98.5 \text{ mm} \))

\[ f_b = -\frac{P \times 98.5}{3400 \times 98.5} \left[ 1 + \frac{1}{0.0837} \frac{y}{98.5+y} \right] \]

or

\[ f_b = -\frac{P}{3400} \left[ 1 + \frac{119.95y}{98.5+y} \right] \]

Maximum bending stress occurs at \( y = 38.5 \text{ mm} \)

\[ \therefore (f_b)_{\text{max}} = -\frac{P}{3400} \left[ 1 - \frac{119.95 \times 38.5}{98.5-38.5} \right] = \frac{P}{510} \]

Also,

\[ f_b = \frac{P}{A} = -\frac{P}{3400} \]

\[ f_{\text{max}} = (f_b)_{\text{max}} + f_b = \frac{P}{510} + \frac{P}{3400} = \frac{P}{443.5} \]

But this is not to exceed the permissible stress of 120 N/mm²

\[ \therefore \frac{P}{443.5} = 120 \]

or

\[ P = 120 \times 443.5 = 53217 \text{ N.} \]

BENDING OF CURVED BARS

Max. compressive stress occurs at \( y = d_2 = 51.5 \text{ mm} \)

\[ f = \frac{P}{3400} \left[ 1 + \frac{119.95 \times 51.5}{98.5+51.5} \right] + \frac{P}{3400} \]

\[ = \frac{53217}{3400} \times \frac{119.95 \times 51.5}{98.5+51.5} \]

\[ = 642 \text{ N/mm}^2 \text{ (comp.).} \]

17.7. STRESSES IN RING SUBJECTED TO CONCENTRATED LOAD

![Diagram of stresses in a closed ring](Fig. 17.14. Stresses in a closed ring)
Consider a ring subjected to a pull (or push) through its centre, as shown in Fig. 17.14 (a). At any radial section, such as $XX$, it is subjected to a bending moment $M$, a radial shear $F$ and a normal pull (or thrust) $P$. From the condition of symmetry, the distribution of stress in two halves of ring will be the same. Due to symmetrical loading, it is evident that there will be no axial load in the ring at the loaded points $A$ and $B$. Cut the ring in two parts, through $A$ and $B$, and fix the ends $A$ and $B$. Each half of the ring carries loads of $\frac{1}{2} P$ and moments $M_0$ at ends $A$ and $B$ [Fig. 17.14 (b)].

The problem of determining moment $M$ at any section is statically indeterminate. The problem can approximately be solved by neglecting the effect of initial curvature in determining an expression for the elastic rotation of any section in terms of the moment at the section. The error introduced due to this simplification is relatively small. However, while calculating the bending stress, after the bending moment has been found, the curvature has to be taken into account since it has significant effect on its value. If the method of strain energy is used in determining the value of $M_0$ (and hence $M$), the impact of the simplification is to neglect the strain energy due to thrust. A straight beam, subjected to transverse loading, is subjected to bending moment and shearing force at any section and the strain energy at any point is predominantly due to bending. A section of a curved beam, however, has strain energy due to both bending as well as thrust. By considering, for the purposes of determining $M$ and $M_0$, the beam to be straight, we indirectly neglect the effect of thrust in the expression the strain energy.

Consider a section $XX$, such that the radius $OX$ makes an angle $\theta$ with $OA$. The section $XX$ is subjected to a bending moment $M$ given by

$$M = M_0 - \frac{1}{2} P (R \sin \theta)$$  

...(17.35)

where $R =$ radius of the centre line of the ring.

In addition to this, the section is subjected to a shearing force of magnitude $\frac{P}{2} \cos \theta$, and a direct pull $\frac{P}{2} \sin \theta$. The most important stresses are those arising from bending and direct stress at the inner and outer edges of the ring at the sections where the bending moments and the direct stress reach their extreme value.

In order to find the value of $M_0$, use the principle of minimum strain energy:

$$\frac{\partial U}{\partial M_0} = 0$$

Consider a small element of length $ds$ of the ring, subtending at angle $d\theta$ at the centre.

$$dU = \frac{1}{2} M^2 \frac{ds}{EI}$$

$$U = \int_0^\pi \frac{1}{2} M^2 \frac{dM}{EI}$$

$$\frac{\partial U}{\partial M_0} = \int_0^\pi \left( M_0 - \frac{1}{2} PR \sin \theta \right) R d\theta = 0$$

But

$$M = M_0 - \frac{1}{2} PR$$

$$\int_0^\pi \left( M_0 - \frac{1}{2} PR \sin \theta \right) R d\theta = 0$$

$$\left( M_0 - \frac{1}{2} PR \cos \theta \right)_0 = 0$$

or

$$M_0 = \frac{PR}{\pi} = 0.318 \frac{PR}{\pi}$$  

...(17.36)

Substituting the value of $M_0$ in Eq. 17.35, we get

$$M = \frac{PR}{\pi} - \frac{1}{2} PR \sin \theta = PR \left( \frac{1}{\pi} - \frac{1}{2} \sin \theta \right)$$  

...(17.37)

It should be noted that the moment $M_0$ is positive, since it reduces the radius of curvature. At the section $CC$, when $\theta = \frac{\pi}{2}$, we have

$$M = PR \left( \frac{1}{\pi} - \frac{1}{2} \right) = -0.182 PR$$  

...(17.38)

The bending moment at $CC$ is, therefore, negative. The bending moment is zero at a section given by

$$M = 0 = PR \left( \frac{1}{\pi} - \frac{1}{2} \sin \theta \right)$$

or

$$\sin \theta = \frac{2}{\pi}$$  

or

$$\theta = 32.9^\circ$$  

...(17.39(a))
The complete B.M. diagram is shown in Fig. 17’14. Knowing $M$, the bending stress at any point can be determined by Eq. 17’12.

The stress at any point will be equal to the algebraic sum of the stress due to direct pull $\frac{1}{2} P \sin \theta$ and the bending stress $f_b$.

Thus $f = f_o + f_b = \frac{P \sin \theta}{2A} + \frac{M}{AR} \left(1 + \frac{1}{m} \frac{y}{R+y}\right)$ ...(17’40)

Let $d_1 =$ distance of the extreme inside edge of the cross-section from the centre line.
$d_4 =$ distance of the extreme outside edge [Fig. 17’14 (a)].

(a) At the intrados of any section, $y = -d_4$

$$f_o = \frac{P \sin \theta}{2A} + \frac{M}{AR} \left(1 - \frac{1}{m} \frac{d_4}{R-d_4}\right)$$ ...(17’41)

where $M = PR \left(\frac{1}{\pi} - \frac{1}{2} \sin \theta\right)$

At $\theta = 0$, Eq. 17’41 reduces to

$$f_o = \frac{PR}{\pi A} \left(1 - \frac{1}{m} \frac{d_4}{R-d_4}\right) = \frac{P}{\pi A} \left(1 - \frac{1}{m} \frac{d_4}{R-d_4}\right)$$ ...[17’42 (a)]

Since $\frac{1}{m} \frac{d_4}{R-d_4}$ is always greater than 1, the above expression is negative. Hence the maximum compressive stress, given by the above equation at $\theta = 0$, is

$$(f_o)_{m=2} = \frac{P}{\pi A} \left(1 - \frac{1}{m} \frac{d_4}{R-d_4}\right)$$ ...(17’42)

Similarly, at $\theta = \frac{\pi}{2}$, Eq. 17’41 reduces to

$$f_o = \frac{PR}{2A} \left(\frac{1}{\pi} - \frac{1}{2}\right) \left(1 - \frac{1}{m} \frac{d_4}{R-d_4}\right)$$ or

$$f_o = \frac{PR}{2A} \left(\frac{1}{\pi} - \frac{1}{2}\right) \left(1 - \frac{1}{m} \frac{d_4}{R-d_4}\right)$$ or

$$f_o = \frac{PR}{2A} \left(\frac{1}{\pi} - \frac{1}{2}\right) \left(1 - \frac{1}{m} \frac{d_4}{R-d_4}\right)$$ ...(17’43)

It can be seen that $f_o$ given by Eq. 17’43 has both the terms positive $\left(\frac{1}{m} \frac{d_4}{R-d_4} > 1\right)$, and hence $f_o$ at $\theta = \frac{\pi}{2}$ is tensile.

Thus, the stress at the intrados changes from a compressive value at $\theta = 0$ to a tensile value at $\theta = \frac{\pi}{2}$.

(b) At the extrados of any section, $y = +d_4$

$$f_o = \frac{P \sin \theta}{2A} + \frac{M}{AR} \left(1 + \frac{1}{m} \frac{d_4}{R+d_4}\right)$$ ...(17’44)

where $M = PR \left(\frac{1}{\pi} - \frac{1}{2} \sin \theta\right)$

At $\theta = 0$, the above expression reduces to

$$f_o = \frac{P}{\pi A} \left(1 + \frac{1}{m} \frac{d_4}{R+d_4}\right)$$ ...(17’45)

This is wholly tensile.

At $\theta = \pi/2$, Eq. 17’44 reduces to

$$f_o = \frac{P}{2A} + \frac{PR}{\pi A} \left(\frac{1}{\pi} - \frac{1}{2}\right) \left(1 + \frac{1}{m} \frac{d_4}{R+d_4}\right)$$ or

$$f_o = \frac{P}{2A} + \frac{1}{182} \frac{P}{A} \left(1 + \frac{1}{m} \frac{d_4}{R+d_4}\right)$$ ...(17’46)

This will be evidently compressive, since the second term of R.H.S. (i.e., bending stress) is always more than the first term.

The absolute maximum tensile stress anywhere in the ring may be given either by Eq. 17’43 or Eq. 17’44 or Eq. 17’45. For a ring whose mean radius $R$ is large compared to the dimensions of cross-section, Eq. 17’43 gives the greatest tensile stress, where if $R$ is small, owing to greater curvature, greatest tension may be given by Eq. 17’45. There is, however, a critical value of $R$ at which the tensile stress given by Eqs. 17’43 and 17’45 are equal.

Example 17’6. A closed ring of mean radius 120 mm is subjected to a pull of 20 kN the line of action of which passes through its centre. The ring is circular in cross-section with a radius equal to 40 mm. Find the maximum value of tensile and compressive stresses in the ring.

Solution. (Fig. 17’14)

The factor $m$ for a circular section is given by Eq. 17’25 (a)

$$m = -1 + 2 \left(\frac{R}{r}\right)^2 - 2 \left(\frac{R}{r}\right) \sqrt{\left(\frac{R}{r}\right)} - 1$$

where $\frac{R}{r} = \frac{120}{40} = 3$

$$m = -1 + 2(3)^2 - 2(3) \sqrt{3} - 1 = 0.02943$$
Alternatively, from Eq. 17'25 (b),

\[ m = \frac{1}{4} \left( \frac{r}{R} \right)^2 + \frac{1}{8} \left( \frac{r}{R} \right)^4 + \frac{5}{64} \left( \frac{r}{R} \right)^6 + \ldots = 0'02943 \]

\[ \therefore \quad \frac{1}{m} = \frac{1}{0'02943} = 33'98 \approx 34 \]

\[ A = \pi r^2 = \pi (40)^2 = 5027 \text{ mm}^2. \]

(a) At \( \theta = 0 \), the stress at intrados is given by Eq. 17'25 (a),

\[ f_t = \frac{P}{\pi A} \left( 1 - \frac{1}{m} \frac{d_1}{R - d_1} \right) \]

\[ = \frac{20000}{\pi (5027)} \left[ 1 - \frac{34}{120} \frac{40}{40} \right] = -20'3 \text{ N/mm}^2 \]

\[ = 20'3 \text{ N/mm}^2 \text{ (compressive)} \quad \ldots (i) \]

The stress at extrados is given by Eq. 17'45,

\[ f_t = \frac{P}{\pi A} \left[ 1 + \frac{1}{m} \frac{d_2}{R + d_2} \right] \]

\[ = \frac{20000}{\pi (5027)} \left[ 1 + \frac{34 \times 40}{120 + 40} \right] \]

\[ = 12'03 \text{ N/mm}^2 \text{ (tensile)} \quad \ldots (ii) \]

(b) At \( \theta = \frac{\pi}{2} \), the stress at intrados is given by Eq. 17'43

\[ f_t = \frac{P}{2A} \left( 1 - \frac{1}{m} \frac{d_1}{R - d_1} \right) \]

\[ = \frac{20000}{2 \times 5027} \cdot \frac{0'182 \times 20000}{5027} \left[ 1 - \frac{34 \times 40}{120 - 40} \right] \]

\[ = 1'99 + 11'59 = 13'58 \text{ N/mm}^2 \text{ (tensile)} \quad \ldots (iii) \]

The stress at the extrados is given by Eq. 17'46,

\[ f_t = \frac{P}{2A} \left[ 1 + \frac{1}{m} \frac{d_2}{R + d_2} \right] \]

\[ = \frac{20000}{2 \times 5027} \cdot \frac{0'182 \times 20000}{5027} \left[ 1 + \frac{34 \times 40}{120 + 40} \right] \]

\[ = 1'99 - 6'88 = -4'89 \]

\[ = 4'89 \text{ N/mm}^2 \text{ (compressive)} \quad \ldots (iv) \]

From (i) to (iv), we get

Max. compressive stress = 20'3 N/mm²

Max. tensile stress = 13'58 N/mm².
For circular portion, 

\[ M = M_0 - \frac{PR}{2} \sin \theta \]

For the straight portion, 

\[ M = M_1 = M_0 - \frac{PR}{2} \]

Substituting in (3), we get

\[ M_0 \theta - \frac{PR}{2} \sin \theta - \frac{PR}{2} \left( M_0 - \frac{PR}{2} \right) \frac{L}{2} = 0 \]

\[ M_0 \left( R^2 + L \right) - \frac{PR}{2} \left( L + 2R \right) = 0 \]

or

\[ M_0 = \frac{PR}{2} \frac{L + 2R}{L + \pi R} \]...(17-47)

Substituting in (1), the moment at any section in the curved portion is given by

\[ M = M_0 - \frac{PR}{2} \sin \theta = \frac{PR}{2} \left( \frac{L + 2R}{L + \pi R} - \frac{PR}{2} \sin \theta \right) \]

\[ = \frac{PR}{2} \left[ \frac{L + 2R}{L + \pi R} \sin \theta \right] \]...(17-48)
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(a) Section AA
Stress at intrados is given by Eq. 17.52,
\[ f_i = \frac{P}{2A} \left[ \frac{L+2R}{L+\pi R} \right] \left[ 1 - \frac{1}{m} \frac{d_1}{R+d_1} \right] \]
\[ = \frac{100000}{2 \times 5027} \left[ \frac{80+160}{80+80\pi} \right] \left[ 1 - \frac{14 \times 40}{80-40} \right] = 93.7 \text{ N/mm}^2 \] (compressive)

The stress at extrados is given by Eq. 17.52,
\[ f_e = \frac{P}{2A} \left[ \frac{L+2R}{L+\pi R} \right] \left[ 1 + \frac{1}{m} \frac{d_1}{R+d_1} \right] \]
\[ = \frac{100000}{2 \times 5027} \left[ \frac{80+160}{80+80\pi} \right] \left[ 1 + \frac{14 \times 40}{80+40} \right] = 40.8 \text{ N/mm}^2 \].

(b) Section BB (curved portion)
Stress at the intrados is given by Eq. 17.55,
\[ f_i = \frac{P}{2A} \left[ \frac{L+2R}{L+\pi R} \right] \left[ 1 - \frac{1}{m} \frac{d_1}{R+d_1} \right] \]
\[ = \frac{100000}{2 \times 5027} \frac{100000 \times 80}{14 \times 40} = 93.7 + 35.64 \text{ N/mm}^2 \] (tensile)

Stress at the extrados is given by Fig. 17.56,
\[ f_e = \frac{P}{2A} \left[ \frac{L+2R}{L+\pi R} \right] \left[ 1 + \frac{1}{m} \frac{d_1}{R+d_1} \right] \]
\[ = \frac{100000}{2 \times 5027} \frac{100000 \times 80}{80+80\pi} \left[ 1 + \frac{14 \times 40}{80+40} \right] = 9.96 - 15.54 = -5.58 \text{ N/mm}^2 \] (compressive).

(c) Straight Portion
The stress at the intrados is given by Eq. 17.58.
\[ f_i = \frac{P}{2A} \left[ \frac{L+2R}{L+\pi R} \right] \left[ 1 - \frac{1}{m} \frac{d_1}{R+d_1} \right] \]

Here \( I = \frac{\pi}{4} r^4 = \frac{\pi}{4} (40)^4 = 201 \times 10^6 \text{ mm}^4 \)
\[ f_i = \frac{100000}{2 \times 5027} \frac{100000 \times 80}{80+80\pi} \left[ 1 + \frac{14 \times 40}{80+40} \right] \]
\[ = 9.96 + 21.94 = 31.9 \text{ N/mm}^2 \] (tensile).
The stress at the extrados is given by Eq. 17.59,

\[ f_x = \frac{P}{2A} - \frac{PR^2}{2I} \left[ \frac{\pi - 2}{L + \pi R} \right] d, \]

where \( f_x \) is the stress at the extrados, \( P \) is the load, \( A \) is the area, \( R \) is the radius, \( I \) is the moment of inertia, \( d \) is the thickness, and \( L \) is a constant.

Since \( d_1 = d_2 = 40 \) mm
\[ f_x = 9.96 - 21.94 = -11.98 \] N/mm\(^2\) (compressive)

Hence the maximum tensile stress in the link is 45.6 N/mm\(^2\), just at the junction of the curved and straight portion, while the maximum compressive stress is 93.7 N/mm\(^2\).

PROBLEMS

1. A curved beam, whose centre line is a circular arc of radius 60 mm, is formed of a tube of radius 20 mm outside and thickness

![Fig. 17-16](image)

Find the dimension \( b_1 \) and \( b_2 \) so that the maximum and minimum stresses developed in the section due to pure bending are numerically equal. Given: \( b_1 + b_2 = 50 \) mm.

3. Calculate the greatest tensile and compressive stresses in the hook shown in Fig. 17.17, if it carries a load \( P = 10 \) kN.

4. A ring with a mean radius of curvature of 25 mm is subjected to a load of 2000 N as shown in Fig. 17.18. The ring is made of circular section of 10 mm radius. Calculate the circumferential stress on the inside of the fibre of the ring at \( A \) and at \( B \).

ANSWERS

1. 65.6 N/mm\(^2\); -91.6 N/mm\(^2\).
2. \( b_1 = 36.7 \) mm; \( b_2 = 13.3 \) mm.
3. 184.2 N/mm\(^2\); -99 N/mm\(^2\).
4. \( f_x = -19.9 \) N/mm\(^2\),
   \( f_y = 29.1 \) N/mm\(^2\).

BENDING OF CURVED BARS

25 mm. Determine the greatest tensile and compressive stresses set up by a bending moment of 200 kN/mm tending to increase the curvature.
18

Stresses Due to Rotation

18.1. ROTATING RING OR WHEEL RIM

Stresses are set up in circular rings, wheel rims, circular discs and cylinders, etc. on account of rotation about their axis of symmetry. The analysis of the stresses set up in a rotating member such as pulleys, flywheels, etc. can be made on the basis of certain simplified assumptions. We shall take first the case of a thin ring rotating about an axis through its centre of gravity and perpendicular to its central plane.

![Diagram of a rotating ring](image)

Let \( r \) be the mean radius of the ring. The rotation will cause hoop stress (hoop tension) in it, due to its inertia. Assuming the cross-sectional dimensions to be small compared to the radius, the hoop tension will be nearly uniform.

Let

- \( \omega = \) Angular velocity in radians.
- \( v = \) Linear velocity.
- \( \omega = \frac{v}{r} \) \( \ldots(18.1) \)

Every point on the rim will have a radial inward acceleration equal to \( \omega^2 r \) or equal to \( \frac{v^2}{r} \). This inward radial acceleration will give rise to an inward radial force, known as centripetal force. The centripetal force is resisted by an equal and opposite force \( P \), known as centrifugal force, produced due to inertia of the ring. This centrifugal force acts like an internal pressure in a thin cylinder trying to burst it out into two halves.

Consider an element \( ABCD \), subtending an angle \( \delta \theta \) at the centre. Then the centrifugal force \( \delta P \) on the elementary volume \( ABCD \) is

\[
\delta P = \text{mass} \times \text{acceleration}
= \left( \frac{pr \cdot r \cdot I \cdot t}{g} \right) \frac{v^2}{r} = \frac{2}{g} \frac{lt v^2}{r} \delta \theta
\]

where
- \( p = \) unit weight of material
- \( g = \) acceleration due to gravity
- \( l = \) length of ring
- \( t = \) thickness of ring.

Let the ring burst about \( XX \).

\( \therefore \) Component of force perpendicular to \( XX \), trying to burst it

\[
= \frac{p}{g} \frac{lt v^2}{r} \delta \theta \sin \theta.
\]

\( \therefore \) Total force = integral of the above over \( \theta \)

\[
= \frac{p}{g} \frac{lt v^2}{r} \int_0^\pi \sin \theta \, d\theta = \frac{2p}{g} \frac{lt v^2}{r} \pi \ldots(1)
\]

Let \( f = \) hoop stress (tensile) produced in the ring to resist this bursting action.

Then total resisting force = stress \( \times \) total resisting area

\[
f = f(2lt) \ldots(2)
\]

Equating (1) and (2), we get

\[
f = \frac{2p}{g} \frac{lt v^2}{r} \ldots(18.2)
\]

or

\[
f = \frac{p}{g} \frac{v^2}{r} = \frac{p}{g} \omega^2 r^2
\]
Thus, the stress induced is independent of the thickness of the ring, and wholly depends upon the velocity. Hence increasing the section of pulleys, flywheels, etc. does not decrease the stress.

For a rim of a given material (i.e., given permissible value of \( f_s \)), the limiting velocity is given by

\[
\nu = \frac{f_s g}{p}
\]

where \( f_s \) = allowable stress.

For pulleys and wheel rims made of cast iron, the limiting speed is in the vicinity of 27 m/sec. Strong wheel rims, made channel shaped and wound round with high tensile steel wire, may have limiting speed as high as 80 m/sec.

**Example 18.1.** The rim of a steel flywheel 1 m diameter and 200 mm wide is rotating at 2400 revolutions per minute. Calculate (i) hoop stress developed, (ii) shrinkage in the width of the rim, and (iii) extension of the circumference due to rotation.

What will be the speed in R.P.M. at which the rim will burst?

**Take**

\[
E = 2 \times 10^5 \text{ N/mm}^2.
\]

**Poisson’s ratio**

\[
\nu = \frac{1}{3}
\]

**Ultimate stress for steel**

\[
= 400 \text{ N/mm}^2
\]

**Unit weight of steel**

\[
= 78.5 \times 10^{-4} \text{ N/mm}^2 (78.5 \text{ kN/m}^3).
\]

**Solution.**

(i) \[
\omega = \frac{2\pi}{60} = \frac{2 \times 2400}{60} = 80 \pi \text{ radians/sec.}
\]

\[
f = \frac{\sigma}{g} \omega^2 = \frac{78.5 \times 10^{-4}}{9810} (80)^2 (500)^2
\]

\[
= 126.4 \text{ N/mm}^2.
\]

(ii) In the above treatment, the effect of radial stress is neglected. Hence

\[
\text{Lateral strain} = -\frac{f}{E} = -\frac{1}{3} \times \frac{126.4}{2 \times 10^5} \text{ mm.}
\]

\[
= -2.1 \times 10^{-4} \text{ (i.e., compressive)}
\]

But lateral strain = \( \frac{\text{change in width}}{\text{original width}} \)

\[
:\text{Change in width} = 2.1 \times 10^{-4} \times 200 \text{ mm} = 0.042 \text{ mm.}
\]

(iii) Circumferential strains

\[
\frac{\sigma d}{E} = \frac{8d}{d} = 6.32 \times 10^{-4}
\]

or

\[
\sigma d = (6.32 \times 10^{-4})(\pi \times 1000) \text{ mm} = 1.98 \text{ mm.}
\]

Let \( N \) be the speed in r.p.m.

Then

\[
\frac{400}{126.4} = \left( \frac{N}{2400} \right)^2
\]

\[
\therefore \quad N = 2400 \sqrt{\frac{400}{126.4}} = 4270 \text{ r.p.m.}
\]

18.2. **ROTATING DISC**

Let us take the case of a circular disc rotating about its axis. It is assumed that the disc is of uniform thickness and that the thickness is so small compared with its diameter that there is no variation of stress along the thickness. At the free flat surfaces there can be no stress normal to these faces and there can be no shear stress on or perpendicular to these faces. Thus the direction of axis is the direction of zero principal stress. The displacement of any point due to strain must be radial. The radial and circumferential stresses, therefore, represent the principal stresses.

**Consider an element** \( A B C D \) of the disc, at a radius \( x \), subtending an angle \( d\theta \) at the centre, and of radial width \( dx \). Volume of element = \( x \cdot d\theta \cdot dx \cdot t \) (approx.)

\[
:\cdot \text{Mass of element} = \frac{\rho}{g} \cdot x \cdot dx \cdot d\theta \cdot t
\]
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Centrifugal force on it, due to rotation is
\[ P = \text{mass} \times \text{acceleration} \]
\[ = \left( \frac{p}{g} \right) x dx \cdot \theta \cdot t \]
\[ = \frac{p}{g} x^2 \omega^2 t \, dx \, d\theta \] ... (1)

Let \( p_x \) = radial stress at radius \( x \)
\( p_x + dp_x \) = radial stress at radius \( (x + dx) \)
\( p_v \) = circumferential stress at radius \( x \).

Resolving the forces along the centre line of the element, we get

\[ p_x \cdot x \, d\theta \cdot t - (p_x + dp_x) \cdot (x + dx) \, d\theta \cdot t + 2p_v \cdot dx \cdot t \sin \frac{d\theta}{2} = P = 0 \]

Taking \( \sin \frac{d\theta}{2} = \frac{d\theta}{2} \), substituting the value of \( P \) from (1) and neglecting infinitesimal quantities of higher order, we get

\[ -p_x \cdot dx \cdot d\theta \cdot t - x \cdot d\theta \cdot dp_x + p_v \cdot dx \cdot t = 0 \]

or

\[ p_v = \frac{p}{g} x^2 \omega^2 + \frac{dx \cdot dp_x}{dx} \] ... (2)

or

\[ p_v = \frac{p}{g} x^2 \omega^2 + \frac{d}{dx} (x p_x) \] ... (2a)

and

\[ p_v - p_x = \frac{dx \cdot dp_x}{dx} + \frac{p}{g} x^2 \omega^2 \] ... (3)

The other relation between \( p_v \) and \( p_x \) can be obtained by the considerations of strains.

Let

\( u = \text{radial displacement of radius} \, OA \)

(i.e. \( x \) increased to \( x + u \))

\( u + du = \text{radial displacement of radius} \, OD \)

(i.e. width \( dx \) increased to \( dx + du \))

\( \therefore \) Circumferential strain at \( x \)

\[ = e_v = \frac{2\pi (x + u) - 2\pi x}{2\pi} = \frac{u}{x} \]

But this is equal to \( \frac{p_v}{E} - \frac{p_x}{mE} \)

\[ \therefore \frac{u}{x} = \frac{p_v}{E} - \frac{p_x}{mE} \] ... (4)

STRESSES DUE TO ROTATION

Also, Radial strain at \( x \)

\[ = e_x = \frac{(dx + du) - dx}{dx} = \frac{du}{dx} \]

But this is equal to \( \frac{p_x}{E} - \frac{1}{m} \frac{p_v}{E} \)

\[ \therefore \frac{du}{dx} = \frac{p_x}{E} - \frac{1}{m} \frac{p_v}{E} \] ... (5)

From (4),

\[ u = \frac{x}{E} \left( p_x - \frac{p_v}{m} \right) \]

Differentiating,

\[ \frac{du}{dx} = \frac{1}{E} \left( p_v - \frac{p_x}{m} \right) + \frac{x}{E} \left[ \frac{dp_v}{dx} - \frac{1}{m} \frac{dp_x}{dx} \right] \]

Substituting this value of \( \frac{du}{dx} \) in (5), we get

\[ \frac{1}{E} \left( p_v - \frac{p_x}{m} \right) + \frac{x}{E} \left[ \frac{dp_v}{dx} - \frac{1}{m} \frac{dp_x}{dx} \right] = \frac{p_x}{E} - \frac{1}{m} \frac{p_v}{E} \]

or

\[ p_x \left( 1 + \frac{1}{m} \right) - p_x \left( 1 + \frac{1}{m} \right) + x \left[ \frac{dp_v}{dx} - \frac{1}{m} \frac{dp_x}{dx} \right] = 0 \]

or

\[ \left( p_v - p_x \right) \left( 1 + \frac{1}{m} \right) + x \left[ \frac{dp_v}{dx} - \frac{1}{m} \frac{dp_x}{dx} \right] = 0 \]

Substituting the value of \( p_v - p_x \) from (3), we get

\[ \left( x \frac{dp_x}{dx} + \frac{p}{g} x^2 \omega^2 \right) \left( \frac{m + 1}{m} \right) + x \frac{dp_v}{dx} - \frac{x}{m} \frac{dp_x}{dx} = 0 \]

or

\[ \frac{dx \frac{dp_x}{dx}}{dx} + \frac{p}{g} x^2 \omega^2 + m + 1 \frac{dp_v}{dx} - \frac{x}{m} \frac{dp_x}{dx} = 0 \]

or

\[ x \frac{dp_x}{dx} \left\{ 1 - \frac{1}{m + 1} \right\} + \frac{mx}{m + 1} \frac{dp_v}{dx} + \frac{p}{g} x^2 \omega^2 = 0 \]

or

\[ \frac{dx \frac{dp_x}{dx}}{dx} + \frac{dp_v}{dx} + \frac{m + 1}{m} \frac{p x^2}{g} = 0 \]

\[ \frac{dx \left( p_x + p_v \right)}{dx} + \frac{m + 1}{m} \frac{p x^2}{g} = 0 \]

Integrating it, we get

\[ (p_x + p_v) + \frac{m + 1}{m} \frac{p x^2}{g} = \text{constant} = 2A \] (say) ... (6)

or

\[ p_x = 2A - p_v \frac{m + 1}{m} \frac{p x^2}{g} \]

... (7)
Substituting this value of \( pv \) in Eq. (3),
\[
2A - px = \frac{m+1}{m} \frac{\rho_0^3}{g} \frac{x^2}{2} - px = x \frac{dpx}{dx} + \frac{p}{g} \frac{\rho_0^3}{g} x^2
\]
or
\[
2px + x \frac{dpx}{dx} = 2A - \frac{m+1}{m} \frac{\rho_0^3}{g} \frac{x^2}{2} - \frac{\rho_0^3}{g} x^2
\]
or
\[
2px + x \frac{dpx}{dx} = 2Ax - \frac{\rho_0^3}{g} \frac{x^2}{2} \left( \frac{3m+1}{2m} \right)
\]
or
\[
\frac{d(x^2 px)}{dx} = 2Ax - \frac{3m+1}{2m} \frac{\rho_0^3}{g} x^2
\]
Integrating,
\[
x^2 px = Ax^2 - \frac{3m+1}{2m} \frac{\rho_0^3}{g} \frac{x^4}{4} + B
\]
\[
px = A - \frac{3m+1}{8m} \frac{\rho_0^3}{g} x^2 + \frac{B}{x^2}
\]
where \( B \) is another constant of integration.

Substituting the value of \( px \) in (7), we get
\[
py = \left( 2A - \frac{m+1}{m} \frac{\rho_0^3}{g} \frac{x^2}{2} \right) - \left[ A - \frac{3m+1}{8m} \frac{\rho_0^3}{g} x^2 + \frac{B}{x^2} \right]
\]
\[
= A - \frac{m+3}{8m} \frac{\rho_0^3}{g} x^2 - \frac{B}{x^2}
\]

To summarise we get the following expression for \( px \) and \( py \) (from 8 and 9):
\[
px = A + \frac{B}{x^2} - \frac{3m+1}{8m} \frac{\rho_0^3}{g} x^2
\]
\[
py = A - \frac{B}{x^2} - \frac{m+3}{8m} \frac{\rho_0^3}{g} x^2
\]

The constants \( A \) and \( B \) are to be evaluated from the various cases of boundary conditions. We will take the following common cases:

(i) Disc with a central hole
(ii) Solid disc.

18.3. DISC WITH A CENTRAL HOLE

Let us take a hollow disc, i.e., disc with a central hole with internal radius \( R_1 \) and external radius \( R_2 \) as shown in Fig. 18.2.
Substituting this value of \( x \) in Eq. 18’9,
\[
(px)_{\text{max}} = \frac{3m+1}{8m} \frac{\rho \omega^2}{g} \left( R_1^2 + R_2^2 - \frac{R_1 R_2}{R_1 + R_2} \right) - \frac{R_1 R_2}{R_1 + R_2}
\]
\[
= \frac{3m+1}{8m} \frac{\rho \omega^2}{g} (R_1^2 - R_1 R_2) \quad \ldots(18’12)
\]

Inspection of Eq. 18’10 shows that \( p_y \) goes on increasing as \( x \) decreases and hence \( p_y \) is maximum at \( x=R_1 \).

\[
(p_y)_{\text{max}} = \frac{\rho \omega^2}{8mg} \left[ 3(m+1)(R_1^2 + R_2^2) + (3m+1)R_2^3 - (m+3)R_1^2 \right]
\]
\[
= \frac{\rho \omega^2}{4mg} \left[ (3m+1)R_2^2 + (m-1)R_1^2 \right] \quad \ldots(18’13)
\]

The variations \( px \) and \( p_y \) with \( x \) are shown in Fig. 18’3.

![Stresses in a hollow disc](image)

Fig. 18’3. Variations of \( px \) and \( p_y \) in a hollow disc.

If \( R_1 \) is very small so that \( R_1^2 \) is negligible as compared to \( R_2^2 \), we get
\[
(px)_{\text{max}} = \frac{(3m+1)\rho \omega^2}{4mg} R_2^2 \quad \ldots(18’14)
\]

This is double the value of \((px)_{\text{max}}\) for a solid disc (see Eq. 18’19). Hence even a small hole in the disc doubles the value of maximum hoop tension.

Also, when \( R_1 \) approaches \( R_2 \), such that \( R_1 \approx R_2 \approx R \), we get
\[
(px)_{\text{max}} = \frac{\rho \omega^2}{4mg} \times 4mR = \frac{\rho \omega^2 R^2}{g} \quad \ldots(18’15)
\]

which is the same as Eq. 26’2 obtained for the case of a ring.

18’4. SOLID DISC

Let the solid disc have radius equal to \( R \).

Then \( R_2=R; R_1=0 \).

At \( x=0 \), \( px \) is having some finite value. But if \( x=0 \) is substituted in Eq. 18’4, it gives infinite value of \( px \). This suggests that the constant \( B=0 \), in Eq. 18’4.

Also, at \( x=R \), \( px=0 \)

\[
0=A - \frac{3m+1}{8m} \frac{\rho \omega^2}{g} R^2
\]
\[
A = \frac{3m+1}{8m} \frac{\rho \omega^2}{g} R^2 \quad \ldots(18’16)
\]

Substituting these values of \( A \) and \( B \), Eqs. 18’4 and 18’5, we get
\[
px = \frac{3m+1}{8m} \frac{\rho \omega^2}{g} R^2 - \frac{3m+1}{3m} \frac{\rho \omega^2}{g} x^2
\]
\[
= \frac{3m+1}{8m} \frac{\rho \omega^2}{g} (R^2 - x^2) \quad \ldots(18’17)
\]

and
\[
p_y = \frac{3m+1}{8m} \frac{\rho \omega^2}{g} R^2 - \frac{m+3}{8m} \frac{\rho \omega^2}{g} x^2
\]
\[
= \frac{\rho \omega^2}{8mg} \left[ (3m+1) R^2 - (m+3)x^2 \right] \quad \ldots(18’18)
\]

By inspection, \( px \) and \( p_y \) are maximum at \( x=0 \)

\[
(px)_{\text{max}} = (py)_{\text{max}} = \frac{(3m+1)\rho \omega^2}{8mg} R^2 \quad \ldots(18’19)
\]

Also at \( x=R \)

\[
p_y = \frac{\rho \omega^2}{8mg} \left[ (3m+1-m-3)R^2 \right] = \frac{(m-1)\rho \omega^2}{4mg} R^2 \quad \ldots(18’20)
\]

The variations of \( px \) and \( p_y \) are shown in Fig. 18’4.

![Stresses in a solid disc](image)

Fig. 18’4. Variations of \( px \) and \( p_y \) in a solid disc.
PERMISSIBLE SPEED OF A SOLID DISC

From Eq. 18.19 the maximum value of \( px \) and \( pv \) is given by
\[
(px)_{\text{max}} = (pv)_{\text{max}} = \frac{(3m+1)\omega^3 R^2}{8mg}
\]

We shall now apply various theories of failure to find the permissible speed \( \omega \). For the treatment that follows, \((pv)_{\text{max}}\) and \((px)_{\text{max}}\) will be designated by \( px \) and \( pv \) respectively, for simplicity.

1. Maximum principal stress theory

Let \( f = \) simple direct stress at elastic failure
\[
px \leq f
\]
\[
\frac{(3m+1)\omega^3 R^2}{8mg} \leq f
\]
From which the maximum speed is given by
\[
\omega = \frac{1}{R} \sqrt[4]{\frac{8mgf}{(3m+1)\rho}} \quad \text{...(18.21)}
\]

2. Maximum principal strain theory

\[
px - \frac{px}{m} \leq f
\]
But \( px = pv \)
\[
px \left( 1 - \frac{1}{m} \right) \leq f
\]
\[
\frac{(3m+1)\omega^3 R^2}{8mg} \cdot \frac{m-1}{m} \leq f
\]
From which
\[
\omega = m \frac{\sqrt{\frac{8gf}{(3m+1)(m-1)\rho}}}{R} \quad \text{...(18.22)}
\]

3. Maximum shear stress theory

\[
f \geq \frac{px + pv}{2} > px
\]
\[
\frac{(3m+1)\omega^3 R^2}{8mg} \leq f
\]
From which
\[
\omega = \frac{1}{R} \sqrt[4]{\frac{8mgf}{(3m+1)\rho}} \quad \text{...(18.23)}
\]

4. Maximum strain energy theory

\[
f^2 \geq px^2 + pv^2 - \frac{2}{m} pxpv
\]
But \( px = pv \)

\[
f^2 \geq \frac{2px^2 \left( 1 - \frac{1}{m} \right)}{m} \leq f^2
\]
\[
\left[ \frac{(3m+1)\omega^3 R^2}{8mg} \right]^2 \leq f^2
\]
From which
\[
\omega = \frac{1}{R} \sqrt[4]{\frac{8mgf}{(3m+1)\rho}} \quad \text{...(18.24)}
\]

5. Maximum shear strain energy theory

\[
px^2 + pv^2 - pxpv \leq f^2
\]
But \( px = pv \)
\[
px \leq f
\]
\[
\frac{(3m+1)\omega^3 R^2}{8mg} \leq f
\]
From which
\[
\omega = \frac{1}{R} \sqrt[4]{\frac{8gf}{(3m+1)\rho}} \quad \text{...(18.25)}
\]

It is to be noted that Eqs. 18.21, 18.23 and 18.25 give the same value of \( \omega \).

Example 18.2. A flat steel disc of uniform thickness and of 1 m diameter rotates at 2400 r.p.m. Determine the intensities of principal stresses.

Take \( \rho = 7.85 \times 10^{-5} \) N/mm² and \( m = 3 \).

Solution.

The intensities of radial and hoop stresses are given by Eqs. 18.29 and 18.5.

\[
px = \frac{A}{x^3} - \frac{3m+1}{8m} \frac{\omega^3}{g} x^3 \quad \text{...(1)}
\]
\[
px = \frac{B}{x^3} - \frac{m+3}{8} \frac{\omega^3}{g} x^3 \quad \text{...(2)}
\]

Since \( px \) cannot have infinite value at \( x = 0 \), we get \( B = 0 \).

Also, at \( x = R, px = 0 \)
\[
0 = A - \frac{3m+1}{8m} \frac{\omega^3}{g} R^2
\]
\[
A = \frac{3m+1}{8m} \frac{\omega^3}{g} R^2
\]

Here \( m = 3; g = 9810 \text{ mm/sec}^2; \rho = 7.85 \times 10^{-5} \) N/mm²
\[
\omega = \frac{2\pi N}{60} = \frac{2\pi}{60} 2400 = 80\pi = 251 \text{ radians/sec}
\]
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\[ \frac{3m+1}{8m} \cdot \frac{P_0 \sigma}{g} = \frac{3 \cdot 3 + 1}{8 \cdot 3} \cdot \frac{785 \times 10^{-5}}{9810} \cdot (251)^2 = 0.21 \times 10^{-3} \] ...

\[ A = 0.21 \times 10^{-3} (500)^2 = 52.5 \] ...

Also, \[ \frac{m+3}{8m} \cdot \frac{P_0 \sigma}{g} = \frac{3+3}{8 \cdot 3} \cdot \frac{785 \times 10^{-5}}{9810} \cdot (251)^2 = 0.126 \times 10^{-3} \] ...

Substituting the values in (1) and (2), we get

\[ P_0 \sigma = 52.5 - 0.21 \times 10^{-3} \cdot x^2 \] ...

At \( x = 0 \), \( P_0 \sigma = 52.5 \) N/mm²

At \( x = R = 500 \),

\[ P_0 \sigma = 52.5 - 0.21 \times 10^{-3} (500)^2 = 0 \]

and

\[ P_0 \sigma = 52.5 - 0.126 \times 10^{-3} (500)^2 = -21 \] N/mm²

It should be noted that the above values of \( P_0 \sigma \) and \( P_0 \sigma \) are also the principal stresses.

Example 18'3. Solve Example 18'2 if the disc has a central hole 200 mm diameter.

Solution.

\[ R_1 = 100 \text{ mm}, \ R_2 = 500 \text{ mm} \]

The stress intensities are given by

\[ P_0 \sigma = A + \frac{B}{x^2} - \frac{3m+1}{8m} \cdot \frac{P_0 \sigma}{g} \cdot x^2 \]

and

\[ P_0 \sigma = A - \frac{B}{x^2} - \frac{m+3}{8m} \cdot \frac{P_0 \sigma}{g} \cdot x^2 \]

where \( \frac{3m+1}{8m} \cdot \frac{P_0 \sigma}{g} = 0.21 \times 10^{-3} \) and \( \frac{m+3}{8m} \cdot \frac{P_0 \sigma}{g} = 0.126 \times 10^{-3} \) from the previous example.

At \( x = 100 \), \( P_0 \sigma = 0 \)

\[ 0 = A + \frac{B}{10000} - 0.21 \times 10^{-3} (10000) \]

or

\[ 10000 \ A + B = 0.21 \times 10^5 \]

At \( x = 500 \), \( P_0 \sigma = 0 \)

Also, \( x = 500 \), \( P_0 \sigma = 0 \)

\[ 0 = A + \frac{B}{250000} - 0.21 \times 10^{-3} (250000) \]

\[ 250000 + B = 131.25 \times 10^5 \]

From (3) and (4), we get

\[ A = 54.6 \text{ and } B = 52.5 \times 10^4 \]

Hence the stresses are given by

\[ P_0 \sigma = 54.6 + \frac{52.5 \times 10^4}{x^2} - 0.21 \times 10^{-3} \cdot x^2 \]

\[ P_0 \sigma = 54.6 + \frac{52.5 \times 10^4}{x^2} \]

At \( x = 100 \), \( P_0 \sigma \) (max) \( = 54.6 + 52.5 \times 2.1 \)

\[ = 105 \text{ N/mm}^2 \]

At \( x = 500 \), \( P_0 \sigma \) (max) \( = 54.6 + \frac{52.5 \times 10^4}{250000} = 0.21 \times 10^{-3} (250000) \)

\[ = 54.6 + 2.1 - 52.5 = 4.2 \text{ N/mm}^2 \]

\( P_0 \sigma \) is maximum at \( x = \sqrt{\frac{R_1 R_2}{100 \times 500}} = 223.6 \text{ mm.} \)

\( (P_0 \sigma) \) (max) \( = 54.6 - \frac{52.5 \times 10^4}{223.6^2} - 0.21 \times 10^{-3} (2236)^2 \)

\[ = 54.6 - 105 = 33.6 \text{ N/mm}^2. \]

18'6. DISC OF UNIFORM STRENGTH

A disc of uniform strength is the one in which the values of radial and circumferential stresses are equal in magnitude for all values \( x \). This suggests that the disc of uniform strength must have a varying thickness, such as shown in Fig. 18'5.

![Fig. 18'5. Disc of uniform strength.](image)

Let \( z \) = thickness at a radial distance \( x \).
Consider an element \( ABCD \) as before.
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Centrifugal force \( P = \left( \frac{r}{g} \right) x \frac{dx}{d\theta} \omega^2 \sin^2 \theta \) \( \cdots (1) \)

Resolving the forces radially, we get

\( p_x d\theta z = (p_x + dp_x)(x + dx) d\theta (z + dz) + 2p_x dx \cdot \sin \frac{\theta}{2} = 0 \)

Taking \( \sin \frac{\theta}{2} = \frac{d\theta}{2} \), substituting the value of \( P \) from (1) and neglecting infinitesimal quantities of higher order, we get

\[ z \frac{p_x}{g} = \frac{r}{g} x^2 \omega^2 z + \frac{d}{dx} (z, x, p_x) \quad \cdots (18'26) \]

Let \( p_x = pv = p \) (constant)

\[ zp = \frac{r}{g} x^2 \omega^2 z + \frac{d}{dx} (x) \]

\[ zp = \frac{r}{g} x^2 \omega^2 z + px + p_z \frac{dz}{dx} \]

or

\[ \frac{dz}{dx} + \frac{p_x}{g} \omega^2 z = 0 \]

or

\[ \frac{dz}{z} + \frac{p_x}{g} \omega^2 \frac{dx}{x} = 0 \]

Integrating, \( \log z = -\frac{p x^2}{8p} \frac{x^2}{2} + \log a \)

where \( \log a \) is a constant of integration.

At \( x = 0 \), \( z = z_0 \)

\[ \log a = 0 + \log z_0 \]

Substituting in (2), we get

\[ \log z = -\frac{p x^2}{8p} \frac{x^2}{2} + \log z_0 \]

or

\[ \log \frac{z}{z_0} = -\frac{p x^2}{8p} \frac{x^2}{2} \]

or

\[ \frac{z}{z_0} = e^{-\frac{p x^2}{8p} \frac{x^2}{2}} \]

or

\[ \frac{z}{z_0} = e^{-\frac{p x^2}{8p} \frac{x^2}{2}} \]

Eq. 18'27 gives the variation of thickness \( z \).

18.7. ROTATING CYLINDER

A cylinder may be defined as a disc of large thickness. Thus the thickness \( t \) corresponds to the length of the cylinder. Let us assume that the length of the axis is great compared to the radius. We shall confine ourselves to the stresses about the region of the central circular section perpendicular to the axis of the cylinder. It is further assumed that the plane sections of the cylinder, when stationary, remain plane even during rotation, i.e., the axial strain \( e_x \) is constant and independent of \( x \). Let the direction of \( x \) be along a radius, direction of \( z \) be along the axis and the direction \( y \) be perpendicular to the two. Let \( p_x, p_y \) and \( p_z \) be the normal stresses in \( x, y \) and \( z \) directions respectively. If we take an element \( ABCD \) (Fig. 18'2) of thickness \( dz \), at the centre of the cylinder axis, there will be no shear stress due to symmetry. Hence the stresses \( p_x, p_y \) and \( p_z \) are the principle stresses.

As in Eq. 2 of § 18'2 the equation for the forces acting radially will be

\[ p_x = \frac{r}{g} \omega^2 x^2 = p_x + \frac{dp_x}{dx} \]

Also, as before, \( e_x = \frac{dx}{dx} \) and \( e_y = \frac{u}{x} \)

Hence we have

Radial strain, \( e_x = \frac{du}{dx} = \frac{1}{E} (p_x - p_y + p_z) \) \( \cdots (2) \)

Circumferential strain,

\[ e_y = \frac{u}{x} = \frac{1}{E} \left( p_y - p_x + p_z \right) \] \( \cdots (3) \)

Axial strain, \( e_z = \frac{1}{E} \left( p_z - p_x + p_y \right) \) \( \cdots (4) \)

If it is assumed that plane sections remain plane, \( e_z \) is constant with respect to \( x \). From (4),

\[ p_z = p_x + p_y + e_x \]

\[ \frac{dp_z}{dx} = \frac{1}{m} \left[ \frac{pdx}{dx} + \frac{dpv}{dx} \right] \] \( \cdots (6) \)
Now, from (3),
\[ uE = x \left( \frac{p_x - px + pz}{m} \right) = xp_x - \frac{x}{m} \left( \frac{dx}{dx} + np_x \right) - \frac{x}{m} \left( \frac{dx}{dx} + np_z \right) \]

\[ E \frac{du}{dx} = p_x + x \left( \frac{dp_x}{dx} - \frac{1}{m} \left( \frac{x}{x} + np_x \right) \right) \]

Substituting the value of \( E \frac{du}{dx} \) from (2), we get
\[ p_x - \frac{px}{m} = px + x \left( \frac{dp_x}{dx} - \frac{1}{m} \left( \frac{x}{x} + np_x \right) \right) \]

or \( (p_x - px) \left( 1 + \frac{1}{m} \right) = x \left[ \frac{dp_x}{dx} - \frac{1}{m} \left( \frac{dp_x}{dx} + \frac{dp_z}{dx} \right) \right] \]

Substituting the value of \( \frac{dp_x}{dx} \) from (6), we get
\[ (p_x - px) \left( 1 + \frac{1}{m} \right) = x \left[ \frac{dp_x}{dx} - \frac{1}{m} \left( \frac{dp_x}{dx} + \frac{dp_x}{dx} \right) \right] \]

or \( (p_x - px) \left( 1 + \frac{1}{m} \right) = x \left[ \frac{dp_x}{dx} \right] \left( 1 - \frac{1}{m} \right) - \frac{1}{m} \left( \frac{dp_x}{dx} \right) \left( 1 + \frac{1}{m} \right) \]

\[ p_x - px = x \left[ \frac{dp_x}{dx} \left( 1 - \frac{1}{m} \right) - \frac{1}{m} \left( \frac{dp_x}{dx} \right) \right] \]

Substituting the value of \( p_x - px \) from (1), we get
\[ x^2 \frac{dp_x}{dx} = \frac{p_x}{g} \omega^2 x^3 = x \left[ \frac{dp_x}{dx} \left( 1 - \frac{1}{m} \right) - \frac{1}{m} \left( \frac{dp_x}{dx} \right) \right] \]

\[ \frac{dp_x}{dx} \left( 1 - \frac{1}{m} \right) + \frac{dp_x}{dx} \left( 1 - \frac{1}{m} \right) = \frac{p_x}{g} \omega^2 x \]

Hence \( \frac{dp_x}{dx} + \frac{dp_x}{dx} = \frac{p_x}{g} \omega^2 x \)

Integrating,
\[ p_x + pv = \frac{p_x}{g} x^\frac{m}{m+1} + 2A \]

where \( 2A = \text{constant of integration} \).

Also, from (1), \( p_x - pv = \frac{p_x}{g} \omega^2 x^3 - \int x \frac{dp_x}{dx} + 2A \)

Adding this to (8), we get
\[ 2x^3 + x^2 \frac{dp_x}{dx} = 2Ax + \frac{p_x}{g} \omega^2 x^3 \left( \frac{m^2 - 2}{m-1} \right) \]

Integrating,
\[ x^2 p_x = Ax^2 - \frac{p_x}{g} \omega^2 x^3 \left( \frac{m^2 - 2}{m-1} \right) + B \]

\[ p_x = A + \frac{B}{x^2} - \frac{p_x}{g} \omega^2 x^3 \left( \frac{m^2 - 2}{m-1} \right) \]

Substituting this in (8), we get
\[ p_x = \left[ - \frac{p_x}{g} \frac{m}{m-1} \omega^2 x^3 \right] - \left[ A + \frac{B}{x^2} - \frac{p_x}{g} \omega^2 x^3 \left( \frac{m^2 - 2}{m-1} \right) \right] \]

\[ = A - \frac{B}{x^2} \omega^2 x^3 \left( \frac{m^2 + 2}{m-1} \right) \]

To summarize, the values of \( p_x, \ p_x \) and \( p_z \) are given by the following equations:

\[ p_x = A + \frac{B}{x^2} - \frac{p_x}{g} \omega^2 x^3 \left( \frac{m^2 - 2}{m-1} \right) \]

\[ p_x = A - \frac{B}{x^2} \omega^2 x^3 \left( \frac{m^2 + 2}{m-1} \right) \]

\[ p_z = \frac{p_x + pv}{m} + E Z \]

Thus, to find the values of \( p_x, \ p_x \) and \( p_z \) we have to first determine the constants \( A, B \) and \( E Z \), depending upon the boundary conditions. Two cases will be considered: (1) hollow cylinder, (2) solid cylinder.

18.8. HOLLOW CYLINDER

Let \( R_i = \text{internal radius} \)
\( R_o = \text{external radius of the hollow cylinder} \).

Boundary conditions are:
\( p_x = 0 \) at \( x = R_i \) and also at \( x = R_o \).

Substituting these in Eq. 18.28, we get
\[ A + \frac{B}{R_i^2} - \frac{p_x}{g} R_i^3 \left( \frac{m^2 - 2}{m-1} \right) = 0 \]

\[ A + \frac{B}{R_o^2} - \frac{p_x}{g} R_o^3 \left( \frac{m^2 - 2}{m-1} \right) = 0 \]

Substituting (2) from (1),
\[ B \left( \frac{R_i^3 - R_o^3}{R_i - R_o} \right) = \frac{p_x}{g} \omega^2 \left( \frac{m^2 - 2}{m-1} \right) \left( R_i - R_o \right) \]
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\[ B = -\frac{\rho a^2}{8g} \cdot \frac{3m-2}{m-1} R_1^2 R_4^2 \] \[ \text{[18'3] (a)} \]

Substituting this value of \( B \) in (1), we get

\[ A = \frac{\rho a^2}{8g} \cdot \frac{3m-2}{m-1} R_2^2 + \frac{\rho a^2}{8g} \cdot \frac{3m-2}{m-1} R_4^2 \]

or

\[ A = \frac{\rho a^2}{8g} \cdot \frac{3m-2}{m-1} \left( R_2^2 + R_4^2 \right) \] \[ \text{[18'3] (b)} \]

Substituting the values of \( A \) and \( B \) in Eq. 18'28, we get

\[ p_x = \frac{\rho a^2}{8g} \left( \frac{3m-2}{m-1} \right) \left( R_1^2 + R_4^2 \right) - \frac{\rho a^2}{8g} \left( \frac{3m-2}{m-1} \right) \left( R_1^2 R_4^2 \right) - \frac{\rho a^2}{8g} \left( \frac{3m-2}{m-1} \right) \]

\[ = \frac{\rho a^2}{8g} \left( \frac{3m-2}{m-1} \right) \left[ R_1^2 + R_4^2 - R_1 R_4 \left( \frac{R_1^2 R_4^2}{\left( \frac{R_1^2 + R_4^2}{2} \right)^2} \right) \right] \] \[ \text{[18'32]} \]

Similarly, substituting the values of \( A \) and \( B \) in Eq. 18'32, we get

\[ p_x = \frac{\rho a^2}{8g} \left( \frac{3m-2}{m-1} \right) \left( R_2^2 + R_4^2 \right) + \frac{\rho a^2}{8g} \left( \frac{3m-2}{m-1} \right) \left( R_2 R_4 \right) \]

\[ = \frac{\rho a^2}{8g} \left( \frac{3m-2}{m-1} \right) \left[ R_2^2 + R_4^2 + R_2 R_4 \left( \frac{R_2 R_4}{\left( \frac{R_2^2 + R_4^2}{2} \right)} \right) \right] \] \[ \text{[18'33]} \]

For maximum value of \( p_x \), \( \frac{dp_x}{dx} = 0 \). This gives, from Eq. 18'32,

\[ \frac{2R_1^2 R_4^2}{x^2} \left[ \frac{3m-2}{m-1} \right] - 2x = 0 \]

or

\[ x = \frac{R_1 R_4}{R_1 - R_4} \] \[ \text{[18'34]} \]

\[ (px)_{max} = \frac{\rho a^2}{8g} \left( \frac{3m-2}{m-1} \right) \left[ R_2^2 + R_4^2 - R_1 R_2 \right] \]

\[ = \frac{\rho a^2}{8g} \left( \frac{3m-2}{m-1} \right) \left( R_2 - R_1 \right)^2 \] \[ \text{[18'35]} \]

By inspection of Eq. 18'23, \( p_y \) will be maximum at minimum value of \( x \), i.e. at \( x = R_1 \).

\[ (p_y)_{max} = \frac{\rho a^2}{8g} \left[ \frac{3m-2}{m-1} \right] \left( R_2 + R_4 \right) - R_1 \left( \frac{m+2}{m-1} \right) \]

\[ = \frac{\rho a^2}{4g(m-1)} \left[ (3m-2)R_3^2 + (m-2)R_1^2 \right] \] \[ \text{[18'36]} \]

STRESSES DUE TO ROTATION

After having known the values of \( p_y \) and \( p_x \), let us now compute the axial stress \( p_z \) from Eq. 18'30 in which the constant \( E ez \) is still to be computed.

Consider a hollow cylinder of internal radius \( R_1 \) and external radius \( R_2 \). Take an elementary ring of radius \( x \) and thickness \( dx \) as shown in Fig. 18'6.

Axial force on elementary ring = \( 2\pi x dx \), \( \rho a \)

\[ \therefore \text{Total axial force} = F_z = \int_{R_1}^{R_2} 2\pi x dx \cdot p_z \]

Substituting the value of \( p_z \) from Eq. 18'30,

\[ F_z = \int_{R_1}^{R_2} 2\pi x dx \left\{ \frac{p_x + p_y}{m} + E ez \right\} \]

Substituting the value of \( p_x \) and \( p_y \) from Eqs. 18'28 and 18'29, we get

\[ F_z = 2\pi \int_{R_1}^{R_2} x dx \left\{ E ez + \frac{1}{m} \left( A + \frac{B}{x^2} - \frac{\rho a^2 x^2}{8g} \left( \frac{3m-2}{m-1} \right) \right) \right. \]

\[ + \left. A - \frac{B}{x^2} - \frac{\rho a^2 x^2}{8g} \left( \frac{m+2}{m-1} \right) \right\} \]

\[ = 2\pi \int_{R_1}^{R_2} x dx \left( E ez + \frac{1}{m} \left( 2A - \frac{\rho a^2 x^2}{8g} \left( m-1 \right) \right) \left( \frac{3m-2}{m-1} \right) \right) \]

or

\[ F_z = 2\pi \int_{R_1}^{R_2} x dx \left( E ez + \frac{1}{m} \left( 2A - \frac{\rho a^2 x^2}{2g(m-1)} \right) \right) \]

\[ = 2\pi \int_{R_1}^{R_2} x dx \left( E ez + \frac{1}{m} \left( 2A - \frac{\rho a^2 x^2}{2g(m-1)} \right) \right) \]

\[ = 2\pi \int_{R_1}^{R_2} x dx \left( E ez + \frac{1}{m} \left( 2A - \frac{\rho a^2 x^2}{2g(m-1)} \right) \right) \]

\[ \therefore \int_{R_1}^{R_2} x dx \cdot E ez + \int_{R_1}^{R_2} \frac{R_2 A x dx}{m} \frac{R_2}{R_1} \frac{p_x x^2}{8g(m-1)} = 0 \]

\[ \therefore \frac{E ez \cdot x^2}{2} + \frac{A x^2}{m} - \frac{\rho a^2 x^4}{8g(m-1)} R_1 = 0 \]
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\[ E_{ez} = \frac{\rho_0^2}{2g(m-1)} \left( R_t^2 + R_s^2 \right) - \frac{2A}{m} \]

Substituting the value of \( A \) from Eq. 26'31 (b),

\[ E_{ez} = \frac{\rho_0^2}{4g(m-1)} \left( R_t^2 + R_s^2 \right) - \frac{3m-2}{8g(m-1)} \left( R_t^2 + R_s^2 \right) \]

\[ = \frac{\rho_0^2}{4g(m-1)} \left( R_t^2 + R_s^2 \right) \left( 1 - \frac{3m-2}{m} \right) \]

\[ = -\frac{\rho_0^2}{2gm} \left( R_t^2 + R_s^2 \right) \]

Substituting the values of \( E_{ez} \) (Eq. 18'38), \( p_x \) (Eq. 18'32) and \( p_x \) (Eq. 18'33) in Eq. 18'30, we get

\[ p_z = \frac{p_x + p_y}{m} + E_{ez} \]

or

\[ p_z = \frac{1}{m} \left[ \frac{\rho_0^2}{4g} \left( \frac{3m-2}{m-1} \right) \left( R_t^2 + R_s^2 \right) - \frac{8g}{m-1} \left( 3m-2 + m + 2 \right) \right] \]

\[ = \frac{\rho_0^2}{2gm} \left[ R_t^2 + R_s^2 \right] - \frac{8g}{m-1} \left( 3m-2 + m + 2 \right) \]

By inspection, \( p_z \) is maximum at \( x = R_t \)

\[ \therefore \quad \left( p_z \right)_{max} = \frac{\rho_0^2}{2gm} \left( R_t^2 - R_s^2 \right) \]

18.9. SOLID CYLINDER \((R_t=0)\)

The values of \( p_x, p_y \) and \( p_z \) are given by Eqs. 18'28, 18'29 and 18'30 respectively. At \( x=0 \), these give infinite values of \( p_x, p_y \) and \( p_z \). Since the stresses are finite, we get \( B=0 \).

To get the value of \( A \), we apply the condition that \( p_x = 0 \) at \( x=R_s=R \) (say).

Hence from Eq. 18'28,

\[ p_x = 0 = A - \frac{\rho_0^2 R^2}{8g} \left( \frac{3m-2}{m-1} \right) \]

\[ A = \frac{\rho_0^2 R^2}{8g} \left( \frac{3m-2}{m-1} \right) \]

Substituting this in Eq. 18'28

\[ p_x = \frac{\rho_0^2 R^2}{8g} \left( \frac{3m-2}{m-1} \right) - \frac{\rho_0^2 x^2}{8g} \left( \frac{3m-2}{m-1} \right) \]

\[ = \frac{\rho_0^2}{8g} \left( \frac{3m-2}{m-1} \right) \left( R^2 - x^2 \right) \]

This will be evidently maximum at the centre of the cylinder.

\[ \therefore \quad \left( p_x \right)_{max} = \frac{\rho_0^2 R^2}{8g} \left( \frac{3m-2}{m-1} \right) \]

Also, substituting the value of \( A \) in Eq. 18'29, we get

\[ p_y = \frac{\rho_0^2 R^2}{8g} \left( \frac{3m-2}{m-1} \right) - \frac{\rho_0^2 x^2}{8g} \left( \frac{m+2}{m} \right) \]

\[ = -\frac{\rho_0^2}{8g(m-1)} \left[ (3m-2) R^2 - (m+2) x^2 \right] \]

This is maximum at \( x=0 \)

\[ \therefore \quad \left( p_y \right)_{max} = \frac{\rho_0^2 R^2}{8g} \left( \frac{3m-2}{m-1} \right) \]

Thus at the centre of the cylinder, \( p_y = p_x \).

To get the value of \( p_z \), consider the axial force on an elementary ring of radius \( x \) and thickness \( dx \) (Fig. 18'6). We have, similar to Eq. 18'37.

\[ 2n \int_0^R x \cdot dx \cdot \left[ E_{ez} + \frac{1}{m} \left( 3m-2 \right) \frac{\rho_0^2 x^2 m}{2g(m-1)} \right] = 0 \]

\[ \therefore \quad \int_0^R x \cdot dx \cdot E_{ez} + \frac{R}{m} \rho_0^2 x^2 \cdot \frac{2A}{2g(m-1)} \cdot \frac{R}{m} \rho_0^2 x^2 \cdot \frac{2}{2g(m-1)} \cdot dx = 0 \]

\[ E_{ez} = \frac{\rho_0^2 R^2}{2gm} \left( \frac{3m-2}{m-1} \right) \]

Substituting the value of \( A \) from Eq. 18'41,

\[ e_{ez} = \frac{\rho_0^2 R^2}{4g(m-1)} - \frac{\rho_0^2 R^2}{4g(m-1)} \left( \frac{3m-2}{m} \right) \]

\[ = -\frac{\rho_0^2 R^2}{2gm} \left( 1 - \frac{3m-2}{m} \right) \]

Substituting the value of \( E_{ez} \) (Eq. 18'46), \( p_x \) (Eq. 18'42) and \( p_y \) (Eq. 18'43) in Eq. 18'30, we get

\[ p_z = \frac{1}{m} \left[ \frac{\rho_0^2}{8g} \left( \frac{3m-2}{m-1} \right) - \frac{\rho_0^2}{8g(m-1)} \left( 3m-2 \right) R^2 \right] \]

\[ = -\frac{\rho_0^2 R^2}{2gm} \left( \frac{3m-2}{m-1} \right) \left( R^2 - x^2 \right) \]
19. Vibrations and Critical Speeds

19'1. INTRODUCTION

If a body, held in equilibrium by elastic constraints, is momentarily disturbed from its equilibrium position, it begins to vibrate. The nature of the vibration depends upon several factors such as inertia of the system, stiffness, elastic forces and the amount of disturbance. When the external forces displaces the body from its equilibrium position, work is done against the external elastic forces resisting deformation. This work is stored momentarily as the strain energy in constraints. When the external force is removed, the body tries to regain its original equilibrium position and thus changes the strain energy into kinetic energy. The body thus vibrates. If the external force does not take part in vibrations except for the initial momentary displacement, the vibrations are called free or natural vibrations. The vibrations continue indefinitely. However, some frictional forces are always present which gradually damp the vibrations. If, however, the external disturbing force is periodically acting on the body, vibrations having the same frequency as that of the disturbing force will be set up. Such vibrations are called forced vibrations.

The vibrations may further be classified as (i) linear and (ii) angular or torsional. Linear vibrations are either longitudinal or transverse.

19'2. LINEAR VIBRATIONS: SIMPLE HARMONIC MOTION

Consider a point P moving along the circumference of a circle, with a constant angular velocity \( \omega \) (Fig. 19'1). Consider a diameter \( AB \) of the circle. Let C be the projection of point P on the line \( AB \). As the point P rotates, point C will oscillate or vibrate between A and B. The motion of C along \( AB \) is called simple harmonic motion.
Let
\[ r = \text{radius of the circle} \]
\[ x = O C = \text{distance of the point } C \text{ from } O = \text{Displacement of } C. \]
\[ x = r \cos \phi \]

![Figure 19.1. Simple Harmonic Motion.](image)

If the time is measured from the instant when \( P \) is at \( B \), \( \phi = \omega t \)
\[ x = r \cos \omega t \]
\[ v = -\frac{dx}{dt} = r\omega \sin \omega t \]
(The minus sign indicates that \( v \) increases as \( x \) decreases)
\[ \therefore \text{ Acceleration } f = \frac{dv}{dt} = r\omega^2 \cos \omega t \]
\[ f = \omega^2 x \quad \text{(19'1)} \]

Thus acceleration of point \( C \) is proportional to its distance or displacement from the mean position (fixed point) \( O \).

From Eq. 19'1,
\[ \omega^2 = \frac{\text{acceleration of } C}{\text{displacement of } C \text{ from the centre}} = \frac{f}{x} \]
\[ \omega = \sqrt{\frac{f}{x}} \quad \text{(19'2 a)} \]

or
\[ \omega = \frac{\sqrt{f}}{x} \quad \text{(19'2 b)} \]

Let \( T = \text{time of one complete 'to and fro' vibration in seconds} \)
\[ \omega = \frac{2\pi}{T} \]
\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{x}{f}} \quad \text{(19'3 a)} \]

or
\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{\text{displacement}}{\text{acceleration}}} = 2\pi \sqrt{\frac{W}{g}} \quad \text{(19'3 b)} \]

19'3. LONGITUDINAL VIBRATIONS
(a) HELICAL SPRING [Fig. 19.2 (a)]

Consider a helical spring [Fig. 19.2 (a)] supporting a weight \( W \). If the weight \( W \) is pulled downwards momentarily by an amount \( x \) and then released, it will have simple harmonic motion. Let \( \delta \) be the extension of the spring due to the static of load \( W \).

Then stiffness \( k = \frac{W}{\delta} \quad \text{(19'4)} \)

Substituting this in Eq. 19'4, we get
\[ n = \frac{1}{T} \sqrt{\frac{k}{W}} = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}} \quad \text{(19'5)} \]

Let
\[ k = \text{stiffness in N/mm} \]
\[ \delta = \text{static extension in mm} \]
\[ g = 9810 \text{ mm/sec/sec} \]
If, however, the stiffness $k$ is expressed in lb/inch, $\delta$ as the extension in inches and $g=32.2\times12$ inch/sec/sec, we get

$$n = \frac{1}{2\pi} \sqrt{\frac{32.2 \times 12}{k}} = 3113 \text{ vibrations per second} \quad \ldots [19'7 (c)]$$

and

$$N = 60n = \frac{187.8}{\sqrt{k}} \text{ vibrations/minute} \quad \ldots [19'7 (d)]$$

(b) **COMPOUND SPRINGS**

(i) Springs in Series [Fig. 19'2 (b)]

Let two springs of stiffness $k_1$ and $k_2$ be arranged in series, and carry a weight $W$ at its end, as shown in Fig. 19'2 (b). Let $\delta$ be the extension of the system, due to the static load $W$.

The stiffness $k$ of the whole system is

$$k = k_1 + k_2 \quad \ldots (i)$$

Then

$$\delta = \delta_1 + \delta_2 = \frac{W}{k_1} + \frac{W}{k_2}$$

$$\therefore \quad \frac{\delta}{k} = W \left( \frac{1}{k_1} + \frac{1}{k_2} \right) \quad \ldots (i)$$

If $k$ is the resultant stiffness of the whole system, we have

$$\delta = \frac{W}{k} \quad \ldots (ii)$$

Equating the two, we get

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

Substituting this in Eq. 27'4, we get

$$n = \frac{1}{2\pi} \sqrt{\frac{kg}{W}} = \sqrt{\frac{g}{W} \left( \frac{k_1k_2}{k_1+k_2} \right)} \text{ (per second)} \quad \ldots [19'7 (e)]$$

(ii) Springs in Parallel [Fig. 19'2 (c)]

If the two springs are arranged in parallel, one end of each spring being fixed and the other end independently connected to a mass of weight $W$, each spring is deformed by an amount $\delta$. Hence

$$W = k_1\delta + k_2\delta = \delta(k_1+k_2) \quad \ldots (i)$$

where $k_1\delta$ is the restoring force for the first spring and $k_2\delta$ is the restoring force for the other spring.

Also $W = k\delta \quad \ldots (ii)$

where $k$ is the stiffness of the composite system.

Equating the two, we get

$$k = k_1+k_2$$

Substituting this in Eq. 19'4, we get

$$n = \frac{1}{2\pi} \sqrt{\frac{kg}{W}} = \frac{1}{2\pi} \sqrt{\frac{g}{W} \left( \frac{k_1k_2}{k_1+k_2} \right)}$$

Note. $\delta =$ static deflection of the system under load $W$.

$x =$ initial displacement of the system to set it in simple harmonic vibrations. The value of $n$ (or $N$) will be independent of the magnitude $x$. The value $\delta$ (or $\delta_1$, $\delta_2$) directly depends upon the stiffness $k$ (or $k_1$, $k_2$).
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1. ELASTIC ROD OF NEGLIGIBLE WEIGHT, LOADED AT THE ENDS

Consider an elastic rod of length \( L \) and uniform cross-sectional area \( A \) subjected to the downward load \( W \). Let the weight of the rod be negligible in comparison to the weight \( W \). If the weight \( W \) is given a displacement of \( x \) from the equilibrium position and then released, vibrations will be set up in the rod.

Let \( P = \) restoring force

From Hooke's law,

\[
x = \frac{PL}{AE}
\]

or

\[
P = \frac{AE}{L} \frac{4\pi^2 x}{T^2}
\]

But

\[
P = \text{mass} \times \text{acceleration}
\]

Also, from Eq. 19'3 (a),

\[
\text{Acceleration} = 4\pi^2 \frac{x}{T^2}
\]

Equating (i) and (ii), we get

\[
\frac{AE}{L} = \frac{W}{g} \frac{4\pi^2 x}{T^2}
\]

or

\[
\frac{1}{T^2} = \frac{EA}{4WL}
\]

or

\[
n = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{EA}{WL}}
\]

If \( E \) is in N/mm\(^2\), \( W \) is in N; \( A \) is in mm\(^2\); \( g \approx 9810 \) mm/sec\(^2\) and \( L \) is in mm, we get

\[
n = \frac{1}{2\pi} \sqrt{\frac{9810 \cdot EA}{WL}} \approx 158 \sqrt{\frac{EA}{WL}} \quad \text{[19'8]}
\]

\[
n \approx 946 \sqrt{\frac{EA}{WL}} \text{ vibrations/min.} \quad \text{[19'9]}
\]

If, however, \( E \) is in lb/in\(^2\); \( W \) is in lbs; \( A \) is in in\(^2\); \( g \) is 32'2 x 12 in inch/sec\(^2\),

\[
n = \frac{1}{2\pi} \sqrt{\frac{32'2 \times 12 \cdot EA}{WL}} = 3'13 \sqrt{\frac{EA}{WL}} \quad \text{[19'9 (a)]}
\]

and

\[
n = 187'8 \sqrt{\frac{EA}{WL}} \quad \text{[19'10 (a)]}
\]

2. EFFECT OF THE WEIGHT OF THE ROD

Consider a rod of length \( L \), fixed at one end. The fixed end forms a node or stationary point. The remainder of the rod has a longitudinal vibratory movement.

Let \( v = \) velocity of the free end.

\[
\text{Velocity at a distance } x \text{ from the fixed end}
\]

\[
\frac{v}{L} x
\]

Consider a small length \( 8x \) of the rod. The kinetic energy of the length \( 8x \) will be

\[
\frac{1}{2} \rhoA8x \left( \frac{vx}{L} \right)^2
\]

where \( \rho = \) unit weight of the rod.

If \( W_t = \) total weight of the rod,

\[
\rho = \frac{W_t}{AL}
\]

\[
\therefore \quad \text{K.E. of } 8x \text{ length}
\]

\[
\frac{1}{2} \frac{W_t}{AL} \cdot A8x \left( \frac{vx}{L} \right)^2 = \frac{W_t 8x}{2gL} \frac{v^2 x^2}{2}
\]

\[
\therefore \quad \text{K.E. of the whole rod}
\]

\[
\int_0^L \frac{W_t v^2}{2gL} x^2 \, dx = \frac{1}{3} \frac{W_t v^2}{2g} \quad \text{[19'11]}
\]

Thus, the K.E. due to the weight of the rod is equal to \( \frac{1}{3} \) of the K.E. due to its weight considered to be concentrated at its end. Hence from Eq. 19'8,

\[
n = \frac{1}{2\pi} \sqrt{\frac{EAG}{4W_tL}} \quad \text{[19'12]}
\]

If a weight \( W \) is also acting at the rod, we get

\[
n = \frac{1}{2\pi} \sqrt{\frac{EAG}{W + \frac{1}{4}W_tL}} \quad \text{[19'13]}
\]
(c) NON-UNIFORM ROD

Let the rod consist of several parts of lengths \( L_1, L_2, L_3, \ldots \) and areas of cross-sections \( A_1, A_2, A_3, \ldots \). Let a weight \( W \) large enough in comparison to the weight of the rod be attached to its place end.

Let \( \delta_1, \delta_2, \delta_3, \ldots \) be the static deflection on displacement of each length \( L \), such that

\[
\delta = \delta_1 + \delta_2 + \delta_3 + \ldots = W \left[ \frac{L_1}{\pi A_1 E} + \frac{L_2}{A_2 E} + \frac{L_3}{A_3 E} + \ldots \right]
\]

or

\[
\delta = \frac{W L}{E \sum A}
\]

Stiffness \( k = \frac{W}{\delta} = \frac{E}{\sum \frac{L}{A}} \) \( \ldots (19.14) \)

Substituting this in Eq. 19.4, we get
\[
n = \frac{1}{2\pi} \sqrt{\frac{Wg}{\delta}} = \frac{1}{2\pi} \sqrt{\frac{Eg}{W^2 \frac{L}{A}}} \text{ per second} \quad \ldots (19.15)
\]

**19.4. TRANSVERSE VIBRATIONS**

A bar or shaft is said to have transverse vibrations when all the particles of the bar move along a straight path perpendicular to the axis of the bar. We shall consider different cases.

(a) CONCENTRATED LOAD AT THE END OF A LIGHT CANTILEVER

![Fig. 19-5](image)

Deflection under the concentrated load is given by

\[
\delta = \frac{WA^2b^2}{3EI}
\]

or

\[
W = \frac{3EI}{A^2b^2}
\]

Stiffness \( k = \frac{W}{\delta} = \frac{3EIL}{A^2b^2} \) \( \ldots (19.20) \)

Substituting this in Eq. 19.4, we get
\[
n = \frac{1}{2\pi} \sqrt{\frac{Eg}{W}} = \frac{1}{2\pi} \sqrt{\frac{3EILg}{A^2b^2W}} \quad \ldots (19.21)
\]

(b) CONCENTRATED LOAD ON A SIMPLY SUPPORTED LIGHT BEAM

![Fig. 19-7](image)

Deflection under the concentrated load is given by

\[
\delta = \frac{Wh^2}{3EI}
\]

or

\[
W = \frac{3EIh}{\delta}
\]

Stiffness \( k = \frac{W}{\delta} = \frac{3EIL}{h^2} \) \( \ldots (19.20) \)

Substituting this in Eq. 19.4, we get
\[
n = \frac{1}{2\pi} \sqrt{\frac{Eg}{W}} = \frac{1}{2\pi} \sqrt{\frac{3EILg}{h^2W}} \quad \ldots (19.21)
\]
Hence taking \( g = 9810 \text{ mm/sec}^2 \) and \( y \) in mm

\[
n = \frac{15.8}{\sqrt{\delta}} \text{ per second} \quad \text{[19'21 (a)]}
\]

and

\[
N = \frac{946}{\sqrt{\delta}} \text{ per minute} \quad \text{[19'21 (b)]}
\]

(c) CONCENTRATED LOAD ON A LIGHT BEAM FIXED AT THE ENDS

![Diagram of a light beam with a concentrated load](Fig. 19.8)

Deflection under the load is given by

\[
\delta = \frac{W d^3}{3 E I L^3}
\]

\[ \therefore \text{ Stiffness} = k = \frac{W d^3}{6 E I L^3} \]

\[ n = \frac{1}{2\pi} \sqrt{\frac{k g}{W}} = \frac{1}{2\pi} \sqrt{\frac{g}{W d^3}} \quad \text{or} \quad = \frac{1}{2\pi} \sqrt{\frac{3 E I L^3 g}{W d^3}} \quad \text{[19'22]}
\]

Hence taking \( g = 9810 \text{ mm/sec}^2 \) and \( y \) in mm

\[
n = \frac{15.8}{\sqrt{\delta}} \text{ per second} \quad \text{[19'22 (a)]}
\]

and

\[
N = \frac{946}{\sqrt{\delta}} \text{ per minute} \quad \text{[19'22 (b)]}
\]

19.5. TRANSVERSE VIBRATIONS OF A UNIFORMLY LOADED BEAM OR SHAFT

Consider a uniformly loaded beam or shaft of length \( L \), subjected to transverse vibrations. The natural frequency of the vibrations can be approximately calculated by equating the strain energy which the beam would have in its static deflected position to the kinetic energy which the system would have in passing through in undeflected position when vibrating with an amplitude equal at every point to the static deflection at that point.

Consider an elementary length \( dx \) at distance \( x \) from one end. Let \( y \) be the static deflection there, given by

\[
y = \frac{W}{24 E I} \left( x^4 - 2Lx^3 + L^2x \right)
\]

Weight on the elementary length = \( w \ dx \)

\[ \therefore \text{ Strain energy of length of } dx = \frac{1}{2}(wdx)(y) \]

\[ \therefore \text{ Strain energy of the whole beam} \]

\[
= \int_0^L \frac{1}{2} w \ dx \ y = \frac{1}{2} w \left[ \frac{L}{24 E I} \left( x^4 - 2Lx^3 + L^2x \right) \right] dx
\]

\[
= \frac{w}{48 E I} \left[ \frac{L}{5} - \frac{2L}{4} + \frac{L^2}{2} \right]
\]

\[
= \frac{w L^2}{240 E I} \quad \text{(1)}
\]

Again, the maximum velocity of the elementary length = \( 2\pi y \) 

where \( n \) = frequency of vibrations.

\[ \therefore \text{ K.E. of elementary length} = \frac{1}{2} \cdot \frac{w dx}{g} (2\pi y)^{n^2} \]

\[ \therefore \text{ K.E. of the whole beam} = \frac{1}{2} \cdot \frac{w}{g} \int_0^L \left( 2\pi y \right)^{n^2} dx \]

\[
= \frac{2\pi n^2 w}{g} \int_0^L \left[ \frac{w}{24 E I} \left( x^4 - 2Lx^3 + L^2x \right) \right]^2 \ dx
\]

\[
= \frac{\pi n^2 w^2}{288 g E I^2} \int_0^L \left( x^8 + 4Lx^7 + 6L^2x^6 + 4L^3x^5 + 2L^4x^4 - 4L^5x^3 \right) dx
\]

\[
= \frac{\pi n^2 w^2}{288 g E I^2} \left[ \frac{L^9}{9} + \frac{4L^9}{7} + \frac{6L^9}{8} + \frac{4L^9}{3} + \frac{2L^9}{6} - \frac{4L^9}{5} \right]
\]

\[
= \frac{31\pi n^2 w^2 L^9}{288 g \times 630 E I^2 g} \quad \text{(2)}
\]

Equating (1) and (2), we get

\[
\frac{w^2 L^6}{240 E I} = \frac{31\pi n^2 w^2 L^9}{288 \times 630 E I^2 g}
\]

\[ n^2 = 2.47 \frac{E I g}{w L^3} \quad \text{or} \]

\[
n = 1.572 \sqrt{\frac{E I g}{w L^3}} \quad \text{(19'23)}
\]
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Substituting \( \delta = \text{central deflection} = \frac{5}{384} \frac{wL^4}{EI} \)

we get

\[ n = 1.572 \sqrt{\frac{5g}{384}} = 0.179 \sqrt{\frac{g}{8}} \text{ per sec} \] 

...(19.24)

Taking \( g = 9810 \text{ mm/sec}^2 \text{ and } \delta \text{ in mm}, \) we get

\[ n = 17.73 \text{ per sec} \] 

...(19.25)

and

\[ N = \frac{1064}{\sqrt{8}} \text{ per minute} \] 

...(19.26)

19.6. TRANSVERSE VIBRATIONS OF A BEAM OR SHAFT WITH SEVERAL POINT LOADS

(1) First Method

We have seen that for a point load, the value of \( N = \frac{946}{\sqrt{8}} \) per minute, while for uniformly distributed load, which is equivalent to infinite number of point loads, \( K = \frac{1064}{\sqrt{8}} \). Hence for any intermediate system of loading, consisting of a number of point loads, \( N \) will be between \( \frac{946}{\sqrt{8}} \) to \( \frac{1064}{\sqrt{8}} \) per minute. A practical value of \( N \) may be taken as

\[ N = 1000 \text{ to } 1050 \text{ per minute} \] 

...(19.27)

where \( \delta_{\text{max}} = \text{maximum deflection of the beam of shaft under a given system of loading.} \)

(2) Second Method (Dunkerley's Method)

This method was suggested by Prof. Dunkerley. Let the shaft be subjected to a number of point loads \( W_1, W_2, W_3, \ldots W_n \), along with a uniformly distributed load (consisting of the self weight). Let \( N_1, N_2, N_3, \ldots N_1, N_2, \) respectively, be the frequencies per minute of the shaft when acted upon by any one load, other loads being absent. Then the frequency \( N \) of the shaft subjected to the combined loading is given by

\[ \frac{1}{N^2} = \frac{1}{N_1^2} + \frac{1}{N_2^2} + \cdots + \frac{1}{N_n^2} = \frac{1}{N_1 \cdot N_2 \cdot \cdots \cdot N_n} \] 

...(19.28)

\( N \) = frequency of the shaft when only load \( W_1 \) is acting

\( N_1 \) = frequency of the shaft when only load \( W_1 \) is acting

\( N_2 \) = frequency of the shaft when only load \( W_2 \) is acting

\( N_n \) = frequency of the shaft when uniformly distributed load \( W \) per unit length is acting.

Now \( N_1 = \frac{946}{\sqrt{8}} ; N_2 = \frac{946}{\sqrt{8}} ; N_3 = \frac{946}{\sqrt{8}} \)

and

\[ N_1 = \frac{1064}{\sqrt{8}} \] 

Hence

\[ N = \frac{946}{\sqrt{8}} \sqrt{\delta_1^2 + \delta_2^2 + \cdots + \delta_n^2 + \delta_0^2} \]

...(19.29)

where \( \delta_0 = \text{maximum deflection due to uniformly distributed load} \)

acting alone on the shaft.

\( \delta_1 = \text{static deflection under load } W_1 \) when \( W_1 \) is acting alone.

\( \delta_2 = \text{static deflection under load } W_2 \) when \( W_2 \) is acting alone.

(3) Third Method

In this method, it is assumed that the shape of the deflection curve of the vibrating shaft is similar to the shape of the static deflection curve.

Then K.E. of whole beam = \( \sum \frac{1}{2} \frac{W^2}{g} (\omega y)^2 \) 

\[ \frac{1}{2} \frac{\omega^2}{g} \sum W y^2 \] 

...(i)

Strain energy of the whole beam = \( \Sigma W y = \sum W y \)

or

\[ \omega^2 = \frac{8 \Sigma W y}{2 \sum W y^2} \]

\[ n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{8 \Sigma W y}{2 \sum W y^2}} \text{ per second} \] 

...(19.30)

or

\[ N = \frac{30}{\pi} \sqrt{\frac{8 \Sigma W y}{2 \sum W y^2}} \text{ per minute} \] 

...(19.31)

where \( y = \text{whole deflection under each load resulting from the action of all the loads.} \)
Example 191. Obtain from first principles an expression for the fundamental natural frequency of transverse vibration of a cantilever of length $L$ and weight $w$ per unit length, it being assumed that the vibration deflection curve is of the same form as the static deflection curve.

Hence find the frequency of transverse vibration of a steel turbine blade of uniform section, 150 mm long, having a weight of 0.02 N/mm length and least moment of inertia of 2540 mm$^4$. Ignore the effect of centrifugal loading. Take $E=2 \times 10^8$ N/mm$^2$.

Solution.

Consider an element of length $dx$ at $X$

\[ EI \frac{d^2y}{dx^2} = W(L-x) \]

\[ \therefore \quad EI \frac{dy}{dx} = W \left( Lx - \frac{x^3}{2} \right) + 0 \]

\[ Ely = W \left( \frac{Lx^2}{2} - \frac{x^3}{6} \right) + 0 \]

\[ \therefore \quad \text{At } x=L, \quad Ely = W \left( \frac{L^2}{2} - \frac{x^3}{6} \right) = \frac{WL^2}{3} \]

Also,

\[ Ely = \frac{W}{6} \left( 3L^2 - x^2 \right) \]
where \( k = \text{stiffness} = \frac{W}{8} \).

Substituting, \( W = \frac{33}{140} wL \), we get

\[
n = \frac{1}{2\pi} \sqrt{\frac{33}{140} \frac{k}{wL}} \text{ per second}
\]

(4)

In the present case, \( k = \frac{W}{8} = \frac{3EI}{L^3} \),

\[
n = \frac{1}{2\pi} \sqrt{\frac{33}{140} \frac{140 EIg}{11 wL^4}} \text{ per second}
\]

...(19'32)

Numerical part:

\( L = 150 \text{ mm} \); \( E = 2 \times 10^4 \text{ N/mm}^2 \),
\( I = 2540 \text{ mm}^4 \); \( g = 9810 \text{ mm/sec}^2 \),
\( w = 0.02 \text{ N/mm} \),

\[
n = \frac{1}{2\pi} \sqrt{\frac{140 \times 2 \times 10^4 \times 2540 \times 9810}{11 \times 0.02 (150)^4}}
\]

= 398 vibrations per second.

**Example 19'2.** A beam of 6 m length is simply supported at the ends and carries a central load of 20 kN. The moment of inertia of the beam is \( 8250 \times 10^4 \text{ mm}^4 \) units. Calculate the natural frequency of vibrations. Neglect the effect of the weight of the beam. Take \( E = 2 \times 10^4 \text{ N/mm}^2 \).

**Solution.**

From Eq. 19'4 (a),

\[
n = \frac{1}{2\pi} \sqrt{\frac{k}{w}}
\]

where \( k = \text{stiffness of the beam} \).

Central deflection \( \delta = \frac{W L^4}{48 EI} \)

\[
k = \frac{W}{8} = \frac{48EI}{L^3}
\]

\[
n = \frac{1}{2\pi} \sqrt{\frac{48EIg}{W L^3}}
\]

Substituting the values, we get

\[
n = \frac{1}{2\pi} \sqrt{\frac{48 \times 2 \times 10^4 \times 8250 \times 10^4 \times 9810}{20000 \times (6000)^3}}
\]

= 675 vibrations per second.

**Example 19'3.** If the beam of Example 19'2 weighs 500 N/m, determine the frequency of vibrations, taking into account the effect the self weight of the beam.

**Solution.**

Let \( w = \text{weight of beam per unit length} \),

Let \( W_1 = \text{dynamical equivalent of the uniformly distributed load} wL \), concentrated at the mid span.

![Fig. 19'11](image)

Let the amplitude of vibration be \( S \) at centre point \( C \), and \( y \) at any section \( X \), above and below the static deflection curve. Due to equivalent point load \( W_1 \).

\( EI \frac{dy}{dx} = - \frac{W_1}{2} \left( \frac{L}{2} - x \right) \)

\[
\therefore \quad EI \frac{dy}{dx} = - \frac{W_1}{2} \left( \frac{L}{2} - x - \frac{x}{2} \right) + B
\]

(\( \text{Since} \ \frac{dy}{dx} = 0 \text{ at } x = 0 \))

and

\( Ely = - \frac{W_1}{2} \left( \frac{L}{2} \frac{x^3}{6} - \frac{x^3}{2} \right) + B \)

At \( x = \frac{L}{2}, \ y = 0 \quad \therefore \ B = \frac{W_1 L^3}{48} \)

\[
\therefore \quad Ely = - \frac{W_1}{2} \left( \frac{L}{2} \frac{x^3}{6} - \frac{x^3}{2} \right) + \frac{W_1 L^3}{48}
\]

\( \cdots (1) \)

At \( x = 0, \ y = S \)

\[
\therefore \ EIS = \frac{W_1 L^3}{48}
\]

\( \cdots (2) \)
Hence from (1) and (2),
\[ \frac{v}{b} = 1 - 6 \left( \frac{x}{L} \right) + 4 \left( \frac{x}{L} \right)^3 \]...
Now consider the original uniformly distributed load.
Let
\[ v = \text{velocity of vibration at } X \]
\[ V = \text{velocity of vibration at centre } C \]
Since \( v \propto y \), we have \( \frac{v}{V} = \frac{y}{b} \)...

Consider an element of length \( dx \) at \( X \)

K.E. of element = \( \frac{1}{2} \left( \frac{wdx}{g} \right) v^2 \)

\[ = \frac{1}{2} \left( \frac{wdx}{g} \right) \left( \frac{yv}{b} \right)^2 \]

\[ = \frac{1}{2} \frac{wdx}{g} \cdot v^2 \left( 1 - 6 \left( \frac{x}{L} \right)^2 + 4 \left( \frac{x}{L} \right)^3 \right)^2 \]

\[ \therefore \text{K.E. of total beam,} \]

\[ = \frac{2wV^2}{2g} \int_0^{L/2} \left[ 1 - 6 \left( \frac{x}{L} \right)^2 + 4 \left( \frac{x}{L} \right)^3 \right]^2 \]

\[ = \left( \frac{17wL}{35} \right) \frac{V^2}{2g} \]...

K.E. of the whole beam due to dynamically equivalent point load \( W_1 \) is equal to \( W_1 \cdot \frac{V^2}{2g} \).

\[ W_1 = \frac{17wL}{35} \cdot \frac{V^2}{2g} \]

Thus the uniformly distributed load can be replaced by an equivalent load \( \frac{17}{35} wL \) placed at the centre of the span.

If \( W \) is the other point load acting at the centre in addition to the self weight of the beam, the total point load at the centre of span = \( W + \frac{17}{35} wL \). Substituting this in Eq. 19'4 (a),

\[ n = \frac{1}{2\pi} \sqrt{\frac{KL}{W + \frac{17}{35} wL}} \]...

where \( k = \text{stiffness of beam} \).
Example 19'4. A helical spring has both ends securely fixed, one vertically above the other, and a mass is attached to the spring at some intermediate point. Show that the frequency of the vibrations is a minimum when the load point is midway between the fixed ends.

A helical spring has a stiffness of 4N/mm when one end is fixed and the load is applied to the free end. Determine the minimum value of the frequency when both ends are fixed and a mass weighing 200 N is applied to the spring.

Solution.

Let the load $W$ be applied at a distance $L_1$ from the fixed end $A$, or at distance $L_2$ from the other fixed end $B$, such that

$$L_1 + L_2 = L$$

Due to load $W$, let $W_1$ be the tension in the upper portion, and $W_2$ be the compression in the lower portion of the spring such that

$$W = W_1 + W_2$$

Now, in general,

$$n = \frac{1}{2\pi} \sqrt{\frac{kg}{W}}$$

where $k$=stiffness.

Hence, for the first length,

$$n_1 = \frac{1}{2\pi} \sqrt{\frac{kg}{W_1}} = \frac{1}{2\pi} \sqrt{\frac{W_1}{\delta_1}} \cdot \frac{g}{W_1}$$

where $\delta_1$=extension of the length

$$C = \text{a constant}$$

Similarly, $n_2 = \frac{1}{2\pi} \sqrt{\frac{g}{C_L W_2}}$

For equal minimum values, $n_1 = n_2$.

$$\therefore \quad (L_1W_1)_{\text{max}} = (L_2W_2)_{\text{max}}$$

Now, if $W_1 > \frac{W}{2}$, $W_2 < \frac{W}{2}$

Also, if $L_1 > \frac{L}{2}$, $L_2 < \frac{L}{2}$
Hence for equal \((LW)_{\text{max}}\) and minimum \(n\) values,
\[ L_1W_1 = L_2W_2 = \frac{L}{2} \times \frac{W}{2} \]
\[ \therefore L_1 = L_2 = \frac{L}{2} \text{ (proved)} \]

Now \(k = 4 \text{ N/mm}\)

For the First Spring,
\[ k = \frac{W}{\delta_1} = \frac{W/2}{\delta_1} = \frac{200/2}{\delta_1} = \frac{100}{\delta_1} \]
\[ \therefore \delta_1 = \frac{100}{k} = \frac{100}{4} = 25 \text{ mm.} \]

Similarly, if \(\delta_2\) is the compression of second spring, we get
\[ k = \frac{W}{\delta_2} = \frac{W/2}{\delta_2} = 200/2 = \frac{100}{\delta_2} \]
\[ \therefore \delta_2 = \frac{100}{k} = \frac{100}{4} = 25 \text{ mm (as expected)} \]
\[ \therefore n_1 = n_2 = n = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}} = \frac{1}{2\pi} \sqrt{\frac{9810}{25}} = 3.15 \text{ vibrations per second.} \]

19.7. CRITICAL OR WHIRLING SPEED OF SHAFT

When a shaft is rotating in bearings, the initial crookedness, the dead weight of the shaft, and vibrations, etc. cause some deflection with the result that the centre line of the shaft do not coincide with the mathematically straight axis of rotation. Due to this, centrifugal forces will be developed producing a bending moment on the shaft tending to deflect it further, until they are balanced by the restoring forces arising from the stiffness of the shaft. As the speed of rotation increases, a limit will be reached when the centrifugal force will exceed the limit of elastic forces. At this stage, instability will follow and the deflection and stress, unless prevented, will increase until fracture occurs. The speed which just gives balance between the two sets of forces, is called the critical speed or whirling speed of the shaft. The centrifugal forces may be regarded as having a neutralising effect upon the elastic forces which tend to return the shaft to its natural shape so that when whirling occurs the effective stiffness of the shaft is reduced to zero. If, however, the speed of the shaft is increased from a value below the critical to a value higher than the critical, in a short time so that the deflections do not get opportunity to increase indefinitely, the shaft restores the stability. Many shafts are designed to run above the whirling speed.

Consider a shaft simply supported at the ends and carrying a central point load \(W\). Let the weight of the shaft be negligible.

Let \(e\) = initial difference between the geometrical axis and the axis of rotation
\[ y = \text{increase in the displacement due to rotation.} \]

The centrifugal force of the rotating mass
\[ \frac{W}{g} = (e+y)\omega^2 \quad \ldots (1) \]
where \(\omega\) = angular speed of the shaft, in radians/sec
Restoring force \(= ky\)
\[ \text{where } k = \text{stiffness of the shaft} \]

Equating the two, we get
\[ \frac{W}{g} = (e+y)\omega^2 = ky \]
or
\[ y \left(\frac{k}{\omega^2} - \frac{W}{g}\right) = \frac{W}{g} e\omega^2 \]
\[ \frac{W\omega^2}{g} = \frac{e}{k - \frac{W}{g}} \quad \ldots (19.34) \]

or
\[ y = \frac{\frac{g}{W}}{\frac{k}{\omega^2}} e = \frac{e}{k - \frac{W}{g}} \omega^2 \]

The above equation gives the deflection due to rotation, at any angular speed \(\omega\). At the whirling speed \(\omega = \omega_1\), the deflection \(y\) becomes infinitely great. This gives
\[ \frac{kg}{Wo_1^2} - 1 = 0 \]
or
\[ \omega_1^2 = \frac{kg}{W} \quad \ldots (19.35) \]

But
\[ \frac{W}{k} = \text{static deflection} = \delta \]
\[ \therefore \omega_1^2 = \frac{g}{\delta} \quad \ldots (19.36) \]
or
\[ \omega_1 = \sqrt{\frac{g}{\delta}} \quad \ldots (19.36) \]
or
\[ n_1 = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}} \text{ revolutions per second} \quad \ldots (19.37) \]
or
\[ n_1 = \frac{30}{\pi} \sqrt{\frac{g}{\delta}} \text{ per minute} \quad \ldots [19.37(a)] \]
Thus we find the critical speed \( n_c \) is equal to the natural frequency of vibration of the system. Similarly it can be shown that the critical speed of an unloaded shaft, taking into account its self weight, or critical speed of a shaft carrying uniformly distributed load is also equal to the natural frequency of vibration. Hence if a shaft carries a number of point loads, the method or expression for finding out the natural frequency \( N \) also apply for the present case of finding the critical speed (i.e. Eqs. 19'27, 19'28 and 19'29 apply for the critical speed also).

Taking \( g=9810 \text{ mm/sec}^2 \), and \( \delta \) in mm, Eq. 19'27 reduces to

\[
n_c = \frac{15'8}{\sqrt{8}} \quad \text{(19'38)}
\]

and

\[
N_e = \frac{946}{\sqrt{8}} \text{ R.P.M.} \quad \text{[19'38 (a)]}
\]

Again, substituting \( \frac{k}{W} = \omega_c^2 \) in Eq. 19'34, we get

\[
y = \frac{e}{\omega_c^2} - \frac{e \omega_c^2}{\omega_c^2 - \omega_c^2} = 1 \quad \text{(19'39)}
\]

It is evident from Eq. 19'37 that \( y \) becomes negative if the speed of rotation \( \omega \) is greater than the critical speed \( \omega_c \). In other words, the shaft tries to straighten out. At very high speed, \( y = -e \), and the geometrical axis and the axis of rotation will coincide. This is the principle of the flexible shaft of the De Laval steam turbine.

Example 19'7. A shaft 20 mm diameter and 500 mm between the long bearing at its ends, carries a wheel weighing 100 N midway between the bearings. Neglecting the increase of the stiffness due to attachment of the wheel to the shaft, find the critical speed of rotation, and the maximum bending stress when the shaft is rotating at 4/5 of this speed, if the centre of gravity of the wheel is 0.4 mm from the centre of the shaft. Take \( E=2 \times 10^8 \text{ N/mm}^2 \).

Solution.

When shaft is supported on long bearings, it has an effect of fixidity at the end. Hence the shaft may be considered to a fixed beam with a central point load. The central deflection \( \delta \) for such a case is given by

\[
\delta = \frac{WL^3}{192EI}
\]

\[
\therefore \text{Stiffness, } k = \frac{W}{\delta} = \frac{192EI}{L^3} \quad \text{(1)}
\]

\[
I = \frac{\pi}{64} (20)^4 = 0.785 \times 10^3 \text{ mm}^4
\]

Again, \( N_e = \frac{60}{2\pi} \sqrt{\frac{k}{gW}} \)

\[
= \frac{30}{\pi} \sqrt{\frac{192EIg}{W^2L^4}}
= \frac{30}{\pi} \sqrt{\frac{192 \times 2 \times 10^4 \times 0.785 \times 10^3 \times 9810}{100 (500)^3}}
= 4645 \text{ revolutions per minute.}
\]

Again, from Eq. 19'39,

\[
y = \frac{e \omega_c^2}{\omega_c^2 - \omega_c^2} \quad \text{[19'38 (a)]}
\]

Putting \( \omega = 4/5 \omega_c \) and \( e = 0.4 \text{ mm} \), we get

\[
\text{Central centrifugal bending force}
= k \cdot y = 0.71 k
\]

\[
\text{B.M.} = M = \frac{1}{8} (0.71 k) L
= \frac{0.71 L \times 192 E I}{8} \times \frac{12 \times 24 E x 10}{L^3}
= 7.1 \times 24 \times 2 \times 10^6
= 136 \times 3 \text{ N/mm}^2
\]

Note: If the bearings are of short length, or if they have spherical seatings, it is taken as simply supported at the ends.

19'8. TORSIONAL VIBRATIONS

(a) SHAFT OR ROD CARRYING A LOAD \( W \) AT ITS END

![Diagram](Fig. 19-14)
Consider a shaft fixed at one end carrying a load \( W \) at the other end. The fixed end prevents any twisting strain at that end and hence form a fixed or stationary node. If the shaft is twisted by the rotation of the weight \( W \), it will have vibratory movement in which every part at a given instant moves in a circle about the axis in the same sense as the applied twist. The torsional rigidity of the shaft will resist this twisting and the resisting couple will try to restore the body to its mean position. If the external torque is removed after giving an initial twist by amount \( \theta \), every point on the shaft will have simple harmonic torsional oscillations or vibrations.

Let \( c = \) torsional rigidity of the shaft

\[
T = c \theta
\]

\[
\text{Work done} = \frac{1}{2} (c \theta) \theta = \frac{1}{2} c \theta^2
\]

Let \( \omega = \) angular velocity in radians/sec = \( 2\pi n \theta \)

where \( n = \) vibrations/second

\[
\therefore \quad \text{Kinetic energy} = \frac{1}{2} I \omega^2
\]

where \( I = \) mass moment of inertia of the weight \( W \)

\[
= \frac{W}{g} K^2 \quad \text{(where} \ K = \text{radius of gyration of the weight} \ W)\]

Equating (2) and (3),

\[
\frac{1}{2} I_0 \omega^2 = \frac{1}{2} c \theta^2
\]

\[
\therefore \quad \omega = \frac{\theta}{\sqrt{\frac{c}{I}}}
\]

But

\[
\omega = 2\pi n \theta
\]

\[
\therefore \quad n = \frac{1}{2\pi} \sqrt{\frac{c}{I}}
\]

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If \( L \) is the length of the shaft, we have

\[
\frac{T}{J} = \frac{N \theta}{L}
\]

But

\[
T = c \theta
\]

\[
\therefore \quad \frac{c \theta}{J} = \frac{N \theta}{L}
\]

or

\[
c = \frac{NJ}{L}
\]

where \( J = \) polar moment of inertia of the shafts = \( \frac{\pi d^4}{32} \)

Substituting this in Eq. 1941, we get

\[
n = \frac{1}{2\pi} \sqrt{\frac{NJ}{IL}}
\]

If the shaft consists of two or more parts of lengths \( L_1, L_2, \) etc. and polar moments of inertia \( J_1, J_2, \) etc., we have

\[
\frac{1}{c} = \frac{1}{N} \left( \frac{L_1}{J_1} + \frac{L_2}{J_2} + \cdots \right)
\]

or

\[
\frac{1}{c} = \frac{1}{c_1} + \frac{1}{c_2} + \cdots \quad \ldots \quad [1942 (a)]
\]

\[
n = \frac{1}{2\pi} \sqrt{\frac{N}{I \left( \frac{L_1}{J_1} + \frac{L_2}{J_2} + \cdots \right)}} \quad \ldots \quad [1942 (b)]
\]

In the above analysis, it has been assumed that the (mass) moment of inertia of the weight is so great that the (mass) moment of inertia of the shaft is negligible.

(b) Effect of the Weight of the Shaft

Let us now take the case of an unloaded shaft, having a self weight of \( w \) per unit length. Let the mass moment of inertia of the shaft be \( I_0 \). If \( \omega \) is the angular frequency of vibration at the free end, the angular frequency of vibration at point distant \( x \) from the fixed end (Fig. 1914) will be \( \omega x/L \).

Mass moment of inertia of length \( dx = \frac{dx}{L} J_1 \)
K.E. of length \( dx = \frac{1}{2} \left( \frac{dx}{L} I_1 \left( \frac{\alpha x}{L} \right)^2 \right) \)

\[
\text{Total K.E. of shaft} = \int_0^L \frac{1}{2} \left( \frac{dx}{L} I_1 \left( \frac{\alpha x}{L} \right)^2 \right) \]

\[
= \frac{1}{2} \left( \frac{I_1 \alpha^2}{L^3} \right) \left( \frac{L^2}{3} \right) = \frac{1}{3} \left( \frac{I_1 \alpha^2}{L} \right)
\]

Thus, the effect of the weight of the shaft is accounted for by adding \( \frac{1}{2} \) to the \( \frac{1}{2} I_1 \) of the weight securely fixed at the end.

Thus, if a shaft having mass moment of inertia \( I_1 \) carries a load \( W \) having mass moment of inertia \( I_2 \), the frequency of vibrations is given by

\[
n = \frac{1}{2\pi} \sqrt{\frac{NJ}{I_1 L}}
\]

\((19'44)\)

**Example 19'8.** A flywheel weighing 20 kN has a radius of gyration \( r \) m, and is fixed at one end of a shaft 100 mm in diameter and 1 metre long. A pulley weighing 12 kN and of radius of gyration of 600 mm is fixed at the other end of the shaft. Calculate the natural frequency of torsional vibrations. Take \( N = 0.82 \times 10^6 \) N/mm² for the material of the shaft. Neglect the mass moment of inertia of the shaft. Find also the position of node.

**Solution.**

\[
W_1 = 20000 \; N \quad K_1 = 1 \; m = 1000 \; mm
\]

\[
I_1 = \frac{W_1 \cdot r^2}{g} = \frac{20000 \cdot 1000}{9810} = 2.04 \times 10^4
\]

\[
W_2 = 12000 \; N \quad K_2 = 600 \; mm
\]

\[
I_2 = \frac{W_2 \cdot r^2}{g} = \frac{12000 \cdot 600}{9810} = 0.44 \times 10^4
\]

\[
I_1, I_2 = I_1L_1
\]

\[
\therefore \quad 2.04 \times 10^4 \cdot L_1 = 0.44 \times 10^4, \quad L_1 = 0.22 \; m
\]

\[
L_1 = 0.22 \; m, \quad L_2 = 0.78 \; m
\]

Hence distance of node = 177.4 mm from the flywheel.
where \( J = \frac{\pi}{32} \) and \( d^4 = \frac{\pi}{32} \). Thus,

\[
\dot{n} = \frac{1}{2\pi} \sqrt{\frac{N}{I_1L_1}}
\]

where \( J = \frac{\pi}{32} \) and \( d^4 = \frac{\pi}{32} \). Thus,

\[
\dot{n} = \frac{1}{2\pi} \sqrt{\frac{0.82 \times 10^4 \times 9.817 \times 10}{2.04 \times 10^6 \times 177.4}}
\]

\[
= 7.51 \text{ per second}
\]

Alternatively, from Fig. 19:16,

\[
\dot{n} = \frac{1}{2\pi} \sqrt{\frac{N}{I_{12}}} (I_1 + I_2)
\]

\[
= \frac{1}{2\pi} \sqrt{\frac{0.82 \times 10^4 \times 9.817 \times 10}{1000 \times 2.04 \times 10^6 \times 44 \times 10^6 \times (2.04 + 0.44) \times 10^6}}
\]

\[
= 7.51 \text{ per second}
\]

Example 19.9. The flywheel of an engine driving a dynamo weighs 300 lb and has a radius of gyration of 10 in.; the armature weighs 220 lb and has a radius of gyration of 8 in. The driving shaft has an effective length of 18 in. and is 2 in. diameter, and a spring coupling is incorporated at one end, having a stiffness of \( 0.25 \times 10^6 \) lb/in. per radian. Neglecting the inertia of the coupling and shaft, calculate the natural frequency of torsional vibration of the system. What would be the natural frequency if the spring coupling were omitted? Take \( N = 11.9 \times 10^6 \) lb/in.².

(U.L.)

Solution.

(a) Coupling omitted

In general assuming that no nodal point occurs between 1 and 2, and have

\[
\dot{n} = \frac{1}{2\pi} \sqrt{\frac{c}{I_1 + I_2}}
\]

where \( c = \text{Torsional stiffness of shaft} \)

\[
= \frac{N}{L} = \frac{11.9 \times 10^6 \times \pi(2)^4}{32 \times 18}
\]

\[
= 1.035 \times 10^6 \text{ lb in./radian}
\]

\[
I_1 = \frac{W_1}{g} K_1^2 = \frac{300}{32^2 \times 12}(10)^3
\]

\[
I_2 = \frac{W_2}{g} K_2^2 = \frac{220}{32^2 \times 12}(8)^3
\]

\[
I_1 + I_2 = \frac{1}{32^2 \times 12} \left[ 300(10)^3 + 220(8)^3 \right] = 114.5
\]

\[
\dot{n} = \frac{1}{2\pi} \sqrt{\frac{1.035 \times 10^6}{114.5}} = 15.15 \text{ per second}
\]

(b) Coupling included

Let \( c = \text{combined stiffness of the shaft and coupling in series} \)

\[
c_s = \text{stiffness of shaft} \)

\[
c_c = \text{stiffness of coupling} \)

\[
\frac{1}{c} = \frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{\frac{1}{10^6} + \frac{4}{10^6}} = 4.966
\]

\[
\dot{n} = \frac{1}{2\pi} \sqrt{\frac{c}{I_1 + I_2}}
\]

\[
= \frac{1}{2\pi} \sqrt{\frac{4.966}{10^6}} = 6.69 \text{ per second}
\]

Example 19.10. An engine shaft is directly coupled to the shaft of a dynamo. The engine shaft has a diameter of 60 mm and an effective length of 300 mm, while the dynamo shaft has a diameter of 30 mm and an effective length of 200 mm. The flywheel weighs 250 \( N \) and has a radius of gyration of 350 mm and the armature weighs 150 \( N \) and its radius of gyration is 250 mm. Neglecting the inertia of the coupling and of the shafts, determine the position of the node and the natural frequency of torsional oscillations. For both the shafts, take \( N = 0.8 \times 10^8 \) N/mm².
The shaft has a diameter \( d_1 = 60 \text{ mm} \) for a length \( L_1 = 300 \text{ mm} \), and a diameter \( d_2 = 50 \text{ mm} \) for a length \( L_2 = 200 \text{ mm} \). Let us first find an equivalent length for the length \( L_2 \) to have a uniform diameter \( d_1 \).

Fig. 19'17(a) shows the actual shaft while Fig. 19'17(b) shows the equivalent shaft of uniform diameter \( d_1 = 60 \text{ mm} \).

If \( c \) is the stiffness, we have
\[
c = \frac{N J}{L} = \frac{N \pi d^4}{32 L} \quad \text{or} \quad c \propto \frac{d^4}{L}
\]
Let the equivalent length of diameter \( d_1 = 6 \text{ cm} \) be \( L_2' \). Then
\[
\frac{d_1^4}{L_2} = \frac{d_2^4}{L_2} \\
\therefore \quad L_2' = \left( \frac{d_1}{d_2} \right)^4 L_2 = \left( \frac{60}{50} \right)^4 \times 200 = 415 \text{ mm}
\]
Thus, the total length \( L' = L_1 + L_2' = 300 + 415 = 715 \text{ mm} \).

Let the node be at a distance \( x \) from \( W_1 \).

Then
\[
\frac{W_1}{K_1} x = \frac{W_2}{K_2} x
\]
which is
\[
2500(350)^a x = 1500(250)^a (715 - x)
\]
or
\[
x = 0.306(715 - x)
\]
From which \( x = 167.5 \text{ mm} \).

Example 19'11. A light elastic shaft \( AB \) of uniform diameter, supported freely in bearings, carries wheel at each end and it is found that the natural frequency of torsional vibrations is 40 per second. A third wheel is mounted on the shaft at a point \( C \), such that \( AC = \frac{1}{3} AB \). If all the wheels have the same (mass) moment of inertia, determine the natural frequency of torsional vibrations.

(U.L.)

Solution.

If the shaft carries only one weight, having mass moments of inertia \( I \), we have
\[
n = \frac{1}{2\pi} \sqrt{\frac{N J}{I}} = \frac{1}{2\pi} \sqrt{\frac{N \pi d^4 g}{32 W_1 K_1 x}}
\]
\[
= \frac{1}{2\pi} \sqrt{0.8 \times 10^5 \times \pi (60)^4 \times 980 \over 32 \times 2500(350)^a (167.5)}
\]
\[
= 22.1 \text{ per second.}
\]

Fig. 19'18

If the shaft carries two weights of equal mass moments of inertia, the node will be at the middle point of the shaft.

Then
\[
n = \frac{1}{2\pi} \sqrt{\frac{N J}{I \frac{1}{2}}} \\
\]
or
\[
n^2 = \frac{N J}{4\pi^2} \cdot \frac{1}{I \frac{1}{2}}
\]
where \( l_1 = \text{distance of node from the mass} = L/2 \)
\[
\frac{N J}{4\pi^2} = K.
\]
of 5 N at the free end. Find the distance of the weight from the fixed end if the frequency of natural vibrations of the strip is 50/sec. Take $E=2.1 \times 10^7$ N/mm$^2$.

2. A 2 in. diameter steel $AB$, 8 ft. 3 in. long, is supported in two short bearings 6 ft. apart, one being at the end $A$ of the shaft. The shaft carries three concentrated loads as under:

<table>
<thead>
<tr>
<th>Load in lb.</th>
<th>180</th>
<th>360</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance from $A$ in ft.</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

Obtain a first approximation to the fundamental frequency of transverse vibration of the loaded shaft. Neglect the weight of the shaft. Take $E=30 \times 10^7$ lb/in$^2$. (U.L.)

3. A uniform vertical bar of steel of length $L$ and cross-sectional area $A$, is fixed at the upper end and is extended a distance $x$ by a load $W$ at the lower end. If the rod is subjected to longitudinal vibrations, show that at any instant, when the additional extension is $x$, the change of potential energy, measured from the rest position of the load is $\frac{1}{2} \frac{AE^2}{L}$ and, from the energy equation, deduce the natural period of vibration.

Find the length of the bar to give a frequency of 100 vibrations per second when $A=100$ sq. mm and $W=100$ N. Take $E=2 \times 10^7$ N/mm$^2$.

4. Solve problem 3 if the weight of the bar is 600 N.

5. A beam of length $L$ is fixed at the ends and weighs $w$ per unit length. Obtain an expression for the natural frequency of vibration if it carries a central point load $W$.

6. A close-coiled helical spring is fixed at its upper end and hangs vertically. A circular metal disc is fixed axially to the lower end of the spring. The times of vertical oscillations and for angular oscillations about the vertical axis are found to be equal.

Show that $\frac{E}{N} = \left( \frac{\text{Diameter of disc}}{\text{Mean diameter of coil}} \right)^2$, where $E$ and $N$ are elastic constants.

If the spring is made of wire 3 mm diameter and has 50 turns of 50 mm mean diameter, find the weight of the time of oscillations being 1 sec. Neglect the weight of the spring. Take $N=0.8 \times 10^7$ N/mm$^2$. 

**PROBLEMS**

1. A horizontal cantilever of length $L$ is clamped at one end and carries a load $W$ at the other. Derive an expression for the time period of vibration of the cantilever when the load is given a small vertical displacement. Neglect the weight of the cantilever.

A horizontal flat steel strip 12 mm wide and 6 mm thick is clamped at one end with 12 mm side horizontal, and carries a weight
20.1. INTRODUCTION

Flat plates are usually supported at its edges and are subjected to loads normal to their flat faces. The bending of such a plate differs from that of a beam in that the plate bend in all planes normal to the flat surface whereas the beam may be assumed to bend in one plane only. In addition to this, the bending of the plate in one plane is greatly influenced by the bending in all other planes; hence the general theory of bending of plates is quite complicated. However, we shall consider here only an approximate theory analogous to the simple Bernoulli-Euler theory of flexure in beams. The cases which are covered by this theory include most of those that are of practical interest.

20.2. SYMMETRICALLY LOADED CIRCULAR PLATE

We shall start with the simplest case: a circular plate loaded symmetrically with respect to the central axis. The treatment that follows has been given by Grashof based on early investigations by Poisson. Grashof took the maximum strain as the measure of elastic strength.

Assumptions. The theory is based on the following assumptions:
1. The plate is of uniform thickness, and the thickness is small in comparison with the diameter.
2. The central deflection is small, and does not exceed say about one-fifth of the thickness of the plate.
3. Loading is symmetrical, so that stress and strain are symmetrical about an axis perpendicular to the plate and through its centre.
4. The plane midway between the faces of the plate is unstressed or unextended, i.e., the middle plane is the neutral plane.

5. The elements of the plate originally straight and perpendicular to the middle plane remain straight and become perpendicular to the middle surface when strained.

6. Only longitudinal and lateral stresses are considered. The normal stresses across planes parallel to the middle surface are neglected.

7. The material is homogeneous and isotropic, and follows Hooke's law.

Fig. 20-1 (a) shows the section of a thin plate after being strained. The concave side of the plate will be in compression while the convex side will be in tension. The middle plane, shown by dotted line, will be unstrained and will be a neutral plane. Let $x$ and $z$ directions be the radial and circumferential directions while $y$ be the direction normal to the neutral plane.

Consider a point $P$, distant $x$ radially from the vertical central axis, before straining. The line $AB$ through $P$, originally vertical, is inclined at $\theta$ to the vertical axis $OV$. Let $y$ be the distance of the point $P$ from the middle plane of the plate. After straining, the radius at $P$ will increase to $x + \delta y$.

Hence circumferential strain $\varepsilon_z$ at a depth $y$ from the neutral plane is

$$\varepsilon_z = \frac{2\pi (x + \delta y) - 2\pi x}{2\pi x} = \frac{\delta y}{x} \quad \text{(1)}$$

Let $\rho = \frac{x}{\delta}$ be the radius of curvature at $P$.

Let $\varepsilon_z = \frac{y}{\rho}$ \quad \text{[1(a)]}

Similarly, if we consider a section at $(x + \delta x)$ radially from $O$, originally vertical but become inclined at $\theta + \delta \theta$ after being strained, the distance $\delta x$ at a depth $y$ is increased to $(\delta x + y \delta \theta)$. Hence the radial strain $\varepsilon_x$ is

$$\varepsilon_x = \frac{y \delta \theta}{\delta x} = y \cdot \frac{d\theta}{dx} = \frac{y}{\rho'} \quad \text{(2)}$$

where $\rho' = \frac{d\theta}{dx}$.

Fig. 20-1. Flat circular plate symmetrically loaded.
If \( p_z \) and \( p_z \) are radial and circumferential stresses, we have

\[
e_{z} = \frac{\theta y}{x} = \frac{1}{E} \left( p_z - \frac{p_x}{m} \right)
\]

and

\[
e_{x} = y \frac{d\theta}{dx} = \frac{1}{E} \left( p_x - \frac{p_z}{m} \right)
\]

where \( m = \) Poisson's ratio.

From Eqs. (3) and (4), we get

\[
p_z = -\frac{mE}{m^2-1} \left( \frac{m \theta}{x} + \frac{d\theta}{dx} \right) y \quad \ldots (20'1)
\]

\[
p_x = \frac{mE}{m^2-1} \left( \frac{\theta}{x} + m \frac{d\theta}{dx} \right) y \quad \ldots (20'2)
\]

From Eqs. 20'1 and 20'2, it is clear that both circumferential stress as well as radial stress vary linearly with \( y \). Thus the stress distribution across a plane is similar to the bending stress distribution in a beam.

Let us now compute resultant circumferential stress and resultant radial stress on an element included between radii \( x \) and \( x + \delta x \), and between two vertical meridians inclined at a very small angle \( \delta \theta \) to each other. Fig. 20'1 (b) shows a pictorial view of such an element while Fig. 20'1 (c) shows the plane of a horizontal plate \( ABCD \) taken at a distance \( y \) below the neutral plane \( A'B'C'D' \) [Fig. 20'1 (b)]. Since the plane \( ABCD \) is below the neutral plane, both circumferential as well as radial stresses have been shown tensile in Fig. 20'1 (c). The distance \( y \) is taken positive when measured below the neutral surface (N.S.) and the tensile stress is taken as positive. Similarly, \( y \) measured above the N.S. is taken negative and the compressive stress is taken as negative.

In Fig. 20'1 (c), the stresses \( p_z \) on faces \( AB \) and \( CD \) are inclined at \( \frac{\pi}{2} - \delta \phi \) to the middle radius \( EHO \). Hence the resultant elementary force in the direction \( EHO \) is equal to

\[
2n_z \cdot \delta a \sin \frac{\delta \phi}{2} = p_z \cdot \delta a \cdot \delta \phi
\]

where \( \delta a = \) elementary area of face \( AB \) or \( CD \) shown hatched in Fig. 20'1 (b).

Since \( p_z \) is of opposite sign on the opposite faces of the neutral surface, the total force parallel to \( EHO \) resulting from \( p_z \) is zero.

The elementary force given by (1) is tensile below the neutral surface and compressive above the neutral surface. Hence total moment \( M_z \) of the couple formed by the above elementary forces, about an axis in the neutral plane and perpendicular to \( EHO \) is

\[
M_z = \delta \phi \Sigma y \cdot p_z \cdot \delta a
\]

Substituting the value of \( p_z \) from Eq. 28'1, we get

\[
M_z = \delta \phi \cdot \frac{mE}{m^2-1} \left( \frac{m \theta}{x} + \frac{d\theta}{dx} \right) \Sigma y^2 \cdot \delta a
\]

Let

\[
t = \text{thickness of the plate}
\]

Then \( \Sigma y^2 \cdot \delta a = \) moment of inertia of rectangular face \( AB \) or \( CD \)

\[
= \frac{1}{12} \cdot t \cdot x^3
\]

\[
M_z = \delta \phi \cdot \frac{mE}{m^2-1} \left( \frac{m \theta}{x} + \frac{d\theta}{dx} \right) \cdot \frac{1}{12} \cdot t \cdot x^3
\]

or

\[
M_z = \frac{1}{12} \cdot \delta a \cdot \delta \phi \cdot \Sigma y \cdot t
\]

or

\[
M_z = R_z \cdot \delta t
\]

where \( R_z = \delta \phi \cdot \Sigma p_z \cdot \delta a = \) total force is direction \( EHO \) on one side of the neutral plane, Fig. 20'1 (d).

\( \delta t = \) lever arm at which the equal and opposite forces \( R_z \) act, Fig. 20'1 (d).

Sign convention. If the vertex \( V \) is formed above the plate, \( \theta \) is taken in positive. Hence \( \frac{d\theta}{dx} \) is positive, making the plate convex downwards. Thus if the element [Fig. 20'1 (b)] is viewed from the \( D_1C_1C_2D_2 \), the moment \( M_z \) due to the circumferential forces is clockwise.

Let us now consider faces \( BC \) and \( AD \) on which radial stresses \( p_z \) act. If an elementary area \( \delta a \) is considered on face \( BC \), the elementary force on this area, resolved in the direction \( EHO \) is approximately equal to \( p_z \cdot \delta a \). This force is tensile if \( \delta a \) is considered below the neutral plane. Thus, the total force on the face \( BC \) due to radial stress is zero. However, the couple formed by equal and opposite elementary force on opposite sides of the neutral plane is given by,

\[
M = \frac{mE}{m^2-1} \left( \frac{m \theta}{x} + \frac{d\theta}{dx} \right) \Sigma y^2 \cdot \delta a
\]

Substituting the value of \( p_z \) from Eq. 20'2, we get

\[
M = \frac{mE}{m^2-1} \left( \frac{m \theta}{x} + \frac{d\theta}{dx} \right) \Sigma y^2 \cdot \delta a
\]
For the face $BC$ of width $x\delta\phi$ and height $t$, we have

$$\Sigma y^2 \delta a = \frac{1}{12} (x\delta\phi) t^3$$

$$M = \frac{mE}{m^3 - 1} \left( \frac{\theta}{x} + m\frac{d\theta}{dx} \right) \cdot \frac{1}{12} x \delta\phi t^3$$

$$M = \frac{x \delta\phi t^3}{12} \cdot \frac{mE}{m^3 - 1} \left( \frac{\theta}{x} + m\frac{d\theta}{dx} \right)$$

If $\theta$ and $\frac{d\theta}{dx}$ are positive, the moment $M$ will be in the counter-clockwise direction as marked in Fig. 201 (d).

Now consider the face $AD$ on which $p_x$ act outwards. Let $(M + \delta M)$ be the moment due to the stress $p_x$ acting on it. This moment $(M + \delta M)$ can be expressed in terms of $(x + \delta x)$ and $(\theta + \delta \theta)$. However,

$$\delta M = \frac{dM}{dx} \cdot \delta x$$

Hence differentiating Eq. 20'4, we get

$$\delta M = \frac{dM}{dx} \cdot \delta x = \frac{1}{12} \cdot \delta x \cdot x \delta\phi \cdot r^3 \cdot \frac{mE}{m^3 - 1} \left( \frac{\theta}{x} + m\frac{d\theta}{dx} + mx \frac{d^2 \theta}{dx^2} \right)$$

If $\theta$ and $\frac{d\theta}{dx}$ are positive, making the plate convex downwards, the moment $(M + \delta M)$ due to $p_x$ for face $AD$ will be clockwise.

Thus, face $BC$ has a moment $M$ in the counter-clockwise direction, while face $AD$ has a moment $(M + \delta M)$ in the clockwise direction. Hence the net moment $\delta M$ (given by Eq. 20'5) due to radial stresses acting on faces $BC$ and $AD$ will be clockwise and will be opposed to the moment $M_1$ due to circumferential stresses. Hence the internal moment of resistance of the element will be the algebraic sum of the moment $M_1$ and the moment $\delta M (= M_2$ say) given by the following expressions:

$$M_1 = \frac{1}{12} \delta x \delta\phi t^3 \frac{mE}{m^3 - 1} \left( \frac{\theta}{x} + m\frac{d\theta}{dx} \right)$$

(counter-clockwise) ...(20'3)

$$M_2 = \delta M = \frac{1}{12} \delta x \delta\phi t^3 \frac{mE}{m^3 - 1} \left( \frac{\theta}{x} + m\frac{d\theta}{dx} + mx \frac{d^2 \theta}{dx^2} \right)$$

(clockwise) ...(11) (20'3)

203. CIRCULAR PLATE FREELY SUPPORTED AT ITS CIRCUMFERENCE

The resultant of the two moments $M_1$ and $M_2$ is balanced by the external forces, included of reactions, acting on the plate. We shall now consider the following cases of circular plates with uniform pressure in its face:

(1) circular plate freely supported at its circumference,
(2) circular plate freely supported at its circumference, with a central circular hole,
(3) circular plate clamped at its circumference.
Similarly, the shear force $F+\delta F$ on face $AD$ is

$$F+\delta F=3p(x+\delta x)^2 \delta \phi$$

These forces $F$ and $F+\delta F$ have been marked on Fig. 20.2 (c).

Neglecting small quantities of second order, the moment of external forces about an axis in the neutral plane and perpendicular to $EH$ is given by

$$M=F_0 x=\frac{1}{2}0x^2 \delta \phi \delta x$$

The moment is in the clockwise direction.

Thus, there are three moments acting on the element: $M_1$, $M_2$, and $M$. For the equilibrium, the algebraic sum of the three must be equal to zero. Since $M_1$ is in the anti-clockwise direction, while $M_2$ and $M$ are in the clockwise direction, we have

$$M_1-M_2-M=0$$

Substituting the values from Eqs. 20.4, 20.5 and 20.6, we get

$$\delta x \delta \phi \frac{1}{12} \frac{mE}{m-1} \left( m \left( \frac{\hat{\alpha}}{x} + \frac{\hat{\beta}}{x} \right) \right) \delta \phi \delta x - \frac{1}{12} \frac{\delta x \delta \phi \delta \phi}{m-1} \left( \frac{\delta \phi}{x} + \frac{\delta \phi}{x} + m \frac{\delta \phi}{x} \right)$$

Dividing by $\frac{1}{12} \frac{\delta x \delta \phi}{m-1}$, we get

$$\frac{\delta \phi}{x} + \frac{1}{m} \frac{\delta \phi}{x} - \frac{1}{m} \frac{\delta \phi}{x} - \frac{\delta \phi}{x} = \frac{\delta \phi}{E} \frac{m-1}{m} \frac{x^2}{p}$$

or

$$x \frac{d^2 \phi}{dx^2} + \frac{\delta \phi}{x} = \frac{6(m^2-1)}{Em^2} \frac{x^3}{p}$$

Putting $k = \frac{3(m^2-1)p}{Em^3} \phi$, we get

$$x \frac{d^2 \phi}{dx^2} + \frac{\delta \phi}{x} = -2k \frac{x^2}{r^2}$$

The complete solution of the above differential equation is given by Eqs. 20.8 and 20.9 as under:

$$\frac{\delta \phi}{x} = A + B \frac{x^2}{r^2}$$

and

$$\frac{\delta \phi}{x} = A - B \frac{x^2}{r^2}$$

where $A$ and $B$ are constants of integration, to be determined by the boundary conditions.

At $x=0$, $\phi=0$. Hence from Eq. 20.8, $B=0$.

From Eq. 20.2, we have

$$p_x = \frac{mE}{m^2-1} \left( \frac{\delta \phi}{x} + \frac{\delta \phi}{x} \right)$$

Substituting the values of $\frac{\delta \phi}{x}$ and $\frac{\delta \phi}{x}$ from Eqs. 20.8 and 20.9,

$$p_x = \frac{mE}{m^2-1} \left( \frac{1}{4} k \frac{x^2}{r^2} + mA - \frac{3m}{4} k \frac{x^2}{r^2} \right)$$

At $x=r$, $p_x=0$

$$p_x = 0 \Rightarrow A = -\frac{1}{4} k \frac{r^2}{r^2} + mA - \frac{3m}{4} k \frac{r^2}{r^2}$$

or

$$A = \frac{3m+1}{m+1} \cdot k \frac{r^2}{r^2}$$

Substituting the values of $B$ and $A$ in Eqs. 20.8 and 20.9,

$$\frac{\delta \phi}{x} = \frac{k (3m+1) - \frac{x^2}{r^2}}{m+1}$$

and

$$\frac{\delta \phi}{x} = \frac{k (3m+1) - \frac{x^2}{r^2}}{m+1}$$

Substituting these values in Eqs. 20.1 and 20.2, we get

$$p_x = \frac{3}{4} \frac{p_x}{p_{max}} \left( 3m+1 \right) \left( \frac{r^2-x^2}{m+1} \right)$$

and

$$p_y = \frac{3}{4} \frac{p_y}{p_{max}} \left( 3m+1 \right) \left( \frac{r^2-x^2}{m+1} \right)$$

By inspection, both $p_x$ and $p_y$ decrease as $x$ increases. Hence maximum values of $p_x$ and $p_y$ occur at the centre of the plate, where $x=0$, on either side of the plate where $y=\pm \frac{r}{2}$.

$$\left( \frac{p_x}{p_{max}} \right) = \left( \frac{p_y}{p_{max}} \right) = \frac{-3p_{max}r^2}{8t} \cdot \frac{-3m}{m+1}$$

The radial and circumferential strains are found as under

$$e_x = x \frac{d \phi}{dx} = \frac{k}{4} \left( \frac{3m+1}{m+1} - \frac{3x^2}{r^2} \right)$$

and

$$e_z = \frac{\delta \phi}{x} = \frac{k}{4} \left( \frac{3m+1}{m+1} - \frac{x^2}{r^2} \right)$$

These are maximum at $x=0$ and $y=\pm \frac{r}{2}$.

$$\left( e_x \right)_{max} = \left( e_z \right)_{max} = \frac{3}{8} \left( \frac{m-1}{m} (3m+1) \right) p r^2$$
Shear stress distribution

The shear stress distribution across the thickness of the slab can be roughly determined following the method followed in the case of straight beams. The expression for the shear stress at plane distant $y$ from the neutral plane may be expressed as

$$q = \frac{FA y}{bf}$$

where $F =$ shear force on the element $= \frac{1}{2} p x^2 \Delta \phi$

$$I = \frac{1}{12} x \Delta \phi \cdot r^3$$

$$b = x \Delta \phi$$

$$A = (3 \Delta \phi) \left( \frac{t}{2} - y \right)$$

$$y = \left( \frac{1}{2} + \frac{y}{2} \right)$$

$$q = \left( \frac{1}{2} \right) \frac{p x^2 \Delta \phi \left( x \Delta \phi \right) \left( \frac{t}{2} - y \right) \left( \frac{t}{4} + \frac{y}{2} \right)}{\left( x \Delta \phi \right) \left( \frac{1}{12} \frac{1}{x} \Delta \phi \cdot r^4 \right)}$$

or

$$q = \frac{3 p x^2}{4} \left( \frac{t^2}{4} - y^2 \right)$$

Maximum shear occurs at the neutral plane, $y = 0$

$$q_{\text{max}} = \frac{3}{4} \frac{p x^2}{t}$$

$$q_{\text{max}} = \frac{3}{4} \frac{p x}{t}$$

$$At x = r, \quad q_{\text{max}} = \frac{3}{4} \frac{pr}{t}$$

It should be noted that we have neglected the shear stress (Eq. 20'18) while finding out expressions for $p_x$ and $p_z$. In addition to this, we have also neglected the vertical direct compressive stress varying from $p$ at the upper face to zero at the lower face of the plate.

Deflection of the plate

Let $\nu = \text{deflection of the neutral surface, at a radius } x$.

$$\tan \theta = -\frac{dv}{dx}$$

$$\Rightarrow \frac{dv}{dx} = \theta$$

Substituting the value of $\theta$ from Eq. 20'8 (a), we get

$$\frac{dv}{dx} = -\frac{k}{4} \left( \frac{3m+1}{m+1} \right) \left( x - \frac{x^3}{r^2} \right)$$

Integrating,

$$v = -\frac{k}{4} \left( \frac{3m+1}{m+1} \right) \left( \frac{x^2}{2} - \frac{x^4}{4r^2} + c \right)$$

where $c$ is a constant of integration.

At $x = r$, $v = 0 = -\frac{k}{4} \left( \frac{3m+1}{m+1} \right) \left( \frac{r^2}{2} - \frac{r^4}{4r^2} + c \right)$

$$\Rightarrow c = -\frac{r^2}{4} \left( \frac{5m+1}{m+1} \right)$$

Substituting this and the value of $k$ in (2), we get

$$v = -\frac{3}{8} \left( \frac{m-1}{m+1} \right) \left( \frac{3m+1}{m+1} \right) \left( \frac{x^2}{2} - \frac{5m+1}{m+1} \right)$$

This is maximum at $x = 0$

$$v_{\text{max}} = \frac{3}{16} \left( \frac{m-1}{m+1} \right) \frac{p r}{E m^2 r^3}$$

20'8. CIRCULAR PLATE WITH CENTRAL HOLE FREELY SUPPORTED AT ITS CIRCUMFERENCE

Let the plate of radius $r$ have a central hole of radius $r_0$. Let $p$ be the intensity of uniformly distributed load.

The radial stress $p_x$ is given by Eq. 20'2,

$$p_x = \frac{m F - \frac{\partial}{\partial x} + \frac{d \theta}{dx}}{m-1} y$$

Substituting the value of $\frac{\partial}{\partial x}$ and $\frac{d \theta}{dx}$ from Eqs. 20'8 and 20'9,

$$p_x = \frac{m F}{m^2-1} \left( A + \frac{B}{x^2} + \frac{1}{4} k \frac{x^2}{r^2} \right) + m \left( \frac{x}{A - \frac{B}{x^2} - \frac{3}{4} k \frac{x^2}{r^2}} \right)$$

where

$$k = -\frac{3(m^2-1)p r^2}{E m^2 r^3} \quad \text{(Eq. 20'7)}$$

The boundary conditions for the present case are

$$p_x = 0 \text{ at } x = r_0$$

and

$$p_x = 0 \text{ at } x = r.$$
Applying boundary conditions in Eq. (1), we get

\[ A(m+1) + \frac{B}{r_0^3} (1-m) - \frac{k}{4} \left( \frac{r_0^2}{r^2} \right) (1+3m) = 0 \] ... (2)

and

\[ A(m+1) + \frac{B}{r_1^3} (1-m) - \frac{k}{4} \left( \frac{r_1^2}{r^2} \right) (1+3m) = 0 \] ... (3)

Solving (2) and (3), we get

\[ A = \frac{3m+1}{4(m+1)} k \left( 1 + \frac{r_0^2}{r^2} \right) \] ... (4)

and

\[ B = \frac{3m+1}{4(m-1)} k\frac{r_0^2}{r^2} \] ... (5)

Substituting these values of \( A \) and \( B \), and of \( k \), we get the following expressions for \( p_x \) and \( p_z \).

\[ p_x = \frac{3}{4} \frac{3m+1}{mt^3} pr^2 y \left[ 1 + \frac{r_0^2}{r^2} - \frac{r_0^2}{r^2} - \frac{x^2}{r^2} \right] \] ...(20'21)

and

\[ p_z = \frac{3}{4} \frac{pr^2}{mt^3} \left[ (3m+1) r^2 \left( 1 + \frac{r_0^2}{r^2} + \frac{r_0^2}{r^2} \right) -(m+3) x^2 \right] \] ... (20'22)

At \( x = r_0 \), \( p_z \) is given by

\[ p_z = \frac{3}{4} \frac{pr^2}{mt^3} \left[ (3m+1) r^2 \left( 1 + \frac{r_0^2}{r^2} \right) \right] - (m+3) r_0^2 \]

If \( r_0 \) is extremely small, so as to have, only a pin hole at the centre of the slab, the maximum circumferential stress at \( y = \pm \frac{1}{2} r \) is given by

\[ (p_z)_{y = \pm \frac{1}{2} r} = \pm \frac{3}{4} \frac{pr^2}{t^3} \frac{3m+1}{m} \] ... (20'23)

Thus the intensity of stress at the centre is twice that for the plate with no hole (Eq. 28'13).

20'5. CIRCULAR PLATE CLAMPED AT ITS CIRCUMFERENCE

\[ \theta = 0 \text{ at } x = 0 \]

and

\[ \theta = 0 \text{ at } x = r \]

Substituting these in Eq. 20'8, we get

\[ B = 0 \]

and

\[ A = \frac{k}{4} \]

Substituting these values in Eqs. 20'8 and 20'9, we get

\[ \frac{\theta}{x} = \frac{k}{4} \left( 1 - \frac{x^2}{r^2} \right) \] ... (20'28)

and

\[ \frac{db}{dx} = \frac{k}{4} \left( 1 - \frac{3x^2}{r^2} \right) \] ... (20'25)

where \( k = \frac{3(m^2-1)pr^3}{En^2t_3} \) (Eq. 20'7)

Substituting the values of \( \theta \), \( \frac{d\theta}{dx} \), and \( k \) in Eqs. 20'1 and 20'2, we get

\[ p_x = \frac{3}{4} \frac{pr^2}{mt^3} \left\{ (m+1)r^2 - (3m+1)x^2 \right\} \] ... (20'26)

and

\[ p_z = \frac{3}{4} \frac{pr^2}{mt^3} \left\{ (m+1)r^2 - (3m+1)x^2 \right\} \] ... (20'27)

At the centre of the plate \( x = 0 \), the radial and circumferential stresses at \( y = \pm \frac{1}{2} r \) are given by

\[ p_x = p_z = \pm \frac{3}{8} \frac{m+1}{m} \frac{pr^2}{t^3} \] ... (20'28)

At the circumference \( x = r \), the stresses are

\[ p_x = -\frac{3}{2} \frac{pr^2}{t^3} y \] ... (20'29 (a))

and

\[ p_z = -\frac{3}{2} \frac{pr^2}{t^3} y \] ... (20'30 (a))

From Eqs. 20'29 (a) and 20'30 (a), it is evident that the circumferential stress \( p_z \) reaches \( \frac{1}{m} \) of the radial stress \( p_x \) at \( x = r \). The maximum values are given at \( y = \pm \frac{1}{2} r \).

\[ p_x = \mp \frac{3}{4} \frac{r^2}{t^3} \frac{pr^2}{t^3} \] ... (20'30 (a))

and

\[ p_z = \mp \frac{3}{4} \frac{r^2}{mt^3} \frac{pr^2}{t^3} \] ... (20'30 (b))
The greatest intensity of bending stress in the plate is thus the radial stress \( p_x \) at \( x = r \) [Eq. 20'30 (a)].

Similarly, the maximum strain is the radial strain at \( x = r \) and \( y = \pm \frac{r}{2} \), given by

\[
e_{x} = \pm \frac{y}{dx} = \mp \frac{3}{4} \cdot \frac{m^2 - 1}{E m^3} \cdot \frac{r^2}{t} \cdot p \quad \text{...(20'31)}
\]

or alternatively,

\[
e_{x} = \frac{p_x}{E} - \frac{1}{m} \cdot \frac{p_x}{E} = \frac{1}{m} \cdot \frac{p_x}{E} \quad \text{...(20'31 (a))}
\]

\[
\sigma = \frac{p_x}{E} \cdot \frac{m^2 - 1}{m^3} \cdot \frac{r^2}{t} \cdot p
\]

Deflection of the plate

As in article 20'3,

\[
-d \frac{d y}{dx} = 0 = \frac{3}{4} \cdot \frac{(m^2 - 1)p}{Em^3} \cdot (r^2 - x^2)
\]

Integrating,

\[
v = - \frac{3}{4} \cdot \frac{(m^2 - 1)p}{Em^3} \cdot \left( \frac{r^4}{2} - \frac{x^4}{4} + C \right) \quad \text{...(1)}
\]

where \( C \) is the constant of integration.

At \( x = r \), \( v = 0 \);

\[
0 = - \frac{3}{4} \cdot \frac{(m^2 - 1)p}{Em^3} \cdot \left( \frac{r^4}{2} - \frac{r^4}{4} + C \right)
\]

\[
C = - \frac{r^4}{4}.
\]

Substituting in (1), we get

\[
v = - \frac{3}{4} \cdot \frac{(m^2 - 1)p}{Em^3} \cdot \left( \frac{r^4}{2} - \frac{x^4}{4} - \frac{r^4}{4} \right) \quad \text{...(20'32)}
\]

The maximum deflection occurs at \( x = 0 \).

\[
v_{max} = \frac{3}{4} \cdot \frac{(m^2 - 1)p}{Em^3} \cdot \frac{r^4}{4} = \frac{3}{16} \cdot \frac{(m^2 - 1)}{Em^3} \cdot pr^4 \quad \text{...(20'33)}
\]

Example 20'1. A cylinder 500 mm internal diameter has a flat end 30 mm thick. Find the greatest intensity of stress in the end if the pressure in the cylinder is 1 N/mm². The end may be taken as freely supported.

Also, find what intensity of simple direct stress would produce (i) the same maximum strain, (ii) the same maximum strain energy and (iii) the same maximum shear strain energy. Take \( m = 3 \).

FLAT CIRCULAR PLATES

Solution.

The greatest intensity of stress is given by Eq. 20'13,

\[
(p_x)_{max} = (p_z)_{max} = \frac{3p_x^2}{8t^2} \left( \frac{3m+1}{m} \right)
\]

\[
= \frac{3 \times 1(250)^3}{8(30)^2} \left( \frac{3 \times 3 + 1}{3} \right)
\]

\[
= 86.8 \text{ N/mm}^2.
\]

(i) Simple stress to produce the same maximum strain

Let

\[
p = \text{simple stress}
\]

\[
e = \text{maximum strain}
\]

\[
p = \frac{p_x - p_z}{m} = px \left( 1 - \frac{1}{m} \right)
\]

Since

\[
p = 86.8 \text{ N/mm}^2.
\]

\[
p = 86.8 \text{ N/mm}^2.
\]

(ii) Simple stress to produce the same shear strain energy

\[
\frac{p^2}{2E} = 2E \left( \frac{p_x^2 + p_z^2 - 2p_x p_z}{m} \right)
\]

or

\[
p = \sqrt{p_x^2 + p_z^2 - 2p_x p_z} \quad \text{=} \quad px \sqrt{2 \left( 1 - \frac{1}{m} \right)}
\]

\[
= 86.8 \sqrt{2(1-1)} = 106.2 \text{ N/mm}^2.
\]

(iii) Simple stress to produce the same shear strain energy

\[
\frac{m+1}{3mE} = \left( \frac{m+1}{3mE} \right)^2 \quad \text{=} \quad px \left( 1 - \frac{1}{m} \right)
\]

or

\[
p = 86.8 \text{ N/mm}^2.
\]

Example 20'2. Solve Example 20'1 if the flat end is assumed to be fixed at the edges.

Solution.

The maximum radial stress is given by Eq. 20'29 (b),

\[
p_x = \frac{3r^4}{4} \quad \text{and} \quad p_z = \frac{3(250)^3}{4} \times 1.0 = 52 \text{ N/mm}^2
\]

\[
p = \frac{1}{m} \quad \text{=} \quad 17.4 \text{ N/mm}^2.
\]

(a) Simple stress to produce the same maximum strain

\[
p = \frac{p_x - p_z}{m} = 52 - \frac{17.4}{3} = 46.2 \text{ N/mm}^2.
\]
21

Unsymmetrical Bending

21.1. INTRODUCTION

In the simple theory of bending using the well known flexure formula \( \frac{M}{I} = \frac{f}{y} \), it is assumed that the neutral axis of the cross-section of the beam is perpendicular to the plane of loading. This condition implies that the plane of loading, or the plane of bending, is coincident with, or parallel to, plain containing a principal centroidal axis of inertia of the cross-section of the beam.
beam. If, however, the plane of loading or that of bending, does not lie in (or parallel to) a plane that contains the principal centroidal axes of the cross-section, the bending is called *unsymmetrical bending*. Fig. 21:1 shows some cases of unsymmetrical bending in which the plane of load *W* is vertical and do not coincide with the principal centroidal axes *UU* and *VV*. In the case of unsymmetrical bending, the direction of the neutral axis will not be perpendicular to the plane of bending.

21'2. CENTROIDAL PRINCIPAL AXES OF A SECTION

The centroidal principal axes of a section are defined as a pair of rectangular axes through the centre of gravity of a plane area such that the product of inertia is zero.

Let *U-U*, *V-V*=Principal centroidal axes  

*X-X*, *Y-Y*=Any pair of centroidal rectangular axes  

\( \alpha = \text{angle between } U-U \text{ and } X-X \text{ axes (Fig. 21'2).} \)

![Fig. 21'2. Principal axes.](image)

ON-SYMMETRICAL BENDING

If *U-U*, *V-V* are the principal axes, the product of inertia \( \Sigma u.v.8a=0 \), where \( 8a \) is an elementary area with co-ordinates \( u \) and \( v \) referred to the principal axes. If a plane area has an axis of symmetry, it is obviously a principal axis, since the axis of symmetry has to satisfy the condition \( \Sigma u.v.8a=0 \) about it. In general, however, a plane area may not have any axis of symmetry. In that ease the principal axes may be located provided its properties about any pair of rectangular axes *X-X*, *Y-Y* are known.

Let \( x, y \) be the co-ordinates of an elementary area \( 8a \), with respect to the *X-Y* axes, and \( u, v \) be the corresponding co-ordinates with respect to the principal axes *U-V*.

By definition, \( I_{ux}=\Sigma u^2\,8a \); \( I_{uy}=\Sigma u\,v\,8a \); \( I_{xy}=\Sigma x\,y\,8a \)

Similarly, \( I_{ux}=\Sigma u^2\,8a \); \( I_{uy}=\Sigma u\,v\,8a \); \( I_{xy}=\Sigma u.v\,8a \)

The relationships between \( x, y \) and \( u, v \) co-ordinates are  

\[
\begin{align*}
  u &= x \cos \alpha + y \sin \alpha \\
  v &= x \sin \alpha - y \cos \alpha 
\end{align*}
\]

Hence  

\[
\begin{align*}
  I_{ux} &= \Sigma (y \cos \alpha - x \sin \alpha)^2 \,8a \\
  &= \cos^2 \alpha \Sigma y^2 \,8a + \sin^2 \alpha \Sigma x^2 \,8a \\
  &= I_{xx} \cos^2 \alpha + I_{yy} \sin^2 \alpha - I_{xy} \sin 2\alpha \\
  I_{uv} &= \Sigma (x \cos \alpha + y \sin \alpha)^2 \,8a \\
  &= \sin^2 \alpha \Sigma y^2 \,8a + \cos^2 \alpha \Sigma x^2 \,8a \\
  &= I_{xx} \sin^2 \alpha + I_{yy} \cos^2 \alpha + I_{xy} \sin 2\alpha \\
  I_{uv} &= \Sigma u^2 \,8a - \Sigma x \,y \,8a \\
  &= I_{xx} \cos \alpha \sin \alpha \Sigma y^2 \,8a - \Sigma x \,y \,8a \\
  &= I_{xx} - I_{yy} \\
  &= \sum (x \cos \alpha + y \sin \alpha)(y \cos \alpha - x \sin \alpha) \,8a \\
  &= \cos^2 \alpha \Sigma x \,y \,8a - \sin^2 \alpha \Sigma x \,y \,8a \\
  &= \cos \alpha \sin \alpha (\Sigma y^2 \,8a - \Sigma x^2 \,8a) \\
  &= (I_{xx} - I_{yy}) \sin 2\alpha + I_{xx} \cos 2\alpha \\
  I_{uv} &= \Sigma u \,v \,8a \\
  &= (I_{xx} - I_{yy}) \sin 2\alpha + I_{xx} \cos 2\alpha \\
  \text{or} \quad \tan 2\alpha &= -\frac{2I_{xy}}{I_{xx} - I_{yy}} \\
  \text{Knowing } I_{xx}, I_{yy} \text{ and } I_{xy}, \text{ the angle } \alpha \text{ can be calculated from Eq. 21'4.} \\
  \text{Substituting } \alpha \text{ in Eqs. 21'1 and 21'2, the moment of inertia about the principal axes can be determined.}
Analytical Solution

Analytical expressions for \( I_{uu} \) and \( I_{vv} \) can be derived by rewriting Eqs. 21'1 and 21'2 in the following alternative forms:

\[
I_{uu} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2} \cos 2\alpha - I_{xy} \sin 2\alpha \quad \ldots \ldots \text{(21'1 (a))}
\]

and

\[
I_{vv} = \frac{I_{xx} + I_{yy}}{2} - \frac{I_{xx} - I_{yy}}{2} \cos 2\alpha + I_{xy} \sin 2\alpha \quad \ldots \ldots \text{(21'2 (a))}
\]

Also, from Eq. 21'4, we have

\[
\sin 2\alpha = \frac{-I_{xy}}{\sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}} \quad \ldots \ldots \text{(21'4 (a))}
\]

and

\[
\cos 2\alpha = \frac{I_{xx} - I_{yy}}{\sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}} \quad \ldots \ldots \text{(21'4 (b))}
\]

Substituting the values of \( \sin 2\alpha \) and \( \cos 2\alpha \) in Eqs. 21'1 (a) and 21'2 (b), we get the following final expression for \( I_{uu} \) and \( I_{vv} \):

\[
I_{uu} = \frac{I_{xx} + I_{yy}}{2} + \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2} \quad \ldots \ldots \text{(21'5)}
\]

\[
I_{vv} = \frac{I_{xx} + I_{yy}}{2} - \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2} \quad \ldots \ldots \text{(21'6)}
\]

Thus knowing \( I_{xx}, I_{yy} \) and \( I_{xy} \), the principal moments of inertia \( I_{uu} \) and \( I_{vv} \) can be calculated from the above analytical expressions. It should be noted that the moments of inertia of a section about its principal axes have maximum and minimum values respectively.

21'3. GRAPHICAL METHOD FOR LOCATING PRINCIPAL AXES

Eqs. 21'5 and 21'6 can also be solved by the following graphical methods:

1. MOHR CIRCLE

A close inspection of Eqs. 21'5 and 21'6 would reveal that the expressions for \( I_{uu} \) and \( I_{vv} \) are similar (or analogous) to the following well known expressions for the principal stresses:

\[
\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}
\]

\[
\tan \theta = \frac{2\tau}{\sigma_x - \sigma_y}
\]

Hence \( I_{uu} \) and \( I_{vv} \), represented by Eqs. 21'5 and 21'6 can be determined by a Mohr-circle of construction—similar to the Mohr stress circle employed for the determination of the principal stresses.

Fig. 21'3 shows the Mohr-circle construction for principal axes and principal moment of inertia, wherein:

\[
OA = I_{xx} ; OB = I_{yy} \\
AD = -I_{xy} \text{ or } BD' = +I_{xy}.
\]

Fig. 21'3. Mohr-circle construction for principal axes.

Bisect \( AB \) at \( C \). With \( C \) as centre and \( CD \) (or \( CD' \)) as radius, draw the circle cutting the horizontal axis \( OC \) at \( E \) and \( F \). Then

\[
\angle ACD = 2\alpha \text{ and } OE \text{ and } OF \text{ represent the minimum and maximum principal moment of inertia.}
\]

**Proof.** From Fig. 21'3, we have

\[
OC = \frac{1}{2} \left( I_{xx} + I_{yy} \right)
\]

\[
CA = \frac{1}{2} \left( I_{xx} - I_{yy} \right)
\]

\[
OE = I_{vv} = OC - EC = OC - D'C
\]

\[
= \frac{1}{2} \left( I_{xx} - I_{yy} \right) \sqrt{I_{xx} - I_{yy}^2}
\]
Similarly,
\[
OF = I_{xy} = OC + CF = OC + CD
\]
\[
= \frac{1}{2} \left( I_{xx} + I_{yy} \right) + \sqrt{\left( \frac{I_{xx} - I_{yy}}{2} \right)^2 + I_{xy}^2}.
\]

These are the same as Eqs. 21'5 and 21'6 respectively. It should be noted that, if \( I_{xy} \) is plotted below the line \( OA \) if it is negative (such as line \( AD \)), and plotted above the line \( OA \) if it is positive. If \( I_{xy} \) is negative, the \( X-X \) axis makes an angle \( 2\alpha = \angle ACD \) with \( U-U \) axis in the clockwise direction (or \( U-U \) axis is inclined at \( 2\alpha \) with \( X-X \) axis in anticlockwise direction). Similarly, if \( I_{xy} \) is positive, \( \angle BCD = 2\alpha \) is the angle with the \( U-U \) axis makes with the \( X-X \) axis in the clockwise direction. In general, therefore, the direction of the principal axis is given by the angle measured from the inclined radial line (such as \( CD \) or \( CD' \)) towards the horizontal line \( CF \) or \( CB \), as the case may be.

2. CIRCLE OF INERTIA

An alternative graphical method to determine \( I_{xy} \) and \( I_{xy} \) is to construct what is commonly known as 'circle of inertia', 'dyadic circle' or 'Mohr-Land construction' (Fig. 21'4).

Let \( O \) be the centroid of the section and \( X-X, Y-Y \) be any set of rectangular axes passing through it. Make \( OA = I_{xx} \) and \( AB = I_{yy} \). Draw a circle with \( BO \) as the diameter. Hence \( BC = CO = \frac{1}{2} (I_{xx} + I_{yy}) \) where \( C \) is the centre of the circle. At \( A \), erect perpendicular \( AD \) to the right if \( I_{xy} \) is positive, or to the left if \( I_{xy} \) is negative. Join \( C \) and \( D \) and prolong it to meet the circle in \( U \) and \( V \). Join \( OU \) and \( OV \). Then \( OU \) is the \( U \)-axis and \( OV \) is the \( V \)-axis. Also, \( UD = I_{xy} \) and \( DV = I_{xy} \).

**Proof.**
\[
CA = CB - AB = \frac{1}{2} \left( I_{xx} + I_{yy} \right) - I_{xy} = \frac{1}{2} \left( I_{xx} - I_{xy} \right)
\]
\[
AD = I_{xy}
\]
\[
CD = \sqrt{\left( \frac{I_{xx} - I_{yy}}{2} \right)^2 + I_{xy}^2}
\]
\[
UD = UC + CD
\]
or
\[
UD = \frac{I_{xx} + I_{yy}}{2} + \sqrt{\left( \frac{I_{xx} - I_{yy}}{2} \right)^2 + I_{xy}^2} = I_{xy}
\]
and
\[
DV = CV - CD
\]
or
\[
DV = \frac{I_{xx} + I_{yy}}{2} - \sqrt{\left( \frac{I_{xx} - I_{yy}}{2} \right)^2 + I_{xy}^2} = I_{xy}
\]
Also, \( \tan 2\alpha = AD \frac{I_{xy}}{AC} = \frac{I_{xy}}{\frac{1}{2} (I_{xx} - I_{yy})} = \frac{2I_{xy}}{I_{xx} - I_{yy}} \) (numerically).

21'4. MOMENTS OF INERTIA REFERRED TO ANY SET OF RECTANGULAR AXES

![Fig. 21'4. Circle of Inertia (Dyadic Circle).](image-url)
The discussions of § 21'2 and § 21'3 can now be generalised to find the moments of inertia referred to any set of rectangular axes $X'-Y'$ inclined at $\beta$ to the principal centroidal axes. Refer Fig. 21'5.

Consider an elementary area $\delta a$. Let its co-ordinates be $u, v$ with respect to $U-V$ axes and $x'-y'$ with respect to $X'-Y'$ axes.

Then
\[ x' = u \cos \beta + v \sin \beta \]
\[ y' = v \cos \beta - u \sin \beta \]

Now
\[ I_{x'} = \Sigma y'^2 \delta a = \Sigma (u \cos \beta + v \sin \beta)^2 \delta a \]
\[ = \cos^2 \beta \Sigma u^2 \delta a + \sin^2 \beta \Sigma v^2 \delta a - 2 \sin \beta \cos \beta \Sigma uv \delta a \]
\[ = I_{uu} \cos^2 \beta + I_{vv} \sin^2 \beta \] (21'7)

(Since $I_{uv} = \Sigma uv \delta a = 0$)

Similarly,
\[ I_{y'} = \Sigma x'^2 \delta a = \Sigma (u \cos \beta + v \sin \beta)^2 \delta a \]
\[ = \cos^2 \beta \Sigma u^2 \delta a + \sin^2 \beta \Sigma v^2 \delta a + 2 \sin \beta \cos \beta \Sigma uv \delta a \]
\[ = I_{uu} \sin^2 \beta + I_{vv} \cos^2 \beta \] (21'8)

Adding Eqs. 21'7 and 21'8, we get
\[ I_{x'} + I_{y'} = I_{uu} + I_{vv} \]

Also, from adding Eqs. 21'1 and 21'2, we get
\[ I_{uu} + I_{vv} = I_{xx} + I_{yy} \]

Hence
\[ I_{xx} + I_{yy} = I_{x'} + I_{y'} = I_{uu} + I_{vv} \] (21'9)

Thus the sum of moments of inertia about any set of rectangular axes is constant.

Example 21'1. Determine the principal moments of inertia for an unequal-angle section 60 x 40 x 6 mm shown in Fig. 21'6.

Solution.

Let $O$ be the centroid of the section. Let the $X$-axis be at a distance $Cx$ from face $PQ$, and $Y$-axis be at a distance $Cy$ from face $PR$.

Area
\[ A = A_1 + A_2 = (40 \times 6) + (54 \times 6) \]
\[ = 240 + 324 = 564 \text{ mm}^2 \]

\[ C_x = \frac{(40 \times 6 \times 3) + (54 \times 6 \times 33)}{564} \]
\[ = 20.2 \text{ mm} \]

\[ C_y = \frac{(40 \times 6 \times 20) + (54 \times 6 \times 2)}{564} \]
\[ = 10.2 \text{ mm} \]

\[ I_{xx} = \frac{(4 \times 6 \times 60^3) + (4 \times 6 \times 34^3)}{2} + 43'44 \times 10^4 \text{ mm}^4 \]
\[ I_{yy} = \frac{(4 \times 6 \times 34^3) + (4 \times 6 \times 60^3)}{2} + 43'44 \times 10^4 \text{ mm}^4 \]
\[ J = I_{xx} + I_{yy} = 13'19 \times 10^4 - 5 \text{ mm}^4 \]
\[ J = 7.33 \times 10^4 \text{ mm}^4 \]

and
\[ J_{xy} = A_1 x_1 y_1 + A_2 x_1 y_2 \]

where $(x_1, y_1)$ are the co-ordinates of C.G. of area $A_1$ and $(x_2, y_2)$ are the co-ordinates of C.G. of area $A_2$.

From Fig. 21'4, the positions of principal axes are given by

\[ \tan 2\alpha = \frac{2J_{xy}}{I_{xx} - I_{yy}} \]
\[ = \frac{2 \times 704 \times 10^4}{(20'34 - 7'33)10^4} = 1.035 \]

\[ 2\alpha = 47^\circ \ 20' \]

or
\[ \alpha = 23^\circ \ 40' \ \text{(anticlockwise)} \]
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\[
\frac{I_{xx} + I_{yy}}{2} = \frac{(20'34 + 7'33)10^4}{2} = 13'84 \times 10^4
\]

\[
\frac{I_{xx} - I_{yy}}{2} = \frac{(20'34 - 7'33)10^4}{2} = 6'55 \times 10^4
\]

\[
\sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 - I_{xy}^2} = \sqrt{(6'55 \times 10^4)^2 + (-7'04 \times 10^4)^2} = 9'58 \times 10^4
\]

Hence from Eqs. 21'5 and 21'6,

\[
I_{uv} = 13'84 \times 10^4 + 9'58 \times 10^4 = 23'42 \times 10^4 \text{ mm}^4
\]

\[
I_{vv} = 13'84 \times 10^4 - 9'58 \times 10^4 = 4'26 \times 10^4 \text{ mm}^4
\]

Check

\[
I_{xx} + I_{yy} = I_{uv} + I_{vv}
\]

\[
(20'34 \times 10^4 + 7'33 \times 10^4) = (23'42 \times 10^4 + 4'26 \times 10^4)
\]

\[
27'67 \times 10^4 = 27'68 \times 10^4
\]

Example 21'2. Find \(I_{uv}\) and \(I_{vv}\) graphically by Mohr-circle method, for the data of previous problem.

Solution.

\[
I_{xx} = 20'34 \times 10^4 \text{ mm}^4; \ I_{yy} = 7'33 \times 10^4 \text{ mm}^4; \ I_{xy} = -7'04 \times 10^4 \text{ mm}^4
\]

as calculated in the previous problem.

Example 21'3. Determine \(I_{uv}\) and \(I_{vv}\) graphically using the dyadic circle method.

Solution. (Fig. 21'8)

Let \(O\) be the centroid of the section.

Make \(OA = I_{xx} = 20'34 \times 10^4\) and \(AB = I_{yy} = 7'33 \times 10^4\) mm.

At \(A\), erect perpendicular \(AD = I_{xy} = 7'04 \times 10^4\) to the left side. From point \(C\) as centre and \(CB\) as radius, draw the dyadic circle. Join \(CD\) and prolong it to both the sides, cutting the circle in \(U\) and \(V\).

Then, by measurement,

\[
UD = I_{uv} = 23'4 \times 10^4 \text{ mm}^4; \ DV = I_{vv} = 4'3 \times 10^4 \text{ mm}^4
\]

and \(\alpha = \angle XOY = 23^\circ 40'.\)
21'5. BENDING STRESS IN BEAM SUBJECTED TO UNSYMMETRICAL BENDING

In the case of simple bending, where the plane of loading (or bending) coincides with one of the principal plane, the neutral axis is perpendicular to the principal plane and passes through the centroid of the section. In the case of unsymmetrical bending, neutral axis is not perpendicular to the plane of bending. The bending stress at any point in the beam subjected to unsymmetrical bending can be determined by following methods:

1. Resolution of bending moment into two components along principal axes.
2. Resolution of bending moment into two components along any rectangular axes through the centroid.
3. Rotating neutral axis of the section.

21'6. RESOLUTION OF BENDING MOMENT INTO TWO COMPONENTS ALONG PRINCIPAL AXES

Let the plane of bending \( M \) be inclined at an angle \( \theta \) with one of the principal planes. The bending moment \( M \) can be resolved into components: \( M \cos \theta \) along plane \( V-V \) and \( M \sin \theta \) along the plane \( U-U \). Having resolved the bending moments in the two components the simple theory of bending can then be applied to bend-

\[
f_b = \frac{M \cos \theta}{I_{uu}} \cdot u + \frac{M \sin \theta}{I_{vv}} \cdot v \quad \ldots (21'10)
\]

The co-ordinates \( u \) and \( v \) will be positive in that quadrant of \( U-V \) planes in which bending moment is applied. Thus, \( u \) is positive in quadrants I and II, while \( v \) is positive in quadrants I and IV. Since the co-ordinates of any extreme point of section are known, the bending stress can be calculated. It should be noted that the component \( M \cos \theta \) causes compression for all points above \( U-U \) axis and tension at points below \( U-U \) axis. Similarly, \( M \sin \theta \) causes compression for points to the left of \( V-V \) axis and tension for the points to the right of \( V-V \) axis. Hence the points of quadrant I are subjected to a resultant bending stress which is wholly compressive, while those in quadrant III to wholly tensile stress. Angle \( \theta \) is taken to be positive when measured in an anticlockwise direction with the +ve \( V \)-axis.

21'7. RESOLUTION OF B.M. INTO ANY TWO RECTANGULAR AXES THROUGH THE CENTROID

The most general method of finding the bending stress at any point is to resolve it along any two rectangular axes passing through the centroid of the section. Let \( X-X \) and \( Y-Y \) be the centroidal axes (Fig. 21'10).

The resolved component of \( M \) along the \( Y-Y \) axis (also called as the moment about \( X-X \) axis) is designated as \( M_{XY} \) and is equal to \( M \cos \theta \). Similarly, the resolved component of \( M \) along \( X-X \) axis
Substituting the values of \( a_1 \) and \( b_1 \) in Eq. 21.11, we get
\[
f_s = \frac{M_{yy}}{I_{xx}} f_{x} - \frac{M_{xx}}{I_{yy}} f_{y} + \frac{M_{xy}}{I_{yy} I_{xx} - I_{xy}^2} \cdot x + \frac{M_{yx}}{I_{xx} I_{yy} - I_{xy}^2} \cdot y
\]
...(21.12)

Thus the bending stress \( f_s \) can be calculated at any point whose co-ordinates \((x, y)\) are known. The method is specially suitable for sections in which the web and flanges are parallel to \( x-x \) and \( y-y \) axes.

21'8. LOCATION OF NEUTRAL AXIS

As stated earlier in the case of unsymmetrical bending, the neutral axis is neither perpendicular to the plane of bending, nor perpendicular to any of the principal planes.

Let \( \theta \) = Inclination of the plane of bending to the \( V-V \) axis.
\( \beta = \) Inclination of the neutral axis, with the \( U-U \) axis.

The neutral axis can be located by two methods:
1. Analytical method
2. Graphical method: Momental ellipse.

Analytical Method

Fig. 21'9 shows the neutral axis \( N-N \), inclined at an angle \( \beta \) with the \( U-U \) axis. At any point (such as \( P \)) on it, the bending stress is equal to zero. Hence equating Eq. 21'10 to zero, we get
\[
f_s = 0 = \frac{M_{yy}}{I_{uu}} v + \frac{M_{xx}}{I_{vv}} u
\]
\[
\therefore \quad v = -u \frac{I_{uu}}{I_{vv}} \tan \theta
\]
...(21'13)

Eq. 21'13 is the equation of the neutral axis \( N-N \) which is a straight line. It is clear that when \( v = 0, u = 0 \); hence the neutral axis passes through the centroid of the section.

From Fig. 21'9, tan \( \beta = \frac{v}{u} \)
\[
\text{But} \quad \frac{v}{u} = \frac{I_{uu}}{I_{vv}} \tan \theta, \text{ from Eq. 21'13.}
\]
Hence
\[
\tan \beta = \frac{I_{uu}}{I_{vv}} \tan \theta
\]
...(21'14)

Thus the N.A. can be located from Eq. 21'14.

Let \( I_{NN} \) = moment of inertia of the beam about the neutral axis.
Thus, from Eq. 21'7, treating 'x'-x' axis as the neutral axis, we have
\[ I_{NN} = I_{UU} \cos^2 \beta + I_{VV} \sin^2 \beta \] ... (21'15)

The neutral axis is inclined at \( \beta \) with the U-axis, while the plane of loading is inclined at \( \theta \) with the V-axis. Hence, the plane of loading is inclined at angle \( (90-\theta+\beta) \) with the neutral axis. If a line is drawn perpendicular to the neutral axis, the plane of bending will be inclined at \( (\beta-\theta) \) to the line. Hence the component of bending moment \( M \) along the axis will be given by
\[ M_{NN} = M \cos (\beta-\theta) \] ... (21'16)

where \( M_{NN} \) = component of bending moment along a line perpendicular to the neutral axis
\( = \) bending moment about the neutral axis.

If \( y_n \) = perpendicular distance of any point from the neutral axis, we have
\[ f_n = \frac{M \cos (\beta-\theta)}{I_{NN}} y_n \] ... (21'17)

The bending stress \( f_n \) will be positive or negative depending upon the position of the point relative to the neutral axis and the direction of bending.

21'9. GRAPHICAL METHOD: MOMENTAL ELLIPSE

From Eq. 21'9, we have
\[ k'x'^2 + k'y'^2 = I_{UU} + I_{VV} \]

This may be written in terms of the radii of gyration as under:
\[ kxx'^2 + kyy'^2 = k_{UU}^2 + k_{VV}^2 \] ... (21'18)

If \( k_{UU} \) and \( k_{VV} \) are known, \( kxx' \) and \( kyy' \) can be determined graphically by the construction of momental ellipse or ellipse of inertia. Refer Fig. 21'11.

Set off the principal axes \( UU \) and \( VV \) through the centroid \( O \) of the section. Draw the inner circle with radius \( OA = k_{VV} \) (assuming \( k_{VV} < k_{UU} \)) and outer circle with radius \( OB = k_{UU} \). Set off axes \( OX' \) and \( OY' \) at inclination \( \theta \) with \( OU \) and \( OV \) respectively. \( OX' \) cuts the circles at \( C \) and \( D \). Through \( C \) and \( D \), draw lines \( CP \) and \( DP \) parallel to \( OV \) and \( OU \) respectively, meeting in a point \( P \) which is a point of an ellipse. Change the value of \( \theta \) (by rotation of axis \( x' \)) and \( y' \) and get a number of such points \( P \). Join them to get an ellipse. From Fig. 21'11 (a),
\[ OP^2 = OE^2 + EP^2 \]

But
\[ OE = OD \sin \theta = k_{UU} \sin \theta \]
and
\[ EP = CF = OC \cos \theta = k_{VV} \cos \theta \]
\[ OP = k_{UU} \sin \theta + k_{VV} \cos \theta \] ... (1)
But, from Eq. 21'17,
\[ k'vy' = ku \sin^2 \theta + k'v \cos^2 \theta \]
or
\[ k'vy'^2 = ku^2 \sin^2 \theta + k'v^2 \cos^2 \theta \]  \hspace{1cm} (2) \hspace{1cm} (21'19)
Comparing (1) and (2), we find
\[ OP = k'v \]
Again, the co-ordinates of points \( P \) are
\[ u = EP = k'v \sin \theta \]  \hspace{1cm} (3)
\[ v = OE = ku \sin \theta \]  \hspace{1cm} (4)
From (3) and (4), we get
\[ \frac{u^2}{ku^2} + \frac{v^2}{kv^2} = \cos^2 \theta + \sin^2 \theta = 1 \]  \hspace{1cm} (21'20)
This is the equation of an ellipse having \( ku \) and \( k'v \) as its semi-major and semi-minor axes.

Fig. 21'11 (b) shows the complete ellipse. In order to find graphically the value of \( k'v' \) corresponding to any value \( \theta \), the axis \( OY' \) is set off at \( \theta \) with the \( OV \) axis. A tangent \( OH \) is then drawn to the ellipse, parallel to the \( OY' \) axis. A line \( OH \) is drawn perpendicular to the tangent. It can be shown that \( OH \) is equal to \( k'v' \).

Proof. Let \( m \) = slope of line \( OY' \) with \( OU \)
\[ = \tan (90 + \theta) = -\cot \theta \]
If the equation of an ellipse is
\[ \frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \]
to its tangent, with a slope \( m \), is given by
\[ v = mu \pm \sqrt{b^2 + a^2 m^2} \]
For the present case, \( m = -\cot \theta ; a = ku \) and \( b = k'v \)
\[ v = -u \cot \theta \pm \sqrt{ku^2 + k'v^2 \cot^2 \theta} \]  \hspace{1cm} (21'21)
The perpendicular distance \( OH \) from \( O \) to this tangent is given by
\[ OH = \frac{\pm \sqrt{h^2 + r^2 m^2}}{\sqrt{1 + m^2}} \]
\[ = \frac{\pm \sqrt{ku^2 + k'v^2 \cot^2 \theta}}{\sqrt{1 + \cot^2 \theta}} \]
\[ OH^2 = ku^2 + k'v^2 \cot^2 \theta \]
\[ \therefore \]
\[ OH = ku \sin^2 \theta + k'v \cos^2 \theta = k'v' \]
But \( ku \sin^2 \theta + k'v \cos^2 \theta = k'v' \)
\[ \therefore \]
\[ OH = k'v' \].

Hence, we draw a very important conclusion: To find the radius of gyration about any axis, draw a tangent to the momental ellipse, in a direction parallel to that axis. Then the perpendicular distance between the tangent and the origin gives the required radius of gyration.

The above conclusion will now be utilized to find the radius of gyration \( k_{SN} \) about the neutral axis \( N-N \). To do this, we must first locate the position of the neutral axis.

Fig. 21'12 shows the momental ellipse. Let \( OM \) represent the plane of loading inclined at an angle \( \theta \) to the \( OV \) axis. Let \( ON \) be the direction of the neutral axis, inclined at \( \beta \) to the \( OU \) axis. It is first required to determine the position of the neutral axis \( ON \) graphically.

From Eq. 21'14, we have
\[ \tan \beta = \frac{lu}{k'v} \tan \theta \]
\[ \therefore \]
\[ \tan \beta = \frac{ku^2}{k'v^2} \tan \theta \]  \hspace{1cm} (21'22)
\[ \therefore \]
\[ \tan \beta \cot \theta = \frac{ku^2}{k'v^2} \]  \hspace{1cm} \([21'22 (a)] \)
Now the slope $m$ of the line $ON = \tan \beta$.
Slope $m'$ of the line $OM = \tan (90 + \theta) = -\cot \theta$

$$m' = -\tan \beta \cot \theta$$

But $\tan \beta \cot \theta = \frac{k_{uu}}{k_{vv}}$, from Eq. 21'22 (a).

Hence

$$m' = -\frac{k_{uu}}{k_{vv}} \quad \ldots (21'23)$$

Eq. 21'23 suggests that the lines $OM$ and $ON$ are the two conjugate diameters of the ellipse. If an ellipse has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

then, from the property of the ellipse, the product $mm'$ of the slopes of any two conjugate diameters is given by

$$mm' = \frac{b^2}{a^2} = -\frac{k_{uu}}{k_{vv}}$$

Thus it is concluded that the neutral axis is in a direction of a diameter which is conjugate to the diameter in the direction loading.

Again, if the direction of any diameter $OM$ of the ellipse is known, the direction of any diameter conjugate to it can be drawn by drawing any diameter $M'M'$ parallel to $OM$, bisecting it at $J$, and joining $O$ and $J$. Thus $OJ$ prolonged gives the direction $ON$ of the neutral axis.

Having located the neutral axis $ON$, a line $N'N'$ is drawn tangentially to the ellipse and parallel to the neutral axis $ON$. The perpendicular $OA$ then gives the radius of gyration $k_{NN}$ about the neutral axis.

Knowing the radius of gyration $k_{NN}$, the moment of inertia $I_{NN}$ about the neutral axis is calculated from the relation

$$I_{NN} = A.k_{NN}^2 \quad \ldots (21'24)$$

where $A =$ area of cross-section of the section.

The bending stress $f_b$ at any point is then calculated from Eq. 21'17,

$$f_b = \frac{M \cos (\beta - \theta)}{l_{NN}} \cdot y_N.$$

Example 21'4. A beam of rectangular section, 80 mm wide and 120 mm deep is subjected to a bending moment of 12 kNm. The trace of the plane of loading is inclined at 45° to the Y-Y axis of the section. Locate the neutral axis of the section and calculate the maximum bending stress induced in the section.

Let the plane of loading (bending) be inclined at an angle $\theta$ with a Y-Y axis and the neutral axis $N'$ be inclined at $\beta$ with the X-X axis.

$\theta = 45^\circ$ (Given)

$M = 12000 \times 1009 = 12 \times 10^6$ N/mm.

$I_{XX} = I_{YY} = \frac{1}{12}bd^3 = \frac{1}{12} \times 80 \times 120^3 = 11,520 \times 10^6$ mm$^4$

$I_{YY} = I_{VV} = \frac{1}{12}bh^3 = \frac{1}{12} \times 120 \times 80^3 = 5,120 \times 10^6$ mm$^4$

From Eq. 29'14, the inclination $\gamma$ of the N.A. is given by

$$\tan \beta = \frac{I_{XX}}{I_{VV}} \tan \theta = \frac{11,520 \times 10^6}{5,120 \times 10^6} \times \tan 45^\circ = 2.25.$$

$$\therefore \beta = 66^\circ$$

This gives the location of the neutral axis.

By inspection, maximum stress will occur either at $B$ or at $D$, whichever is more distant from the N.A.
The stress is given by Eq. 21.10
\[ f_x = \frac{M \cos \theta}{I_{XX}} \cdot x + \frac{M \sin \theta}{I_{YY}} \cdot y \]
\[ f_y = \frac{M \cos \theta}{I_{YY}} \cdot y + \frac{M \sin \theta}{I_{XX}} \cdot x \]
where \( x \) and \( y \) are the coordinates of the point. The coordinates \( x \) and \( y \) will be positive in that quadrant of \( X-Y \) plane in which bending moment is applied. From this point of view, both the coordinates of point \( D \) will be positive, while those of point \( B \) will be negative.

Thus, for point \( B \), \( x = -40 \) and \( y = -60 \)
For point \( D \), \( x = +40 \) and \( y = +60 \)
\[ (f_x)_B = -\frac{12 \times 10^6 \cos 45^\circ}{11'52 \times 10^4} \times 60 - \frac{12 \times 10^6 \sin 45^\circ}{5'12 \times 10^4} \times 40 = -110'5 \text{N/mm}^2 \]
(i.e. tensile)
\[ (f_y)_D = +\frac{12 \times 10^6 \cos 45^\circ}{11'52 \times 10^4} \times 60 + \frac{12 \times 10^6 \sin 45^\circ}{5'13 \times 10^4} \times 40 = +110'5 \text{N/mm}^2 \]
(i.e. compressive)

Alternative Solution
\[ I_{NN} = I_{XX} \cos^2 \beta + I_{YY} \sin^2 \beta \] (Eq. 21.15).
\[ = 11'52 \times 10^4 \cos^2 66^\circ + 5'12 \times 10^4 \sin^2 66^\circ \]
\[ = 1'91 \times 10^4 + 4'28 \times 10^4 = 6'19 \times 10^4 \text{mm}^4 \]
Also, \( (y)_B = x \sin \beta + y \cos \beta \)
\[ = -40 \sin 66^\circ - 60 \cos 66^\circ = -61 \text{mm.} \]

Hence from Eq. 21.17,
\[ (f_x)_B = -\frac{M \cos (\beta - \theta)}{I_{NN}} \cdot x \]
\[ = -\frac{12 \times 10^6 \cos (66^\circ - 45^\circ)}{6'19 \times 10^4} \times 61 \]
\[ = -110'4 \text{N/mm}^2 \] (i.e. tensile).

**Example 21.5.** A 60 mm x 40 mm x 6 mm unequal angle is placed with the longer leg vertical, and is used as a beam. It is subjected to a bending moment of 12 kN-cm acting in the vertical plane through the centroid of the section. Determine the maximum bending stress induced in the section.
\[
\begin{align*}
C_N &= 20.2 \text{ mm} ; C_V = 10.2 \text{ mm} \\
A &= 564 \text{ mm}^3 \\
I_{Nn} &= 20.34 \times 10^4 \text{ mm}^4 ; I_{VV} = 7.33 \times 10^4 \text{ mm}^4 ; \\
I_{Ny} &= +7.04 \times 10^4 \text{ mm}^4 \\
I_{vN} &= 23.42 \times 10^4 \text{ mm}^4 ; I_{NV} = 4.26 \times 10^4 \text{ mm}^4 ; \alpha = 23^\circ 40' \\
\text{The plane of loading is vertical. Hence } Y' \text{ axis and } Y \text{ axis coincide.} \\
\end{align*}
\]

\[\theta = \alpha = 23^\circ 40'.\]

(a) **Analytical Solution**

The inclination \(\beta\) of the neutral axis \(N-N\) with the \(U-U\) axis is given by

\[
\tan \beta = \frac{I_{vN}}{I_{NN}} \tan \theta = \frac{23.42 \times 10^4}{4.26 \times 10^4} \tan 23^\circ 40' = 2.4
\]

\[\beta = 67^\circ 24'.\]

\[
I_{NN} = I_{vN} \cos^2 \beta + I_{NN} \sin^2 \beta
\]

\[= 23.42 \times 10^4 \cos^2 67^\circ 24' + 4.26 \times 10^4 \sin^2 67^\circ 24'
\]

\[= 3.46 \times 10^4 + 3.64 \times 10^4 = 7.1 \times 10^4 \text{ mm}^4
\]

Since point \(S\) is farthest from the N.A., it will be stressed maximally. The distance \(S\) from N.A. is given by

\[
A_{yN} = u \sin \beta + v \cos \beta
\]

where \(u\) and \(v\) are the coordinates of point \(S\) referred to \(U-V\) axes.

If \((x, y)\) are the coordinates of \(S\) referred to \(x-y\) axes, we have

\[u = y \sin \alpha - x \cos \alpha \]

and

\[v = y \cos \alpha + x \sin \alpha \]

where

\[x = -10.2 - 6 = -16 \text{ mm} \]

and

\[y = -60 - 20.2 = -39.8 \text{ mm}.
\]

(Both \(x\) and \(y\) are negative since \(S\) is in the second quadrant with respect to the \(X-Y\) axes, the plane of loading being reckoned as situated in the first quadrant).

\[u = -39.8 \sin 23^\circ 40' + 4.2 \cos 23^\circ 40' = -12.2 \text{ mm} \]

\[v = -39.8 \cos 23^\circ 40' - 4.2 \sin 23^\circ 40' = 38.2 \text{ mm} \]

\[(J)_{yN} = u \sin \beta + v \cos \beta = -12.2 \sin 67^\circ 24' - 38.2 \cos 67^\circ 24'
\]

\[= -25.9 \text{ mm}.
\]

Hence from Eq. 21'17,

\[\left( \phi \right)_S = \frac{M \cos (\beta - \theta)}{I_{NN}} J_{yN}
\]

\[= 12 \times 10^4 \cos (67^\circ 24' - 23^\circ 40') \times 25.9 \\
= -31.7 \text{ N/mm}^2.
\]

**Graphical Solution** [Fig. 21'14 (b)]

\[
k_{uu} = \sqrt{\frac{I_{uu}}{A}} = \sqrt{\frac{23.42 \times 10^4}{564}} = 20.4 \text{ mm}
\]

\[
k_{vv} = \sqrt{\frac{I_{vv}}{A}} = \sqrt{\frac{4.26 \times 10^4}{564}} = 8.8 \text{ mm}
\]

Draw the \(U-U\) axis inclined at 23° 40' with \(X-X\) axis in clockwise direction. Similarly, set off \(V-V\) axis. Draw the momental ellipse, making \(OB = k_{uu} = 20.4 \text{ mm}\) and \(AO = k_{vv} = 8.8 \text{ mm} \).

To find the direction of neutral axis, draw any vertical line \(MM'\), and find its middle point \(J\). Then \(OJ\) gives the direction of neutral axis \(NN\). By measurement, \(\beta = 67^\circ\).

Draw tangent \(N'N'\) parallel to the neutral axis. Draw \(OH\) perpendicular to \(N'N'\).

Then \(OH = k_{NN} = 11.3 \text{ mm} \) (by measurement).

\[I_{NN} = A \cdot k_{NN}^2 = 564 (11.3)^2 = 7.2 \times 10^4 \text{ mm}^4
\]

From Fig. 21'14 (a), \((J)_{yN} = 26 \text{ mm} \) (by measurement)

\[\left( \phi \right)_S = \frac{M \cos (\beta - \theta)}{I_{NN}} (J)_{yN}
\]

\[= 12 \times 10^4 \cos (67^\circ - 23^\circ 40') \times 26 \\
= 31.5 \text{ N/mm}^2 \text{ (tensile)}.
\]

\[21'10. \ \text{THE Z-POLYGON}
\]

In the case of simple bending, the strength of a beam depends upon its section modulus \(Z\). In the case of unsymmetrical bending,
the section modulus \( Z \) for any point in the section depends also on the position of the plane of loading. It is interesting to study the variation of \( Z \) for any point as the direction of the plane of loading varies. It will be proved below that the variation of \( Z \) for a point, is linear with the varying value of \( \theta \), and if such variations are plotted for some key points of the section, a polygon is obtained. Such a polygon is known as \( Z \)-polygon, and is very useful in finding out the minimum value of \( Z \) for the section and the corresponding position of the plane of loading.

From Eq. 21'10, the bending stress at any point \( A \) having co-ordinates \( \nu_A \) and \( \nu_A \) with reference to the principal axes, is given by

\[
f_A = \frac{M \cos \theta}{I_{UU}} \nu_A + \frac{M \sin \theta}{I_{VV}} \nu_A = \frac{M}{Z} \left( \frac{\nu_A \cos \theta}{I_{UU}} + \frac{\nu_A \sin \theta}{I_{VV}} \right)
\]

(1)

(where \( \theta \) is the angle of plane of loading \( OM \) with \( OV \) axis)

\[
\therefore \quad f_A = \frac{M}{Z} \left( \frac{\nu_A \cos \theta}{I_{UU}} + \frac{\nu_A \sin \theta}{I_{VV}} \right)
\]

where \( Z \)-section modulus of the sections for the point \( A \), given by

\[
\frac{1}{Z} = \frac{\nu_A \cos \theta}{I_{UU}} + \frac{\nu_A \sin \theta}{I_{VV}}
\]

(21'26)

or

\[
\frac{Z \cos \theta}{I_{UU}} + \frac{Z \sin \theta}{I_{VV}} = 1.
\]

Putting \( X \cos \theta = \nu \) and \( Z \sin \theta = \nu \), we get

\[
\nu = \frac{\nu_A}{I_{UU}} + \frac{\nu_A}{I_{VV}} = 1
\]

(21'27)

This is the equation of straight line which gives the variation of \( Z \) with \( \theta \). The straight line \( A_1A_2 \) (Fig. 21'15) is called the \( Z \)-line for the point \( A \). The intercepts of this straight line \( UV \) and \( VV \) axes are \( \frac{I_{VV}}{v_A} \) and \( \frac{I_{UU}}{v_A} \) respectively. Hence in order to draw the \( Z \)-line for \( A \), set off \( OA_1 = \frac{I_{VV}}{v_A} \) on \( UV \) axis, and \( OA_2 = \frac{I_{UU}}{v_A} \) on the \( VV \) axis. Join \( A_1A_2 \), which is the \( Z \)-line for the point \( A \). The minimum value of \( Z \) is given by the perpendicular \( OA_+ \), inclined at an angle \( \theta \) with the \( OV \) axis.

\[
Z_{\text{min}} = OA_+
\]

Maximum bending stress at \( A \) will be \( \frac{M}{OA_+} \) when the plane of bending is inclined at \( \theta \) to the principal axis \( OV \).

It will be useful to plot such \( Z \)-lines for some key points of a given section, getting what is known as the \( Z \)-polygon. We shall take the case of a rectangular section \( ABCD \) of width \( b \) and depth \( d \), to plot the \( Z \)-polygon. (Fig. 21'16).

The principal axes \( U-U \) and \( V-V \) of a rectangular section coincide with the usual \( XX \) and \( YY \)-axes passing through its centroid.

For the \( Z \)-line for \( A \), the distance \( OP = \frac{I_{VV}}{u_A} \) and \( OQ = \frac{I_{UU}}{v_A} \).

But

\[
I_{VV} = I_{YY} = \frac{1}{12} bd^3; \quad u_A = \frac{b}{2}
\]

\[
I_{UU} = I_{XX} = \frac{1}{12} bd^3; \quad v_A = \frac{d}{2}
\]

\[
\therefore \quad OP = \frac{1}{12} bd^3 \times \frac{2}{b} = \frac{1}{6} bd^2 = \text{usual } Z_{YY}
\]

\[
OQ = \frac{1}{12} bd^3 \times \frac{2}{d} = \frac{1}{6} bd^2 = \text{usual } Z_{XX}
\]

Similarly, the \( Z \)-lines for \( B, C \) and \( D \) are respectively obtained as the line \( QR, RS \) and \( SP \). Then \( PQRS \) is the required \( Z \)-polygon,
for the rectangular section. The Z-polygon provides, at a glance, the position of the plane of bending for the maximum and minimum strength.

For the rectangular section, the position of plane of loading for maximum strength is along YY axis since the value of Z along this axis is $\frac{1}{6} bd^2$ and is the maximum. Similarly, the position of plane of loading for minimum strength is along $A'C'$ (or $B'D'$), inclined at $\phi$ with the YY axis, given by,

$$\tan \phi = \frac{A'O}{A'O} = \frac{QQ}{OP} = \frac{Z_{XX}}{Z_{YY}} = \frac{1}{6} \frac{bd^2}{bd^2} = \frac{1}{6} \frac{bd^2}{db^2}$$

or

$$\tan \phi = \frac{d}{b} \quad \ldots(21'28)$$

Now

$$Z_{max} = OA' = OO \cos \phi$$

$$= Z_{XX} \cos \phi$$

$$= \frac{1}{6} \frac{bd^2}{b} \frac{b}{\sqrt{b^2 + d^2}} = \frac{b^2 d^2}{6 \sqrt{b^2 + d^2}} \quad \ldots(21'29)$$

21'11. DEFLECTION OF BEAM UNDER UNSYMMETRICAL BENDING

Fig. 21'17 shows the plane of loading $OM$ inclined at $\theta$ to the $OV$ axis. Let the neutral axis be inclined at $\beta$ with the $OU$ axis. The resolved component of bending moment in the $VV$ direction is $M \cos \theta$, while in the $U$ direction it is equal to $M \sin \theta$. 

Plane of Resultant Deflection

$\delta$V

$\delta$U

$\delta$N

$\delta$M

Fig. 21'17
The deflection of the beam in any direction, due to a bending moment $M_f$, is given by

$$\delta = \int_0^L \frac{M_f m_1}{EI} \, dx$$

where $m_1 = \text{moment due to unit load at the point in the direction of the desired deflection}$

$dx = \text{elementary length of beam, measured along the span of the beam.}$

Hence the deflection of the beam in the direction of axis $VV$ is given by

$$\delta_v = \int_0^L \frac{M \cos \theta}{EI_{uv}} \cdot m_v \, dx \quad \text{...(1) \ [21 \cdot 30 (a)]}$$

The deflection in the direction of axis $UU$ is given by

$$\delta_u = \int_0^L \frac{M \sin \theta}{EI_{uv}} \cdot m_u \, dx \quad \text{...(2) \ [21 \cdot 30 (b)]}$$

The resultant deflection $\delta$ is then given by

$$\delta = \left[ \delta_u^2 + \delta_v^2 \right]^{1/2} \quad \text{...(21 \cdot 31)}$$

In Eqs. (1) and (2) above, $m_u = m_v = m$.

Let $\beta_1 = \text{angle which the resultant deflection in the direction } N'N' \text{ makes with the } UU \text{ axis.}$

Then, $\tan \beta_1 = - \frac{\delta_u}{\delta_v}$

$$= - \int_0^L \frac{M \sin \theta}{EI_{uv}} \, dx \quad \text{...(3)}$$

From Eq. 21'14,

$$\tan \beta = \frac{I_{uv}}{I_{uv}} \tan \theta \quad \text{...(4)}$$

Comparing (3) and (4), we get

$$\tan \beta_1 = - \tan \beta = \tan (90 + \beta)$$

Hence the resultant deflection occurs in a direction $N'N'$, which is perpendicular to the neutral axis $NN$ for any given direction of loading.

Let us take the case of a simply supported beam subjected to uniformly distributed load.

Then,

$$\delta_v = \frac{5}{384} \frac{w \sin \theta \cdot L^4}{EI_{uv}} \quad \text{...(5)}$$

$$\delta_u = \frac{5}{384} \frac{w \cos \theta \cdot L^4}{EI_{uv}} \quad \text{...(6)}$$

$$\delta = \left[ \delta_u^2 + \delta_v^2 \right]^{1/2}$$

$$= \frac{5}{384} \frac{wL^4}{EI_{uv}} \left[ \sin^2 \theta + \cos^2 \theta \right] \left[ 1 + \frac{I_{uv}}{I_{uv}} \tan \theta \right]^2 \quad \text{...(7)}$$

$$= \frac{5}{384} \frac{wL^4}{EI_{uv}} \left[ 1 + (1 + \tan \beta \tan \theta) \right] \quad \text{...(8)}$$

(Here $I_{uv} \tan \theta = \tan \beta$, from Eq. 21'14)

$$\delta = \frac{5}{384} \frac{wL^4}{EI_{uv}} \cos \theta \sec \beta \quad \text{...(9)}$$

$$= \frac{5}{384} \frac{wL^4}{EI_{uv}} \cos \theta \cos \beta \cos (\beta - \theta) \quad \text{...(10)}$$

$$= \frac{5}{384} \frac{wL^4}{EI_{uv}} \cos \beta \cos (\beta - \theta) \quad \text{...(11)}$$

$$= \frac{5}{384} \frac{wL^4}{EI_{uv}} \cos \beta \cos \theta \left( 1 + \tan \beta \tan \theta \right) \quad \text{...(12)}$$

Substituting the value of $\tan \theta = \frac{I_{uv}}{I_{uv}} \tan \beta$, we get

$$\theta = \frac{5}{384} \frac{wL^4}{EI_{uv}} \cos \beta \cos (\beta - \theta) \left[ 1 + \frac{I_{uv}}{I_{uv}} \tan \beta \right] \quad \text{...(13)}$$

$$= \frac{5}{384} \frac{wL^4}{EI_{uv}} \cos \beta \cos (\beta - \theta) \left[ I_{uv} \cos \beta + I_{uv} \sin \beta \sin \theta \right] \quad \text{...(14)}$$

$$= \frac{5}{384} \frac{wL^4}{EI_{uv}} \cos \beta \cos (\beta - \theta) \left[ I_{uv} \cos \beta + I_{uv} \sin \beta \sin \theta \right] \quad \text{...(15)}$$

$$= \frac{5}{384} \frac{wL^4}{EI_{uv}} \cos \beta \cos (\beta - \theta) \left[ I_{uv} \cos \beta + I_{uv} \sin \beta \sin \theta \right] \quad \text{...(16)}$$

$[I_{uv} \cos \beta + I_{uv} \sin \beta \sin \theta] = I_{uv}$

Thus, $\delta = \frac{5}{384} \frac{w \cos (\beta - \theta) \cdot L^4}{EI_{NN}} \quad \text{...(21 \cdot 32)}$

In the above expression, $w \cos (\beta - \theta)$ is the component of the resultant uniformly distributed load along the direction $N'N'$ perpendicular to the neutral axis.
Example 21.6. A 60 mm × 40 mm × 6 mm unequal angle is placed with the longer leg vertical and is used as a beam simply supported at the ends, over a span of 2 m. If it carries a uniformly distributed load of such magnitude as to produce the maximum bending moment of 0.12 kN-m determine the maximum deflection of the beam.

Take \( E = 2.1 \times 10^5 \) N/mm².

Solution.

The properties of the section are known from example 21.6.

Thus, \( I_{SN} = 7.1 \times 10^4 \) mm⁴; \( \beta = 67^\circ 24' \); \( \theta = \alpha = 23^\circ 40' \)

Note, for a simply supported beam, maximum B.M. at the centre of the span, is given by

\[
M = \frac{wL^2}{8}
\]

\[
w = \frac{8M}{L^3} = \frac{8 \times 0.12 \times 10^6}{(2000)^3} = 0.24 \text{ N/mm} \quad \ldots (i)
\]

From Eq. 21.32, the maximum resultant deflection is given by

\[
\delta = \frac{5}{384} \times \frac{w \cos (\beta - \theta)}{E I_{SN}} \frac{L^3}{(2000)^3}
\]

\[
= \frac{5}{384} \times \frac{0.24 \cos (67^\circ 24' - 23^\circ 40')}{(2000)^3} \times 10^3 \times (7.1 \times 10^4)
\]

\[
= 2.54 \text{ mm.}
\]

The maximum (resultant) deflection takes place in a direction perpendicular to the neutral axis. If \( \beta_i \) is the inclination of the plane of maximum deflection, we have, [Fig. 21.14 (a)],

\[
\beta_i = 67^\circ 24' - 23^\circ 40' = 43^\circ 44'.
\]

Example 21.7. Draw Z-polygon for a rolled steel joist (RSJ) having the following properties [Fig. 21.18 (a)]:

- Depth of section \( h = 200 \) mm
- Width of flange \( b = 100 \) mm
- Thickness of flange \( t = 7.3 \) mm
- Thickness of web \( = 5.4 \) mm

\[
I_{UU} = 16966 \times 10^4 \text{ mm}^4
\]

\[
I_{VV} = 1154 \times 10^4 \text{ mm}^4
\]

\[
Z_{UU} = 16971 \times 10^3 \text{ mm}^3
\]

\[
Z_{VV} = 231 \times 10^8 \text{ mm}^2
\]

Hence find the maximum bending stress due to a bending moment of 1800 N-m. What is the inclination of the plane of loading to give the maximum bending stress?
Since the key points A, B, C, and D form the corners of a rectangle of size 100 mm x 200 mm, the Z-polygon for a rectangular beam will be similar in shape to the Z-polygon for a rectangular beam.

Hence, to get the points P, Q, R, S of the Z-polygon, drop perpendiculars OA', OB', OC', OD' from O to the sides of the rectangle.

The inclination of OA' or OB' be \( \phi \) with the V-V axis. Then, for \( \phi = 82^\circ \) and \( \sin \phi = 0.991 \),

The maximum strength is obtained along the direction V-V, having a section modulus \( Z_{uu} = 169 \times 10^3 \) mm^3.

Example 21.8.

For the angle section of Example 21.5, draw the Z-polygon. Hence determine (i) maximum bending stress due to a bending moment of 6 kN-m acting in the vertical plane through the centroid of the section, (ii) the absolute maximum bending stress due to a bending moment of 6 kN-m, and the corresponding position of plane of loading.

Solution. (Fig. 21.4). From example 21.5, we have \( \alpha = 23^\circ \) 40'. Hence set of U-U axes at 23° 40' with the X-axis, in a direct manner. Make Y-axis perpendicular to U-U axes.

The U and V coordinates of key-points A, B, C, and D are determined by graphical method, or determined by direct measurement. Since the Z-polygon method is essentially a graphical method, it is advisable to determine these coordinates by graphical method from the drawing (Fig. 21.4).

Regarding the sign of U and V, it should be noted that U and V are positive in the quadrant in which the plane of loading lies. The four quadrants I, II, III, and IV, have been marked in Fig. 21.4, taking the quadrant of the plane of loading as the first quadrant. According to this, the coordinates of A, B, C, and D are found to be as follows:

\[ (U) \text{ and } (V) \text{ are positive in quadrant I.} \]

The plane of bending is inclined at \( \phi = 82^\circ \) 15' with the V-V axis.

The maximum bending stress is obtained along the direction V-V, to give the above maximum bending stress.

The plane of bending is inclined at \( \phi = 82^\circ \) 15' with the V-V axis, to give the above maximum bending stress.
Solving equations (3) and (4), the co-ordinates of point $P_3$ are:

$$u = -23.4; \quad v = -29.2$$

Solving equations (4) and (5), the co-ordinates of point $P_4$ are:

$$u = -4.8; \quad v = -52.7$$

Solving equations (5) and (1), the co-ordinates of point $P_d$ are:

$$u = +38.2; \quad v = -93.0$$

Knowing these co-ordinates, the Z-polygon $P_1P_2P_3P_4P_d$ can be plotted, as shown in Fig. 21.19.

(i) For the loading in the vertical plane, the section modulus $Z$ for $A$ is given by $Oa = 57.8 \times 10^3 \text{mm}^3$ and that of $D$ is given by $Od = 37.5 \times 10^3 \text{mm}^3$. Hence the minimum section modulus $= 37.5 \times 10^3 \text{mm}^3$. The maximum bending stress is, therefore, given by:

$$f_{\text{max}} = \frac{M}{Z} = \frac{0.12 \times 10^6}{37.5 \times 10^3} = 32 \text{ N/mm}^2$$

(ii) For absolute maximum bending stress, we have to find the absolute minimum section modulus. If perpendiculars are drawn from $O$ to the various Z-lines, we find that the minimum $Z$ is equal to $OC$.

$$Z_{\text{min}} = OC = 18.8 \times 10^3 \text{ mm}^3$$

$$f_{\text{max}, \text{min}} = \frac{0.12 \times 10^6}{18.8 \times 10^3} = 63.8 \text{ N/mm}^2$$

The corresponding direction of loading is at 102° with the vertical line YY, as marked in Fig. 21.19.

PROBLEMS

1. Determine the principal moments of inertia for an unequal angle section 200 mm $\times$ 150 mm $\times$ 10 mm.

2. A 4 in. $\times$ 4 in. $\times$ $\frac{1}{4}$ in. steel angle is used as a cantilever of length 3 ft. and carries an end load. One leg of the angle is horizontal, and the load at the end is vertical with its line of action passing through the centroid of the section. Determine the maximum allowable load if the bending stress is not to exceed $75 \text{ tons/in.}^2$ and find also the vertical deflection at the end due to this load. Assume all corners of the angle to be left square. $E = 13500 \text{ tons/in.}^2$. (U.L.)

3. A cantilever consists of 3 in. $\times$ 3 in. $\times$ $\frac{1}{4}$ in. angle with the top face $AB$ horizontal (Fig. 21.20). It carries a load of 400 lb. at a distance of 3 ft. from the fixed end, the line of action of the load
passing through the centroid of the section and inclined at 30° to the vertical. Determine the stress at the corners A, B and C at the fixed end and also the position of the neutral axis. Given the following:

\[ A = 2'753 \text{ in.}^2; \quad I_{xx} = I_{yy} = 2'18 \text{ in.}^4; \quad I_{uu} = 3'44 \text{ in.}^4; \quad I_{vv} = 0'92 \text{ in.}^4. \]

**ANSWERS**

1. \[ I_{uu} = 17 \times 10^6 \text{ mm}^4; \quad I_{vv} = 3'51 \times 10^6 \text{ mm}^4. \]
2. \[ W_{max} = 705 \text{ lb}; \quad 6\nu = 0'0993 \text{ in.} \]
3. \[ (f_0)_\alpha = 19,650 \text{ lb/in.}^2 \]
\[ (f_0)_\beta = 10,140 \text{ lb/in.}^2 \]
\[ (f_0)_\gamma = 14,740 \text{ lb/in.}^2. \]

Neutral axis inclined at 40° 34' to the XX-axis in the anticlockwise direction.

**Elementary Theory of Elasticity**

221. **STATE OF STRESS AT A POINT: STRESS TENSOR**

Force systems acting on an elastic body in equilibrium are of two kinds: body forces and surface forces. Forces distributed over the surface of the body, such as the pressure of one body on another or hydrostatic pressure are called surface forces. Such forces are applied externally at the boundaries of the body, and dimensionally, a surface force is defined as force per unit area. Forces distributed over the volume of a body, such as gravitational forces, magnetic forces, seepage forces, or in the case of a body in motion, inertia forces, are called body forces. Dimensionally, a body force is taken as a force per unit volume.

The total stress field on any three dimensional element is determined by the following stresses:

\[
\begin{pmatrix}
\sigma_{XX} & \tau_{XY} & \tau_{XZ} \\
\tau_{YX} & \sigma_{YY} & \tau_{YZ} \\
\tau_{ZX} & \tau_{ZY} & \sigma_{ZZ}
\end{pmatrix}
\]

These nine stress components, as given by this group of square matrix of stresses, are the components of a mathematical entity called the stress tensor, with a symmetrical matrix relative to its main diagonal (upper left to lower right). The main diagonal elements of the stress tensor are the normal stress components and the off-diagonal elements are shear stresses.

Each stress component in it is represented by its magnitude, direction as well as the position of the plane on which it is acting. For example, \( \sigma_{XX} \) (or simply \( \sigma_X \)) signifies the normal stress acting on the face of the element that is perpendicular to x-axis, and the stress is acting in the x-direction. Similarly, the shearing stress \( \tau_{XY} \) denotes a stress acting on the face of an element that is perpendicular to
the stress acting in the direction of \( y \)-axis. The stress \( \tau_{yz} \) denotes a stress acting on the face of an element that is perpendicular to \( y \)-axis, the stress acting in the direction of \( z \)-axis. Thus, at a point, there are three normal stresses: \( \sigma_x \) (or \( \sigma_{xx} \)), \( \sigma_y \) (or \( \sigma_{yy} \)) and \( \sigma_z \) (or \( \sigma_{zz} \)) and six shearing stresses: \( \tau_{xy} \), \( \tau_{yx} \); \( \tau_{yz} \), \( \tau_{zy} \); \( \tau_{xz} \) and \( \tau_{zx} \) as shown in Fig. 22.1 (a).

Fig. 22.1 (a) shows the stresses acting at the centre of an elemental volume of size \( dx \), \( dy \) and \( dz \). Figs. 22.1 (b), (c) and (d) show the views in \( z-y \), \( x-z \) and \( x-y \) planes respectively. Considering the equilibrium of the elemental volume, and applying equation \( \Sigma M_z = 0 \) (where \( M_z \) represents the moment about \( z \)-axis), we get from Fig. 22.1 (d),

\[
(\tau_{yx}.dx \cdot dz)\,dy = (\tau_{xy}.dy \cdot dz)\,dx
\]

or

\[ \tau_{yx} = \tau_{xy} \]  \[ (22.1) \]

Similarly, from Figs. 22.1 (b) and (c), we get

\[ \tau_{yx} = \tau_{xy} \]  \[ (22.2a) \]

and

\[ \tau_{xz} = \tau_{zx} \]  \[ (22.2b) \]

Thus, out of six shearing stresses there are only three independent shearing stresses and total independent stresses are therefore six only (i.e., three normal stresses and three shearing stresses).

22.2. EQUILIBRIUM EQUATIONS

Fig. 22.2 shows an elemental volume of size \( dx \), \( dy \) and \( dz \), with the nine stress components acting at the centre of the element. The stress on each face will be equal to the stress at the centre increased or reduced by the distance from the centre to the face times the spatial derivative of the stress. For example, normal stress component \( \sigma_x \) acting at the centre will be increased to \( \left( \sigma_x + \sigma_x \, \frac{\partial \sigma_x}{\partial x} \cdot \frac{dx}{2} \right) \) at the face \( ABB_1A_1 \) and decreased to \( \left( \sigma_x - \sigma_x \, \frac{\partial \sigma_x}{\partial x} \cdot \frac{dx}{2} \right) \) at the face \( CDD_1C_1 \).

If \( X \), \( Y \) and \( Z \) denote the components of body forces per unit volume, in the three corresponding directions, then the equation of
equilibrium obtained by summing all the forces acting on the element in the x-direction is:

\[
\left\{ \left( \sigma_x + \frac{\partial \sigma_x}{\partial x} \cdot \frac{dx}{2} \right) dy \; dz - \left( \sigma_y - \frac{\partial \sigma_y}{\partial x} \cdot \frac{dx}{2} \right) dy \; dz \right\} + \left\{ \left( \tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} \cdot \frac{dy}{2} \right) dx \; dz - \left( \tau_{yx} - \frac{\partial \tau_{yx}}{\partial x} \cdot \frac{dy}{2} \right) dx \; dz \right\} + \left\{ \left( \tau_{xz} + \frac{\partial \tau_{xz}}{\partial z} \cdot \frac{dz}{2} \right) dx \; dy - \left( \tau_{zx} - \frac{\partial \tau_{zx}}{\partial x} \cdot \frac{dz}{2} \right) dx \; dy \right\} + X \cdot dx \cdot dy \cdot dz = 0
\]

Dividing all the terms by \(dx \cdot dy \cdot dz\) and simplifying, we get

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \quad \text{(I)} \quad \text{[22'3 (a)]}
\]

Similarly, in the y-direction, the balance of forces requires that

\[
\left\{ \left( \sigma_y + \frac{\partial \sigma_y}{\partial y} \cdot \frac{dy}{2} \right) dx \; dz - \left( \sigma_x - \frac{\partial \sigma_x}{\partial y} \cdot \frac{dy}{2} \right) dx \; dz \right\} + \left\{ \left( \tau_{yx} + \frac{\partial \tau_{yx}}{\partial x} \cdot \frac{dx}{2} \right) dy \; dz - \left( \tau_{xy} - \frac{\partial \tau_{xy}}{\partial y} \cdot \frac{dx}{2} \right) dy \; dz \right\} + \left\{ \left( \tau_{yz} + \frac{\partial \tau_{yz}}{\partial z} \cdot \frac{dz}{2} \right) dy \; dx - \left( \tau_{zy} - \frac{\partial \tau_{zy}}{\partial z} \cdot \frac{dz}{2} \right) dy \; dx \right\} + Y \cdot dx \cdot dy \cdot dz = 0
\]

which on simplification reduce to:

\[
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \quad \text{(II)} \quad \text{[22'3 (b)]}
\]

Lastly, in the z-direction, we have

\[
\left\{ \left( \sigma_z + \frac{\partial \sigma_z}{\partial z} \cdot \frac{dz}{2} \right) dx \; dy - \left( \sigma_x - \frac{\partial \sigma_x}{\partial z} \cdot \frac{dz}{2} \right) dx \; dy \right\} + \left\{ \left( \tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} \cdot \frac{dx}{2} \right) dy \; dz - \left( \tau_{zx} - \frac{\partial \tau_{zx}}{\partial x} \cdot \frac{dx}{2} \right) dy \; dz \right\} + \left\{ \left( \tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} \cdot \frac{dy}{2} \right) dz \; dx - \left( \tau_{zy} - \frac{\partial \tau_{zy}}{\partial y} \cdot \frac{dy}{2} \right) dz \; dx \right\} + Z \cdot dx \cdot dy \cdot dz = 0
\]

which on simplification reduces to

\[
\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z = 0 \quad \text{(III)} \quad \text{[22'3 (c)]}
\]

However, we have seen in §22'1 that there are six independent stress components acting at a point and the complete solution of the problem requires the determination of these six stress components. Thus, there are six unknowns, and only three equations of equilibrium are available. Thus the problem of elasticity is strictly of indeterminate nature. These equations of static equilibrium must be supplemented with equations of compatibility of deformations (§22'4) to get the complete solution. In addition to this, the final solution should satisfy the boundary conditions (§22'5).

22'3. STRAIN COMPONENTS: STRAIN TENSOR

Let u, v and w be the displacements in x, y and z directions respectively. For a three dimensional case, there are six strain components:

\[
e_x, e_y, e_z, \gamma_{xy}, \gamma_{yz} \text{ and } \gamma_{zx}.
\]

The three linear strain components are defined by:

\[
e_x = \frac{\partial u}{\partial x} \quad \text{[22'7 (a)]}
\]

\[
e_y = \frac{\partial v}{\partial y} \quad \text{[22'7 (b)]}
\]

\[
e_z = \frac{\partial w}{\partial z} \quad \text{[22'7 (c)]}
\]

In order to find the other strain components (called the shearing strain components), consider a plane lamina of size \(dx, dy\) in the \(x-y\) plane. The lines \(OA\) and \(OB\), originally orthogonal to each other, are displaced to positions \(OA'\) and \(OB'\) respectively. The shearing strain is equal to the change in the angle at O.

Displacement of \(A\) in x-direction = \(u + \frac{\partial u}{\partial x} \cdot dx\)

Displacement of \(B\) in y-direction = \(v + \frac{\partial v}{\partial y} \cdot dy\)
Displacement of $A$ in $y$-direction $= v + \frac{\partial v}{\partial x} \cdot dy$.

Displacement of $B$ in $x$-direction $= u + \frac{\partial u}{\partial y} \cdot dy$.

$\therefore$ Total change in the angle at $O$

$= \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \ldots [22'8 (a)]$

Similarly,$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \ldots [22'8 (b)]$

and $\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \ldots [22'8 (c)]$

Fig. 22.3. Shear Strains.

It can be shown that linear strain of a diagonal is equal to half the shearing strain. Thus, if $\varepsilon_{xy}$, $\varepsilon_{yz}$ and $\varepsilon_{xz}$ represent the linear strains of the diagonals of a plane lamina, we have

$\varepsilon_{xy} = \frac{1}{2} \gamma_{xy}; \varepsilon_{yz} = \frac{1}{2} \gamma_{yz}; \varepsilon_{xz} = \frac{1}{2} \gamma_{zx} \ldots (22'9)$

Therefore, the strain tensor, consisting of nine strain components, can be represented as under:

$\begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{pmatrix}$ or $\begin{pmatrix} \varepsilon_{xx} & \frac{1}{2} \gamma_{xy} & \frac{1}{2} \gamma_{xz} \\ \frac{1}{2} \gamma_{xy} & \varepsilon_{yy} & \frac{1}{2} \gamma_{yz} \\ \frac{1}{2} \gamma_{xz} & \frac{1}{2} \gamma_{yz} & \varepsilon_{zz} \end{pmatrix}$

22.4. COMPATIBILITY EQUATIONS

The equations resulting from the application of strain equations are known as the compatibility equations, or Saint-Venant's equations.

Differentiating Eq. 22'7 (a) twice with respect to $y$, Eq. 22'7 (b) twice with respect to $x$ and Eq. 22'8 (a) once with respect to $x$ and then with respect to $y$, we get

$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y^2}$ \ldots (i)

$\frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 v}{\partial x^2}$ \ldots (ii)

$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial y}$ \ldots (iii)

By inspection from (i), (ii) and (iii), we get

$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}$ \ldots [22'10 (a)]

Similarly,

$\frac{\partial^2 \varepsilon_{xy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{yz}}{\partial x \partial z} = \frac{\partial^2 \varepsilon_{yx}}{\partial z \partial y}$ \ldots [22'10 (b)]

and

$\frac{\partial^2 \varepsilon_{ex}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = \frac{\partial^2 \varepsilon_{zx}}{\partial x \partial z}$ \ldots [22'10 (c)]

Again, from Eq. 22'7 (a),

$\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial^2 u}{\partial y \partial z}$ \ldots (iv)

From Eq. 22'8 (a), $\frac{\partial^2 \gamma_{xy}}{\partial z \partial x} = \frac{\partial^2 v}{\partial z \partial x}$ \ldots (v)

From Eq. 22'8 (b), $\frac{\partial^2 \gamma_{yz}}{\partial x \partial y} = \frac{\partial^2 w}{\partial x \partial y}$ \ldots (vi)

and from Eq. 22'8 (c), $\frac{\partial^2 \gamma_{zx}}{\partial y \partial z} = \frac{\partial^2 u}{\partial y \partial z}$ \ldots (vii)

From (iv), (v), (vi) and (vii), we have

$2 \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial^2 \gamma_{xy}}{\partial z \partial x} - \frac{\partial^2 \gamma_{yz}}{\partial x \partial y} + \frac{\partial^2 \gamma_{zx}}{\partial y \partial z}$ \ldots [22'10 (d)]

or

$2 \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial z} \left( - \frac{\partial \gamma_{xy}}{\partial x} + \frac{\partial \gamma_{yz}}{\partial y} + \frac{\partial \gamma_{zx}}{\partial z} \right)$ \ldots [22'10 (d)]

Similarly,

$2 \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial x} \left( - \frac{\partial \gamma_{xy}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial z} + \frac{\partial \gamma_{zx}}{\partial x} \right)$ \ldots [22'10 (e)]
and
\[ 2 \frac{\partial^2 s}{\partial x \partial y} = \frac{\partial}{\partial z} \left( \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} - \frac{\partial \tau_{yz}}{\partial z} \right) \]  ...[22'10 (f)]
Eqs. 22'10 are the six compatibility equations.

22'5. BOUNDARY CONDITION EQUATIONS

The solution of an elasticity problem is obtained by the solution of equilibrium and compatibility equations, but the final solution must also satisfy the boundary condition equations. In order to derive the boundary condition equations, consider a boundary plane ABC [Fig. 22'4 (a)] with \( l, m \) and \( n \) as the direction cosines of the external normal to its surface at any point. \( \bar{X} \), \( \bar{Y} \) and \( \bar{Z} \) be the components of surface forces per unit area on the elementary area ABC. Fig. 22'4 (b) shows the nine stress components on the face OBC, OAC and OAB. If the elemental volume is considered to be shrunk to a point, these nine stress components are assumed to act at the point.

Let area \( ABC = ds \)
Area \( OBC = ds \cos (N, x) = ds \cdot l \)
\( OAB = ds \cos (N, y) = ds \cdot m \)
\( OAC = ds \cos (N, z) = ds \cdot n \)

Fig. 22'4. Boundary conditions.
finite value of 36, while the considerations of isotropy further reduce these constants to only 2. The generalised Hooke's law equations (Eqs. 22.12) then reduce to

\[ \varepsilon_x = C_{11}\sigma_x + C_{12}(\sigma_y + \sigma_z) \]
\[ \varepsilon_y = C_{11}\sigma_y + C_{12}(\sigma_x + \sigma_z) \]
\[ \varepsilon_z = C_{11}\sigma_z + C_{12}(\sigma_x + \sigma_y) \]
\[ \gamma_{xy} = 2(C_{11} - C_{12})\tau_{xy} \]
\[ \gamma_{yz} = 2(C_{11} - C_{12})\tau_{yz} \]

and

\[ \gamma_{xz} = 2(C_{11} - C_{12})\tau_{xz} \]

In order to evaluate the values of the two constants \( C_{11} \) and \( C_{12} \) let us take the case of uniaxial stress in the x-direction, with

\[ \sigma_y = \sigma_z = 0 \]

From which \( C_{11} = \frac{\sigma_x}{\sigma_x} = \frac{1}{E} \)

where \( E \) = Young's modulus of elasticity.

Also,

\[ C_{12} = \frac{1}{F} \quad \frac{\varepsilon_y}{\varepsilon_x} = -\frac{\mu}{E} \]

where \( \mu \) = Poisson's ratio \( = \frac{1}{v} \).

Substituting these in Eq. 22.13, we get the final equations as under:

\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x - \mu(\sigma_y + \sigma_z) \right] \quad \ldots[22.14 \; (a)] \]

\[ \varepsilon_y = \frac{1}{E} \left[ \sigma_y - \mu(\sigma_x + \sigma_z) \right] \quad \ldots[22.14 \; (b)] \]

\[ \varepsilon_z = \frac{1}{E} \left[ \sigma_z - \mu(\sigma_x + \sigma_y) \right] \quad \ldots[22.14 \; (c)] \]

\[ \gamma_{xy} = \frac{2(1 + \mu)}{E} \tau_{xy} \quad \ldots[22.14 \; (d)] \]

\[ \gamma_{yz} = \frac{2(1 + \mu)}{E} \tau_{yz} \quad \ldots[22.14 \; (e)] \]

\[ \gamma_{xz} = \frac{2(1 + \mu)}{E} \tau_{xz} \quad \ldots[22.14 \; (f)] \]

22.7. TWO DIMENSIONAL PROBLEMS

(a) Plane Stress. If a thin plate is uniformly loaded by forces applied at the boundary, parallel to the plane (say xy plane) of the plate, the stress components \( \sigma_y, \tau_{xy} \) and \( \tau_{xz} \). These components are independent of \( y \) (i.e. they do not vary with \( y \)). The Hooke's law equations for plane stress case are:

\[ \varepsilon_x = \frac{1}{E} \left( \sigma_x - \mu \sigma_z \right) \]

\[ \varepsilon_z = \frac{1}{E} \left( \sigma_z - \mu \sigma_x \right) \]

\[ \sigma_y = -\frac{\mu}{E} \left( \sigma_x + \sigma_z \right) \]

\[ \gamma_{xz} = \frac{2(1 + \mu)}{E} \tau_{xz} \]

and

\[ \gamma_{xy} = \gamma_{yz} = 0 \quad (\therefore \tau_{xy} = \tau_{yz} = 0) \]

(b) Plane Strain. There are many problems in which one dimension (say, y-direction) is very large in comparison to the other two. If such bodies are loaded by forces which are perpendicular to the longitudinal elements (in the long direction y) and do not vary along the length, all cross-sections will be in the same condition. The state of affairs existing in the xz plane through a point holds for all planes parallel to it. Such a case is known as plane strain case. The strain components \( \varepsilon_x, \gamma_{xy} \) and \( \gamma_{xz} \) will each be zero. The other strain components \( \varepsilon_y, \gamma_{yz} \) and \( \gamma_{xy} \) are given by the following Hooke's law equations:

\[ \sigma_y = -\frac{1}{E} \left[ \sigma_y - \mu(\sigma_x + \sigma_z) \right] \]

\[ \sigma_y = -\frac{1}{E} \left[ \sigma_x - \mu \sigma_y \right] \]

\[ \sigma_y = -\frac{1}{E} \left[ \sigma_z - \mu \sigma_x \right] \]

\[ \gamma_{xz} = \frac{2(1 + \mu)}{E} \tau_{xz} \]

or

\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x - \frac{\mu}{1 - \mu} \sigma_z \right] \quad \ldots[22.17 \; (a)] \]
Similarly, \( e = \frac{1 - \mu^2}{E} \left[ \sigma_x - \frac{\mu}{1 - \mu} \sigma_z \right] \) \[
\text{...(22'17 b)}
\]
and
\[ \gamma_{xz} = \frac{2(1 + \mu)}{E} \tau_{xz} \] \[
\text{...(22'17 c)}
\]

**Equilibrium Equations.** For both plane stress as well as plane strain case, the equilibrium equations are:
\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + X = 0
\]
\[
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zx}}{\partial z} + Y = 0
\]

**22.8. Compatibility Equation in Two Dimensional Case**

For the two dimensional case, the six compatibility equations (Eq. 22'10) evidently reduce to one single equation:
\[
\frac{\partial^2 \epsilon_x}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial x^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z}
\]

**Compatibility equation in terms of Stress**

The above compatibility equation in terms of strains can be converted into the compatibility equation in terms of stress. We shall consider both the cases: plane stress case and plane strain case.

(1) **Plane Stress Case**

Substituting the value of \( \epsilon_x, \epsilon_z \) and \( \gamma_{xz} \) from the Hooke's law equations (Eq. 22'15 to Eq. 22'19), we get
\[
\frac{\partial^2 \sigma_x}{\partial x^2} - \mu \frac{\partial^2 \sigma_z}{\partial x^2} + (1-\mu) \frac{\partial^2 \sigma_x}{\partial z^2} = 2(1+\mu) \frac{\partial^2 \gamma_{xz}}{\partial x \partial z}
\]

Again, differentiating first of the equilibrium equations [Eq. 22'18 (a)] with respect to \( x \), and the second [Eq. 22'18 (b)] with respect to \( z \) and adding them together, we get
\[
2 \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} = - \left( \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_z}{\partial z} \right) + \left( \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_z}{\partial z} \right)
\]

Substituting the value of \( \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} \) in (i), we get
\[
\frac{\partial^2 \tau_{xx}}{\partial x \partial z} + \frac{\partial^2 \tau_{xz}}{\partial x \partial z} = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_z}{\partial z}
\]

or
\[
\gamma_{xz} = \frac{2(1 + \mu)}{E} \tau_{xz}
\]

or
\[
\frac{\partial^2 \tau_{xx}}{\partial x \partial z} \left( \sigma_x + \sigma_z \right) = - \frac{1}{1 - \mu} \left( \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_z}{\partial z} \right)
\]

If the body forces are absent, or constant, we have
\[
\gamma_{xz} = 0
\]

which is the same as Eq. 22'21 found for the plane stress case. Thus in case of constant body forces (or no body forces), same compatibility equation holds both for the case of plane stress and for case of plane strain. Hence the stress distribution is the same in both the cases, provided the shape of the boundary and the external forces are...
the same. Also, the stress distribution is the same for all the isotropic materials, since Eq. 22.21 or 22.23 do not contain any elastic constant. The photo-elastic method of determination of stress distribution is based on this conclusion.

22.9. STRESS FUNCTION

The solution of a two dimensional problem of elasticity reduces to the integration of the differential equations of equilibrium together with the compatibility equation and the boundary condition equations. The equations are:

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0 \] \[ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \]

and

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) (\sigma_x + \sigma_z) = 0 \]

(Where Z is the body force per unit volume in the Z-direction. It is assumed that the weight of the body is directed towards Z-axis, so that X is zero.)

It is usual to reduce the above three equations into one single equation in terms of the so-called 'stress function' \( \phi \) defined by

\[ \sigma_x = \frac{\partial \phi}{\partial x} \] \[ \sigma_z = \frac{\partial \phi}{\partial z} \]

and

\[ \tau_{xz} = -\frac{\partial^2 \phi}{\partial x \partial z} - Z \]

The above function \( \phi \) is called the Airy's Stress Function and was introduced by G.B. Airy in 1862. It can be easily seen that the stress function \( \phi \) defined by Eq. 22.25 satisfies the equilibrium equations [Eq. 22.24 (a), (b)]. Performing differentiation on Eq. 22.25, we get

\[ \frac{\partial \sigma_x}{\partial x} = \frac{\partial^2 \phi}{\partial x^2} ; \quad \frac{\partial \tau_{xz}}{\partial z} = -\frac{\partial^2 \phi}{\partial x \partial z} \]

\[ \frac{\partial \tau_{xz}}{\partial x} = \frac{\partial^2 \phi}{\partial z^2} ; \quad \frac{\partial \sigma_z}{\partial z} = -\frac{\partial^2 \phi}{\partial x \partial z} - Z \]

Again, substituting the proper derivatives of stress components as function of Airy's stress function in the compatibility equation, [Eq. 22.24 (c)], we get

\[ \frac{\partial \phi}{\partial x} + 2 \frac{\partial \phi}{\partial x \partial z} = 0 \]

or

\[ \nabla^4 \left[ \frac{\partial \phi}{\partial x^2} + \frac{\partial \phi}{\partial z^2} \right] = 0 \]

or

\[ \nabla^4 \phi = 0 \]

This is a biharmonic differential equation of fourth order, and is known as the compatibility equation in terms of stress function. The solution of Eq. 22.26 must also satisfy the boundary conditions.

In most of the problems, it is usual to find the stresses due to body forces and those due to boundary forces separately. In that case, Eq. 22.25 (c) is modified to

\[ \tau_{xz} = -\frac{\partial \phi}{\partial x \partial z} \]

The solution of two dimensional problems is thus reduced to the integration of the differential equation 22.26 having regard to the boundary conditions. In case of long rectangular strips, the solution of the biharmonic equation is done in the form of polynomials of various degrees.

22.10. EQUILIBRIUM EQUATIONS IN POLAR COORDINATES

In most of the problems, it is more convenient to use polar coordinates. An infinitesimal element in the polar coordinate system is shown in Fig. 22.5.

Let

\[ \sigma_r = \text{normal stress component in radial direction.} \]

\[ \sigma_\theta = \text{normal stress component in circumferential direction.} \]

\[ \tau_{r\theta} = \text{shear stress component acting tangentially to the four surfaces.} \]
Let the stresses on the four faces be further defined by suffixes 1, 2, 3 and 4 respectively.

Resolving the forces in $\theta$ direction and noting that
\[
\sin \frac{d\theta}{2} = \frac{d\theta}{2} \quad \text{and} \quad \cos \frac{d\theta}{2} = 1
\]
and assuming body forces in $\theta$ direction to be zero, we get
\[
((\sigma_0)_2 - (\sigma_0)_4) \, dr \cos \frac{d\theta}{2} + ((\tau_0)_2 + (\tau_0)_4) \, dr \sin \frac{d\theta}{2} + ((\tau_0)_1 - (\tau_0)_3) \, r \, d\theta = 0
\]
or
\[
((\sigma_0)_2 - (\sigma_0)_4) \, dr + ((\tau_0)_2 + (\tau_0)_4) \, dr \cdot \frac{d\theta}{2} + ((\tau_0)_1 - (\tau_0)_3) \, r \, d\theta = 0.
\]

Fig. 22.5. Stresses in polar co-ordinates.

Dividing by $dr \, d\theta$, we get
\[
\frac{((\sigma_0)_2 - (\sigma_0)_4)}{d\theta} + \frac{((\tau_0)_2 + (\tau_0)_4)}{d\theta} + \frac{((\tau_0)_1 - (\tau_0)_3) \, r}{dr} = 0
\]

Assuming the element to be shrunk to a point, we have
\[
\frac{((\sigma_0)_2 - (\sigma_0)_4)}{d\theta} = \frac{\partial \sigma_0}{\partial \theta}
\]
\[
\frac{((\tau_0)_2 + (\tau_0)_4)}{d\theta} = \tau_0
\]
\[
\frac{((\tau_0)_1 - (\tau_0)_3) \, r}{dr} = \frac{\partial (\tau_0)}{\partial r} = \tau_0 + r \frac{\partial \tau_0}{\partial r}
\]

and
\[
\frac{((\tau_0)_1 - (\tau_0)_3) \, r}{dr} = \frac{\partial (\tau_0)}{\partial r} = \tau_0 + r \frac{\partial \tau_0}{\partial r}
\]

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Substituting in (i), we get
\[
\frac{\partial \sigma_0}{\partial \theta} + \tau_0 + \left(\tau_0 + r \frac{\partial \tau_0}{\partial r}\right) = 0
\]
which gives
\[
\frac{1}{r} \frac{\partial \sigma_0}{\partial \theta} + \frac{\partial \tau_0}{\partial \theta} + \frac{\partial \tau_0}{\partial r} = 0 \quad \ldots (I) \quad ([2227 (a)]
\]
which is the equilibrium equation in $\theta$ direction.

Similarly, resolving in $r$-direction, we get
\[
\frac{((\sigma_1)_1 - (\sigma_1)_3) \, r \, d\theta - ((\sigma_0)_2 + (\sigma_0)_4) \, dr \cdot \sin \frac{d\theta}{2}}{d\theta} + ((\tau_0)_1 - (\tau_0)_3) \, dr \cdot \cos \frac{d\theta}{2} = 0
\]
or
\[
\frac{((\sigma_1)_1 - (\sigma_1)_3) \, r \, d\theta - ((\sigma_0)_2 + (\sigma_0)_4) \, dr \cdot \frac{d\theta}{2} + ((\tau_0)_1 - (\tau_0)_3) \, dr \cdot \frac{d\theta}{2} = 0
\]

Dividing by $dr \, d\theta$, we get
\[
\frac{((\sigma_1)_1 - (\sigma_1)_3) \, r \cdot \frac{d\theta}{2} - ((\sigma_0)_2 + (\sigma_0)_4) \cdot \frac{d\theta}{2} + ((\tau_0)_1 - (\tau_0)_3) \cdot \frac{d\theta}{2} = 0
\]
or
\[
\frac{\partial (\sigma_1)}{\partial r} - \sigma_0 + \frac{\partial \tau_0}{\partial r} = 0
\]
or
\[
\frac{\partial \tau_0}{\partial r} + \sigma_0 = 0
\]

which may be rewritten as
\[
\frac{\partial \sigma_0}{\partial r} + \frac{\partial \tau_0}{\partial r} + \frac{1}{r} \frac{\partial \tau_0}{\partial \theta} = 0 \quad \ldots (II) \quad ([2227 (b)]
\]
which is the second equilibrium equation. If, however, $R$ is the body force per unit volume in the $r$-direction, one more term $R \, r \, d\theta \, dr$ will be added to the right hand side, and Eq. (II) will be modified as
\[
\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_0}{r} + \frac{1}{r} \frac{\partial \tau_0}{\partial \theta} + R = 0 \quad \ldots (II a) \quad ([2227 (c)]
\]

22'11. Compatibility Equation and Stress Function in Polar Coordinates

It can be shown that the compatibility equation in terms of stress components, in polar co-ordinates, is given by
\[
\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\right) (\sigma_r + \sigma_\theta) = 0 \quad \ldots (2228)
\]
The stress function $\Phi$ is defined in terms of stress components in polar co-ordinates as follows:

$$
\sigma_r = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \quad \text{[22.29 (a)]}
$$

$$
\sigma_\theta = \frac{\partial^2 \Phi}{\partial \theta^2} \quad \text{[22.29 (b)]}
$$

$$
\sigma_\rho = \frac{1}{r^2} \frac{\partial \Phi}{\partial r} - \frac{1}{r} \frac{\partial \Phi}{\partial \theta} - \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right) \quad \text{[22.29 (c)]}
$$

To yield a possible stress distribution, this stress function must ensure that the condition of compatibility (Eq. 22.28) is satisfied. In cartesian co-ordinates, this condition is $\nabla^2 \Phi = 0$ (Eq. 22.26). A corresponding condition in polar co-ordinates can be obtained by the substitutions: $r^2=x^2+z^2$ and $\theta=\tan^{-1} \frac{z}{x}$. Thus, the compatibility equation, in terms of $\Phi$, in polar co-ordinates becomes:

$$
\left( \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \right) \left( \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \right) = 0
$$

or

$$
\nabla_r^2 (\nabla_\theta^2 \Phi) = 0 \quad \text{[22.30]}
$$

**Example 22.1.** Given the following stress function

$$
\Phi = \frac{H}{\pi} z \tan^{-1} \frac{x}{z}
$$

determine the stress components $\sigma_x$, $\sigma_z$ and $\tau_{xz}$.

**Solution.**

By successive differentiation of the stress function, we get

$$
\frac{\partial \Phi}{\partial z} = \frac{H}{\pi} \left[ -\frac{zx}{x^2+z^2} + \tan^{-1} \frac{x}{z} \right] \quad \text{[i]}
$$

$$
\frac{\partial^2 \Phi}{\partial z^2} = \frac{H}{\pi} \frac{1}{(x^2+z^2)^3} \left[ 2x^2z-x^2z-3x^3-x^3 \right] = -\frac{2H}{\pi} \frac{x^3}{(x^2+z^2)^3} \quad \text{[ii]}
$$

$$
\frac{\partial \Phi}{\partial z} = \frac{H}{\pi} \frac{8x^2z}{(x^2+z^2)^3} \quad \text{[iii]}
$$

$$
\frac{\partial^2 \Phi}{\partial z^2} = \frac{H}{\pi} \frac{8x^3-40x^2z^2}{(x^2+z^2)^3} \quad \text{[iv]}
$$

$$
\frac{\partial \Phi}{\partial z} = \frac{H}{\pi} \frac{3x^2z-3x^4}{(x^2+z^2)^3} \quad \text{[v]}
$$

$$
\frac{\partial \Phi}{\partial x} = -\frac{2H}{\pi} \frac{x^4-z^4}{(x^2+z^2)^3} \quad \text{[vi]}
$$

Similarly,

$$
\frac{\partial \Phi}{\partial x} = \frac{H}{\pi} \frac{24x^2z^4-24x^4z^2}{(x^2+z^2)^3} \quad \text{[vii]}
$$

$$
\frac{\partial \Phi}{\partial x} = \frac{2H}{\pi} \frac{x^2z^2}{(x^2+z^2)^3} \quad \text{[viii]}
$$

$$
\frac{\partial \Phi}{\partial x} = \frac{H}{\pi} \frac{24x^2z^4-24x^4z^2}{(x^2+z^2)^3} \quad \text{[ix]}
$$

Now

$$
\sigma_x = \frac{\partial \Phi}{\partial z} = -\frac{2H}{\pi} \frac{x^3}{(x^2+z^2)^3} \quad \text{(Answer)}
$$

$$
\sigma_z = \frac{\partial \Phi}{\partial z} = \frac{2H}{\pi} \frac{x^2z^2}{(x^2+z^2)^3} \quad \text{(Answer)}
$$

and

$$
\tau_{xz} = \frac{\partial \Phi}{\partial z} = -\frac{2H}{\pi} \frac{x^2z}{(x^2+z^2)^3} \quad \text{(Answer)}
$$

**Check for equilibrium equations**

The above solution will be acceptable only if the stress components satisfy the equilibrium equations. The equilibrium equations are

$$
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0 \quad \text{[i]}
$$

$$
\frac{\partial \sigma_x}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} = 0 \quad \text{[ii]}
$$

Now

$$
\frac{\partial \sigma_x}{\partial x} = \frac{H}{\pi} \frac{2x^4-6x^2z^2}{(x^2+z^2)^3}
$$

$$
\frac{\partial \tau_{xz}}{\partial z} = \frac{H}{\pi} \frac{24x^4-24x^2z^2}{(x^2+z^2)^3}
$$

$$
\frac{\partial \sigma_z}{\partial z} = \frac{H}{\pi} \frac{4x^2z-4x^2z^3}{(x^2+z^2)^3}
$$

$$
\frac{\partial \tau_{xz}}{\partial x} = \frac{H}{\pi} \frac{4x^2z-4x^2z}{(x^2+z^2)^3}.
Substituting the above values in the equilibrium equations, we see that they are satisfied.

Check for compatibility equation

\[ \frac{\partial^2 \Phi}{\partial x^2} + 2\frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \]

Substituting the values, we get

\[
\frac{H}{(x^2+y^2)^{1/2}} \left[ 24xz - 24x^2z^2 + 64x^3z^2 - 24x^4z^2 - 8x^3 + 8x^2 - 40x^2z^2 \right] = 0
\]

Thus the compatibility equation is also satisfied.

Hence the values of \( \sigma_x, \sigma_y \) and \( \tau_{xy} \) found above are acceptable.

Example 22.2. Given the following stress function

\[ \Phi = \frac{P}{\pi} r \theta \cos \theta \]

determine the stress components : \( \sigma_r, \sigma_\theta \) and \( \tau_{r\theta} \).

Solution.

The stress components, by definition of \( \Phi \), are given as follows:

\[ \sigma_r = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial r^2} \quad \ldots (i) \]

\[ \sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2} \quad \ldots (ii) \]

\[ \tau_{r\theta} = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right) = \frac{1}{r^2} \frac{\partial \Phi}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \Phi}{\partial r \partial \theta} \quad \ldots (iii) \]

The various derivatives are as follows:

\[ \frac{\partial \Phi}{\partial r} = \frac{P}{\pi} \theta \cos \theta \]

\[ \frac{\partial^2 \Phi}{\partial r^2} = 0 \]

\[ \frac{\partial \Phi}{\partial \theta} = \frac{P}{\pi} r (-\theta \sin \theta + \cos \theta) \]

\[ \frac{\partial^2 \Phi}{\partial \theta^2} = \frac{Pr}{\pi} (\theta \cos \theta + 2 \sin \theta) \]

\[ \frac{\partial^2 \Phi}{\partial r \partial \theta} = \frac{P}{\pi} (-\theta \sin \theta + \cos \theta) \]

Substituting the values in (i), (ii) and (iii), we get

\[ \sigma_r = \frac{1}{r} \frac{P}{\pi} \theta \cos \theta - \frac{1}{r^2} \frac{P}{\pi} r (\theta \cos \theta + 2 \sin \theta) \]

\[ = \frac{1}{r} \frac{P}{\pi} \theta \cos \theta - \frac{1}{r} \frac{P}{\pi} \theta \cos \theta - \frac{1}{r} \frac{P}{\pi} 2 \sin \theta \]

\[ = -\frac{2}{r} \frac{P}{\pi} \sin \theta \quad \text{(Answer)} \]

\[ \sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2} = 0 \quad \text{(Answer)} \]

\[ \tau_{r\theta} = \frac{1}{r^2} \frac{P}{\pi} r (-\theta \sin \theta + \cos \theta) - \frac{1}{r} \frac{P}{\pi} (-\theta \sin \theta + \cos \theta) \]

\[ = 0 \quad \text{(Answer)} \]

It can be shown that the above values satisfy both equilibrium equations as well as compatibility equation.

22.12. SOLUTION OF TWO DIMENSIONAL PROBLEMS BY POLYNOMIALS

In the case of two dimensional problems, the solution of problem (when the body forces are absent or are constant) consists of the integration of the differential equation

\[ \frac{\partial^2 \Phi}{\partial x^2} + 2 \frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \quad \ldots (22.26) \]

The solution so obtained must satisfy the boundary conditions.

Eq. 22.26 can be solved by means of polynomials of various degrees by suitably adjusting their coefficients. We shall consider here second and third degree polynomials.

(i) Second degree polynomial

Let the solution of Eq. 22.26 be represented by the following polynomial of second degree

\[ \Phi_2 = a_2 x^2 + b_2 x y + c_2 y^2 \quad \ldots (22.31) \]

where suffix 2 denotes the second degree of the polynomial, and \( a, b \) and \( c \) are constants. It can be seen that \( \Phi_2 \), given above satisfies Eq. 22.26. The corresponding stress components (Eq. 22.25) are
STRENGTH OF MATERIALS AND THEORY OF STRUCTURES

\[ \sigma_x = \frac{\partial^2 \Phi}{\partial x^2} = c_2 \]
\[ \sigma_z = \frac{\partial^2 \Phi}{\partial z^2} = a_z \]

and

\[ \tau_{xz} = \frac{\partial^2 \Phi}{\partial x \partial z} = -b_z. \]

This shows that the above stress components do not depend upon the co-ordinates \( x \) and \( z \), i.e. they are constant throughout the body. Thus, the stress function represented by Eq. 2221 represents a state of uniform tensions (or compressions) in two perpendicular directions accompanied with uniform shear, as shown in Fig. 226.

(ii) Third degree polynomial

Let the solution of Eq. 2226 be represented by the following polynomial of third degree:

\[ \Phi = \frac{a_2}{3} x^3 + \frac{b_2}{2} x^2 + c_2 x + d_2 x z + e_2 z^3 \]  \hspace{1cm} \ldots (2232)

It can be shown that \( \Phi \) given above satisfies Eq. 2226. The corresponding stress components are:

\[ \sigma_x = \frac{\partial^2 \Phi}{\partial x^2} = c_2 x + d_2 z \]
\[ \sigma_z = \frac{\partial^2 \Phi}{\partial z^2} = a_2 x + b_2 \]
\[ \tau_{xz} = \frac{\partial^2 \Phi}{\partial x \partial z} = -b_2 \]

and

\[ \tau_{xz} = \frac{\partial^2 \Phi}{\partial x \partial z} = -b_2 x. \]

It should be noted that we are completely free in choosing the magnitudes of the coefficients \( a_2, b_2, c_2 \) and \( d_2 \) since Eq. 2226 is satisfied whatever values these coefficients have. Choosing all coefficients except \( d_2 \) equal to zero, we get from (a),

\[ \sigma_x = d_2 z \]
\[ \sigma_z = 0 \]
\[ \tau_{xz} = 0. \]

Eq. 2233 evidently represents a case of pure bending, as shown in Fig. 227. At \( z = -h, \sigma_x = -d_2 h \) and at \( z = +h, \sigma_x = +d_2 h \). The variation of \( \sigma_x \) with \( z \) is linear.

Similarly, if all the coefficients except \( b_2 \) are zero, we get from (a),

\[ \sigma_x = 0 \]
\[ \sigma_z = b_2 x \]
\[ \tau_{xz} = -b_2. \]

The stresses represented by Eq. 2234 vary as shown in Fig. 228.

The \( \sigma_z \) stress is constant with \( x \) (i.e. constant along the span \( L \) of the beam), but varies with \( z \) at a particular section. At \( z = +h \), \( \sigma_z = b_2 h \) (i.e. tensile), while at \( z = -h, \sigma_z = -b_2 h \) (i.e. compressive). \( \sigma_x \) is zero everywhere. Shear stress \( \tau_{xz} \) is zero at \( x = 0 \) and is equal to \(-b_2 h\) at \( x = L \). At any other section, the shear stress is proportional to \( x \).
Example 22.3. Given the following stress function:

\[ \Phi = - \frac{F}{c^2} x^2 (3d - 2z) \]

Determine the stress components and sketch their variations in a region included in \( z = 0, z = d, x = 0 \), on the side \( x \) positive.

**Solution**

The given stress function may be rewritten as

\[ \Phi = - \frac{3F}{d^2} x^2 + \frac{2F}{d^3} x^3 \quad \text{...(i)} \]

\[ \frac{\partial^2 \Phi}{\partial z^2} = - \frac{6Fx}{d^2} + \frac{12F}{d^3} xz \]

\[ \frac{\partial^2 \Phi}{\partial x^2} = 0 \]

and

\[ \frac{\partial^2 \Phi}{\partial x \partial z} = - \frac{6Fz}{d^2} + \frac{6F}{d^3} z^2 \]

Hence

\[ \sigma_x = \frac{\partial^2 \Phi}{\partial z^2} = - \frac{6Fx}{d^2} + \frac{12F}{d^3} xz \quad \text{...(ii)} \]

\[ \sigma_z = \frac{\partial^2 \Phi}{\partial x^2} = 0 \quad \text{...(iii)} \]

and

\[ \tau_{xz} = - \frac{\partial^2 \Phi}{\partial x \partial z} = \frac{6Fx}{d^2} - \frac{6F}{d^3} z^2 \quad \text{...(iv)} \]

Eqs. (ii), (iii) and (iv) give the values for the three stress components. Let us find their values at certain boundary points.

(i) Variation of \( \sigma_x \)

From (ii), it is clear that \( \sigma_x \) varies linearly with \( x \), and at a given section it varies linearly with \( z \).

At \( x = 0 \) and \( z = \pm d \), \( \sigma_x = 0 \)

At \( x = L \) and \( z = 0 \), \( \sigma_x = - \frac{6FL}{d^2} \)

At \( x = L \) and \( z = +d \), \( \sigma_x = - \frac{6FL}{d^2} + \frac{12F}{d^3} Ld = - \frac{6FL}{d^2} \)

At \( x = L \) and \( z = -d \), \( \sigma_x = - \frac{6FL}{d^2} - \frac{12F}{d^3} Ld = - \frac{18FL}{d^2} \)

The variation of \( \sigma_x \) is shown in Fig. 22.9.

(ii) Variation of \( \sigma_z \)

\( \sigma_z \) is zero for all values of \( x \).

(iii) Variation of \( \tau_{xz} \)

From (iv), it is clear that variation of \( \tau_{xz} \) is parabolic with \( z \). However, \( \tau_{xz} \) is independent of \( x \), and is thus constant along the length, corresponding to a given value of \( z \).

At \( z = 0 \), \( \tau_{xz} = 0 \)

At \( z = +d \), \( \tau_{xz} = \frac{6F}{d^2} \cdot d - \frac{6F}{d^3} d^2 = 0 \)

At \( z = -d \), \( \tau_{xz} = - \frac{6F}{d^2} \cdot d - \frac{6F}{d^3} (-d)^2 = - \frac{12F}{d} \)

The variation of \( \tau_{xz} \) is shown in Fig. 22.9.
Example 22.4. Investigate what problem of plane stress is satisfied by the stress function

$$\Phi = \frac{3F}{4d} \left( xz - \frac{x^2}{3d^2} \right) + \frac{P}{2} \frac{z^2}{2^2}$$

applied to the region included in \( z=0, z=d, x=0 \) on the side \( x \) positive.

Solution.

The given stress function may be written as

$$\Phi = \frac{3F}{4d} xz - \frac{1}{4} \frac{F}{d^2} \frac{z^2}{2} + \frac{P}{2} \frac{z^2}{2^2}$$

... (i)

and

$$\frac{\partial^2 \Phi}{\partial x^2} = 0$$

$$\frac{\partial^2 \Phi}{\partial z^2} = -\frac{3 \times 2}{4d} \frac{F}{d^2} + \frac{2P}{2} = P - 1.5 \frac{F}{d^2} xz$$

Hence the stress components are:

$$\sigma_x = \frac{\partial^2 \Phi}{\partial z^2} = P - 1.5 \frac{F}{d^2} xz$$

... (ii)

$$\sigma_z = \frac{\partial^2 \Phi}{\partial x^2} = 0$$

... (iii)

$$\tau_{xz} = -\frac{\partial^2 \Phi}{\partial x \partial z} = \frac{3}{4} \frac{F}{d^2} \frac{z^2}{2} - \frac{3}{4} \frac{F}{d^2}$$

... (iv)

(i) Variation of \( \sigma_x \)

$$\sigma_x = P - 1.5 \frac{F}{d^2} xz$$

When \( x=0 \) and \( z=0 \) or \( \pm d \), \( \sigma_x = P \) (i.e. constant across the section)

When \( x=L \) and \( z=0 \), \( \sigma_x = P \)

When \( x=L \) and \( z=+d \), \( \sigma_x = P - 1.5 \frac{FL}{d^2} \)

When \( x=L \) and \( z=-d \), \( \sigma_x = P + 1.5 \frac{FL}{d^2} \)

Thus, at \( x=L \), the variation of \( \sigma_x \) is linear with \( z \).

The variation of \( \sigma_x \) is shown in Fig. 22.10.

Fig. 22.10

22.13. BENDING OF A CANTILEVER LOADED AT THE END

![Diagram of a cantilever loaded at the end]
Consider a cantilever of span \( L \), subjected to a point load \( P \), which may be considered to be the resultant of the shearing forces across the section at \( x=0 \). Let the depth of the section of the cantilever be \( 2h \) and width unity.

The normal stress \( \alpha_z \) at any point in the cantilever may be taken to be proportional to \( x \) and \( z \).

\[
\alpha_z = \frac{\partial^3 \Phi}{\partial x^3} = c_1 \alpha_z
\]

where \( c_1 \) is a constant, to be determined.

Integrating the above equation twice with respect to \( z \), we get

\[
\frac{d\Phi}{dz} = c_1 \frac{x^2}{2} + f_{ix}
\]

and

\[
\Phi = c_1 \frac{x^3}{6} + z f_{ix} + f_{ix}
\]

where \( f_{ix} \) and \( f_{ix} \) are the functions of \( x \).

Eq. (2) gives the stress function \( \Phi \). It should satisfy the compatibility equation \( \nabla^4 \Phi = 0 \).

\[
\frac{\partial^4 \Phi}{\partial x^4} + 2 \frac{\partial^4 \Phi}{\partial x^2 \partial z^2} + \frac{\partial^4 \Phi}{\partial z^4} = 0
\]

From (2),

\[
\frac{\partial^4 \Phi}{\partial x^4} = z \frac{d^2 f_{ix}}{dx^2} + \frac{d^2 f_{ix}}{dx^2}
\]

\[
\frac{\partial^4 \Phi}{\partial x^2 \partial z^2} = 0 \quad \text{and} \quad \frac{\partial^4 \Phi}{\partial z^4} = 0
\]

Substituting these in (3), we get

\[
z \frac{d^2 f_{ix}}{dx^2} + \frac{d^2 f_{ix}}{dx^2} = 0
\]

This must be satisfied for all values of \( x \) and \( z \). If \( z \) is not to be zero, we get from (4),

\[
\frac{d^2 f_{ix}}{dx^2} = 0, \quad \text{or} \quad f_{ix} = c_5 x^2 + c_7 x^2 + c_9 x + c_6
\]

and

\[
\frac{d^2 f_{ix}}{dx^2} = 0, \quad \text{or} \quad f_{ix} = c_5 x^2 + c_7 x^2 + c_9 x + c_9
\]

Thus, we have to determine nine constants \( c_1, c_6, \ldots, c_9 \).
From which \( c_1 = \frac{P}{\frac{4}{3}h^3} = \frac{P}{I_{YY}} \) ...(8)

where \( I_{YY} = \frac{1}{12} \times 1 \times (2h)^3 = \frac{8}{3}h^3 \)

**Final stresses**

Thus, the final stresses are as follows:

(1) \( \sigma_x = c_1 \frac{P}{I_{YY}} = \frac{Mz}{I_{YY}} \) ...(IV)

This is the same as used in the simple theory of bending.

(2) \( \sigma_y = 0 \)

(3) \( \tau_{xz} = \frac{c_1}{2} \left( \frac{h^2 - z^2}{h^2} \right) = \frac{P}{2I_{YY}} \left( \frac{h^2 - z^2}{h^2} \right) \) ...(V)

This shows that the external force must be distributed parabolically on the external face.
**Welded Joints**

**23.1. GENERAL**

Welding is a process of joining two similar pieces of metal by fusion or pressure. A metallic bond is established between the two pieces. This bond has the same mechanical and physical properties as the parent metal. A number of methods are used for the process of fusion. The oxyacetylene or gas welding and electric arc welding are the most important of these methods. The metal at the joint is melted by the heat generated from either an electric arc or an oxyacetylene flame and fuses with metal from a welding rod. After cooling, the parent metal (base metal) and the weld metal form a continuous and homogeneous joint. The weld connections have become so reliable that they are replacing riveted joints, both in structural as well as machine design. Before proceeding further, we shall discuss here, in short, the advantages and disadvantages of welded connections.

**Advantages**

1. Welded joints are economical, from the points of view of cost of labour and materials, both. The filler plates, gusset plates, connecting angles, etc. are eliminated in welded joints. The smaller sizes of members, compared to those which may be used in riveted connections from the practical point of view, may be used here.

2. The efficiency of the welded joint is 100% as compared to an efficiency of 75 to 100% in case of riveted joints.

3. The fabrication of a complicated structure is easier by connection as in case of a circular steel pipe. The alterations or additions in existing structure are facilitated by it.

4. The welding provides very rigid joints. This is in keeping with the modern trend of providing rigid frames.

5. The noise associated with the riveting work is a source of a great nuisance. This is avoided in welding operation.

6. When riveting is done in populated localities, safety precautions to protect the public from the flying rivets has to be taken. No such precautions are necessary in case of welding operation.

7. The welded structures look more pleasing in comparison to the riveted ones.

**Disadvantages**

Not withstanding the advantages narrated above, the welded connections have a number of disadvantages in comparison to the riveted connections. The same have been narrated below:

1. No provision for expansion and contraction is kept in welded connection and, therefore, there is possibility of cracks developing in such structures.

2. Due to uneven heating and cooling of the members during welding the members may distort resulting in additional stresses.

3. The inspection of welding work is more difficult and costlier than the riveting work. The welding work requires a skilled person, while, a semi-skilled person can do the riveting work.

4. On account of extreme heat, fatigue may take place.

**23.2. TYPES OF WELDS**

In structural practice, the following types of welds are used:

1. Butt weld or groove weld.

2. Fillet weld.

3. Plug or slot weld.

1. **Butt Weld or Groove Weld**

When the joining members are to be jointed in such a way that they form a T or butt against each other, butt weld is used.

The butt weld is usually made convex on either sides. This extra area is called reinforcement. The reinforcement varies from 1 to 3 mm.

The common types of butt welds have been shown in Fig. 23.1. The square butt joints shown in Fig. 23.1 (a) and (b) are used for thickness less than 8 mm. The effective thickness of the weld, called throat thickness, is less than the thickness T of the plates jointed. It is taken as \( \frac{1}{6} T \). In single V-butt joint [Fig. 23.1 (c)], the throat thickness is taken at \( \frac{1}{2} T \). The butt welds of Fig. 23.1 (d) and (e) are fully
effective. In Fig. 23.1 (e), a backing strip has been used. As a rule, in single-$V$, single-$U$, single-$J$ butt welds, with backing strips, and in which the welding has been done from one side, full penetration is not possible, and the effective throat thickness is taken equal to $\frac{1}{2} T$.

The sides containing the right angles are called legs. The size of a fillet weld is specified by the minimum leg length. The throat or throat distance of the fillet weld is equal to the perpendicular distance of the corner from the hypotenuse, the reinforcement being neglected [Fig. 23.2 (a)]. The throat thickness $t$ is, therefore, equal to $\frac{\text{min. leg length}}{\sqrt{2}} = 0.707 \times \text{minimum leg length}$.

For angles other than $90^\circ$, throat thickness is given by:

$\text{Maximum throat thickness} = k \times \text{minimum leg length}$.

The value of $k$ is given in the table below:

<table>
<thead>
<tr>
<th>Angle</th>
<th>60$^\circ$ to 90$^\circ$</th>
<th>91$^\circ$ to 100$^\circ$</th>
<th>101$^\circ$ to 106$^\circ$</th>
<th>107$^\circ$ to 113$^\circ$</th>
<th>114$^\circ$ to 120$^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>0.7</td>
<td>0.65</td>
<td>0.6</td>
<td>0.55</td>
<td>0.50</td>
</tr>
</tbody>
</table>

The maximum size of a fillet weld at the square edge of a plate [Fig. 23.2 (b)] is 1.5 mm less than the plate thickness and in case of a weld at the rounded edges of flanges or the toe of an angle is kept three-fourths the thickness of the edge [Fig. 23.2 (c)].

When the fillet weld is placed parallel to the direction of the force, on both sides of the member, it is called a side fillet weld. When the weld is placed at the end of the member, such that it is perpendicular to the direction of the force, it is called an end fillet weld (Fig. 23.3).

In all the above mentioned welds with backing strips and in case of double-$V$, double-$U$ and double-$J$ butt welds, full penetration is possible and the effective thickness of the throat is taken equal to the thickness of the plates joined. Whenever two plates of different thickness are joined, the thickness of the thinner plates must be taken into account.

2. Fillet Welds

When the lapped plates are to be joined, fillet welds are used. These are generally of right-angled triangle shape. The outer surface is generally made convex.
The effective length of the fillet weld is taken as the overall length minus twice the weld size. The effective length should not be less than four times the size of the weld, otherwise the weld size must be taken as one-fourth of its effective length. If only the side welds are used, the length of each side fillet weld must not be less than the perpendicular distance between the two.

3. Plug or Slot Weld

Whenever sufficient space is not available for providing the necessary length of the fillet weld, the plug or slot weld is provided to strengthen the connection. The slot or plug weld is also used for equalising the stress in plates and to prevent buckling in case of wide plates.

A circular or a slotted hole is made into one of the members to be jointed. The weld metal is then filled in the hole. Otherwise a fillet weld is provided along the edge of the hole [Fig. 23/4 (a)].

(a) FILLET WELD (b) PLUG WELD (c) SLOT WELD

![Diagram](image)

Fig. 23/4

The minimum diameter and the width of the hole should not be less than the thickness of the part containing the hole plus 8 mm, the maximum diameter and width being limited to 2.25 times the thickness of the plate punched.

23.3. STRENGTH OF WELDS

1. BUTT WELDS

For full penetration butt welds [Fig. 23/1 (c), (d), (e), (g)] the strength of the welds is equal to that of the parent metal. In such cases, no calculations are needed. Even then if the strength is to be found, it can be found by the following formula:

\[ P = f_w \times t \times L \]

where \( P \) = strength of weld,
\( t \) = effective thickness, including reinforcement,
\( L \) = length of the weld,
\( f_w \) = working stress in tension or compression.

For full penetration welds, the effective thickness is equal to the thickness of the thinner plate. If the full penetration weld has not been used the effective thickness must be determined.

WELDED JOINTS

It must be noted here that a butt weld is generally subjected to either tensile or compressive stress.

The allowable stress in butt welds is taken as that for the parent metal. Thus for mild steel conforming to IS: 226-1962 as parent metal and with electrodes conforming to IS: 814-1963, the allowable stresses in the welds, as recommended in IS: 816-1956 are as under:

<table>
<thead>
<tr>
<th>Kind of stress</th>
<th>Max. permissible value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Tension on section through throat of butt weld</td>
<td>142 N/mm² (1420 kg/cm²)</td>
</tr>
<tr>
<td>2. Compression on section through throat of butt weld</td>
<td>142 N/mm² (1420 kg/cm²)</td>
</tr>
<tr>
<td>3. Fibre stress in bending (a) Tension</td>
<td>157.5 N/mm² (1575 kg/cm²)</td>
</tr>
<tr>
<td>(b) Compression</td>
<td>157.5 N/mm² (1575 kg/cm²)</td>
</tr>
<tr>
<td>4. Shear on section through throat of butt and fillet welds</td>
<td>102.5 N/mm² (1025 kg/cm²)</td>
</tr>
</tbody>
</table>

Maximum permissible value of stresses for shear and tension are reduced to 80% of those given above if the welding is done at site (field). When effects of wind and/or earth-quake forces are considered, maximum permissible values of stress are increased by 20%.

2. FILLET WELDS

(a) Side fillets

When a side fillet weld is subjected to a load \( P \), it is subjected to shear stresses only. The maximum stress will develop at the throat and failure will occur by shear along the throat. If we assume a uniform distribution of shear stress,

\[ P = \text{Stress} \times \text{Area} \]

\[ \text{Area} = \text{Throat thickness} \times \text{Length} \]

\[ P = 0.707 \times h \times L \]

or

\[ \text{Stress} = \frac{P}{0.707 \times h \times L} \]

(b) End fillets

If a load \( P \) is applied to an end fillet weld as shown in Fig. 23/5 it will be subjected to different types of stresses on different
planes. The plane DB will be subjected to tensile stress and the plane AC to shearing stress. In both the cases the stress will be numerically equal to \( \frac{P}{hL} \). The section AD is subjected to tensile and shear stresses both, each equal to \( \frac{P}{Lh\sqrt{2}} = \frac{P}{Lh} \).

![Fig. 23-5](image)

It can be proved that the maximum tension on this section is \( \frac{106P}{\sqrt{2}} \) and the maximum shear stress is \( \frac{112P}{Lh} \). As the material is weaker in shear we shall consider the shear stress only. It may be noted that the shear stress developed in this case is less than that developed in case of side fillets. So for convenience in design, no distinction is made between the end fillets.

The allowable stress in shear is taken to 10.25 N/mm².

3. PLUG OR SLOT WELDS

The strength of a plug or slot weld is equal to the cross-sectional area of the plug or slot, multiplied by the allowable stress.

If fillet weld is provided in the plug or slot, its effective length is taken equal to the average length of the throat of the fillet. Generally, it is taken equal to the length of a line running parallel to the vertical leg of the weld at one-fourth the leg dimension \( h \) from it.

For a hole of diameter \( d \) the length of the fillet is \( \pi \left( \frac{d - h}{2} \right) \). The maximum length is limited to 10 times the thickness of the material.

Example 23'1. A 100 mm×10 mm plate is to be welded to another plate 150 mm×10 mm by the fillet welding on three sides as shown in Fig. 23'6. The size of the weld is 6 mm. Find out the necessary overlap of the plate if the smaller plate is to develop full strength. The allowable stress in plates 142 N/mm².

Solution.

The total load taken by the smaller plate

\[ 100 \times 10 \times 142 = 142000 \text{ N} = 142 \text{ kN} \]
thick plate will carry 1.5P. Therefore, to keep the stresses same in both the end welds, we must keep the size of the welds in proportion to the thickness of the respective plates.

Let the size of the lower weld = h
∴ Size of the upper weld = 1.5h

The length of the welds in each case = 120 mm

Strength of lower weld = 0.707 × h × 120 × 102.5 = 8696h N

Strength of upper weld = 0.707 × 1.5h × 120 × 102.5 = 13044h N

Total load to be carried by the tie bar = 160000 N.

\[ \frac{290}{400582} \times 1380 \text{ mm} \]

Length of the end weld = 200 cm

Therefore, length of side welds = 1380 - 200 = 1180 mm

\[ \text{Overlap} = \frac{1180}{2} = 590 \text{ mm.} \]

Example 23.3. A tension member consisting of two channel section 200 mm × 75 mm @ 22.1 kg/m back to back is to be connected to gusset plate. Design the welded joint for the condition that the section is loaded to its full strength. A = 2821 sq. mm, thickness of the flange = 11.4 mm and the thickness of the web = 6.1 mm.

Solution.

In case of rolled section, the size of the weld is taken as three-fourth of the thickness.

∴ Size of the weld = \( \frac{3}{4} \times 6.1 = 4.6 \text{ mm} \)

We shall provide 4 mm weld.

\[ \text{Strength of weld per linear cm} = 0.707 \times 4 \times 102.5 = 290 \text{ N} \]

The load to be carried by each channel = 2821 × 142 = 400582 N

Total length of weld required for one channel

\[ \frac{400582}{290} = 1380 \text{ mm} \]

Length of the end weld = 200 cm

Therefore, length of side welds = 1380 - 200 = 1180 mm

\[ \text{Overlap} = \frac{1180}{2} = 590 \text{ mm.} \]

Example 23.4. An I-section is build-up by welding a 250 mm × 15 mm web plate to two 150 mm × 15 mm flange plates by 8 mm filler welds. Find out maximum shearing force which may be permitted if the mean shearing stress in web and maximum shear stress in weld are not to exceed 100 N/mm².

Solution.

Shear stress q at the section passing through welds is given by

\[ q = \frac{F}{I_{xx} b \left( A y \right)} \]

where

\[ b = \text{Effective thickness} = 2 \times \text{throat thickness} = 2 \times 8 = 16 \text{ mm} \]

\[ A y = \text{Moment of the flange area about xx} = 150 \times 15(125 + 75) = 298125 \text{ mm}² \]

\[ q = \text{Allowable shear stress} = 100 \text{ N/mm}² \]

\[ 100 = \frac{298125}{9862 \times 10^6 \times 1.3} \]
STRENGTH OF MATERIALS AND THEORY OF STRUCTURES

From which \( F = 373,805 \text{ N} = 373'8 \text{ kN} \).

The maximum shear force limited on the web
\( = 250 \times 15 \times 100 = 375,000 \text{ N} = 375 \text{ kN} \).

The max. allowable shearing force = 373'8 kN.

23'4. FILLET WELDING OF UNSYMMETRICAL SECTIONS, AXIALLY LOADED

Uptil now we have considered the fillet welding of symmetrical sections. In case of unsymmetrical sections like angles and Tee which are loaded along the axis passing through their centroid, the weld lengths are so arranged that the gravity axis of the weld coincides with the neutral axis. This will avoid eccentricity of loading and hence the bending moment.

Let us consider an angle section subjected to load \( P \), welded to a gusset plate as shown in Fig. 23'10.

![Fig. 23'10](image)

Let \( L_1 \) and \( L_2 \) be the required lengths of welds on the two faces, and \( P_1 \) and \( P_2 \) be the resisting forces exerted by the respective welds. These are assumed to act along the edges of the angle.

Taking moments about the line of action of \( P_2 \) we obtain,
\[(a+b) \times P_1 = Pa\] \( \therefore P_1 = \frac{Pa}{a+b} \) ... (1)

Similarly, taking moments about the line of action of \( P_1 \),
\[(a+b) \times P_2 = Pb\] \( \therefore P_2 = \frac{Pb}{a+b} \) ... (2)

If \( s \) is the strength of the weld per unit length of the weld,
\[L_1 = \frac{P_1}{s(a+b)} \quad \text{and} \quad L_2 = \frac{P_2}{s(a+b)}\]

Sometimes it is not possible to accommodate the required length of the weld on the sides of the section. In such cases, the end fillets are also provided. The procedure for analysing such a

Example 23'5. An equal angle \( 65 \text{ mm} \times 65 \text{ mm} \) @ 9'4 kg/m of thickness 10 mm carries a load of 160 kN, applied along its centroidal axis. The angle is to be welded to a gusset plate. Find out the lengths of the side fillet welds required at the heel and the toe of angle. Its C.G. is at 19'7 mm from its heel.

Solution.

Taking moments about the line of action of \( P_2 \),
\[65P_2 = 160 \times 45.5 \quad \therefore P_1 = 111'5 \text{ kN}\]
\[P_2 = P - P_1 = 160 - 111'5 = 48'5 \text{ kN}\]

The maximum size of the weld for a rounded edge at the toe of the angle is three-fourths of the thickness, i.e. \( \frac{3}{4} \times 10 = 7'5 \text{ mm} \).

![Fig. 23'11](image)

Strength of weld per linear mm = 0'707 \times 7'5 \times 102'5 = 543'5 N
\[L_1 = \frac{P_1}{543'5} = \frac{111'5 \times 1000}{543'5} = 205 \text{ mm}\]

and
\[L_2 = \frac{P_2}{543'5} = \frac{48'49 \times 1000}{543'5} = 89'2 \text{ mm}\]

These values are effective and must be increased by twice the weld size, i.e. \( 2 \times 7'5 = 15 \text{ mm} \) to get the actual lengths of the welds.

Example 23'6. A tie bar consisting of a single angle \( 60 \text{ mm} \times 60 \text{ mm} \times 10 \text{ mm} \) is to be welded to a gusset plate. The tie bar carries a load of 150 kN along its centroidal axis. Design the joint if both the side fillets and end fillets are to be provided. The centroidal axis lies at 18'5 mm from the heel of the angle.

Solution.

![Fig. 23'12](image)
The maximum size of the fillet weld at the end, along the square edge of the angle will be 1.5 mm less than the thickness of the angle. Therefore, the maximum size of the end fillet = 10 - 1.5 = 8.5 mm.

The maximum size of the side fillets, along the rounded edge

= \frac{3}{4} \times 10 = 7.5 \text{ mm}

We shall provide 7.5 mm weld throughout.

Strength of the weld per cm length

= 0.707 \times 7.5 \times 10^2 = 543.5 \text{ N}

The end weld will be placed symmetrical about the line of action of the load in order to avoid eccentricity. The maximum length of the end weld is, therefore, equal to 2 \times 18.5 = 37 \text{ mm}.

The strength of the end weld

= 543.5 \times 37 = 20100 \text{ N}

= 20.11 \text{ kN}

Taking moments about the line of action of force \( P \), we get

60P_2 = 150 \times 41.5 - 20.11 \times 41.5

From which \( P_2 = 89.94 \text{ kN} \)

\( P_1 = 150 - 89.94 - 20.11 = 39.95 \text{ kN} \)

\( \therefore \text{ Length } L_1 = \frac{P_1}{543.5} = 39.95 \times 1000 \frac{543.5}{543.5} = 73.5 \text{ mm} \)

and \( \text{ Length } L_2 = \frac{P_2}{543.5} = \frac{89.94 \times 1000}{543.5} = 165.5 \text{ mm} \).

Twice the size of the weld, i.e., 2 \times 7.5 = 15 mm must be added to above lengths to get the actual lengths of the side fillets.

235. WELDED JOINT SUBJECTED TO BENDING MOMENT

If the load \( P \) acting on a welded joint does not pass through the centroid of weld lines, it will subject the welded joint to bending moment in addition to the direct load. Consider I-beam welded to the flange of a column and subjected to a load \( P \) at eccentricity \( e \), shown in Fig. 23'13.

Let \( L \) be the length of the weld and \( t \) be the throat thickness. The joint is subjected to a load \( P \) and a moment \( P \times e \).

The area resisting direct stress = \( 2 \times L \times t \).

\( \therefore \) Direct throat stress \( f_d = \frac{P}{2Lt} \). It will act in the direction of the load, i.e, vertically in this case.

Section modulus of both the weld lines = \( \frac{2 \times t \times L^3}{6} = \frac{L^3}{3} \)

Bending stress, \( f_b = \frac{M}{Z} = \frac{P \times e \times 3}{L^3 e} \). The stress acts in horizontal direction.

Therefore, the direct and bending stresses act at right angles to each other. The resultant is given by

\( f = \sqrt{f_a^2 + f_b^2} \).

Example 237. A bracket consisting of an I-section is connected to the flange of a vertical column by means of two side fillets 250 mm deep and 8 mm thick, as shown in Fig. 23'13. The bracket carries a load of 160 kN at an eccentricity of 60 mm. Calculate the throat stress in the weld.

Solution.

Throat thickness = \( \frac{8}{\sqrt{2}} = 5.67 \text{ mm} \)

Direct stress \( f_a = \frac{160000}{2 \times 250 \times 5.67} = 56.4 \text{ N/mm}^2 \)

Bending moment = 160000 \times 60 = 96 \times 10^5 \text{ N-mm}

\( Z_{xx} \) of the weld lines = \( 2 \times \frac{1}{6} \times 5.67(250)^3 \)

\( = 118125 \text{ mm}^3 \)
Example 23.8. A bracket carrying a load of 120 kN is connected to a column by means of two horizontal fillet welds, each 150 mm long and 10 mm thick. The load acts at 80 mm from the face of the column. Find the throat stress.

Solution.

Fig. 23'14 shows the arrangement of the bracket.

Throat thickness = \( \frac{10}{\sqrt{2}} = 7.07 \) mm

Direct force = 120 kN

Direct stress \( f_d = \frac{120 \times 1000}{2 \times 150 \times 7.07} = 56.6 \) N/mm\(^2\), acting in vertical direction.

Moment = 120000 \times 80 = 96 \times 10^5 \) N-mm

The forces due to bending in the two welds will form a resisting couple = 150 \times 7.07 \times f_d \times 120 = 1.27 \times 10^5 \) N-mm.

From which \( f_d = 75.4 \) N/mm\(^2\) acting in vertical direction.

Resultant stress = \( \sqrt{(75.4)^2 + (56.6)^2} = 94.3 \) N/mm\(^2\).

23'6. WELDED JOINTS SUBJECTED TO TORSION

If a bracket carrying an eccentrically applied load \( P \) is connected to the column as shown in Fig. 23'15 the welds will be subjected to torsion. The difference between this sort of bracket connection and that shown in Fig. 23'13 must be noted. In this case the moment is acting in a plane containing the welds, while in previous case, the moment was acting in a plane perpendicular to the welds. The method of analysis for welded joint subjected to torsion is similar to that for riveted joints.

WELDED JOINTS

Let the load be transmitted to the stanchion by means of weld of throat thickness \( t \) on top, bottom and one side of the bracket.

Total throat area = \( t(d + 2b) \)

Direct throat stress = \( \frac{P}{t(d + 2b)} \).

To find out the torsional stress in the welds, the position of centroid of the welds must be determined. Let \( x \) be the distance of centroid from the face of the stanchion CD.

\[ x = \frac{2b \times t \times \frac{b}{2} + d \times t \times b}{2b \times t + d \times t} = \frac{b^2 + bd}{2b + d} \]

Eccentricity \( e = (x + \overline{r}) \cos \theta \)

Twisting moment \( T = P \times e \)

It will be assumed that the stresses developed due to torsion at any point of the weld will be proportional to its distance from the centroid and that it will act at right angles to the radius vector.

If \( T \) is the twisting moment and \( f_t \) is the stress in weld caused due to twisting,

\[ f_t = \frac{T \times r}{J} \]

where \( J \) is the polar moment of inertia of the weld sections out the centroid \( O \).

\[ J = J_{zz} + J_{xy} \]

Here

\[ J_{zz} = 2bh \times \frac{ix}{4} + \frac{td^3}{12} \]

and

\[ J_{xy} = \frac{2tb^3}{12} + 2b \left( x - \frac{b}{2} \right)^2 + td \left( b - x \right)^2 \]
The maximum stress will occur at C and D. The direct stress will act in the direction of the load and the stress due to torsion will act at right angles to the direction of the radius vector. The resultant of the two will give the total stress at the point.

Example 23'9. A circular shaft of diameter 120 mm is welded to a rigid plate by a fillet weld of size 6 mm. If a torque of 8 kN-m applied to the shaft find the maximum stress in the weld.

Solution.

We shall derive relation between the stresses developed and the torque applied for such a shaft. Let \( d \) be its diameter and \( h \) be the size of the weld.

Considering a small area \( \delta a \) of the weld as shown in Fig. 23'16,
\[
J = \sum \delta a = \left( \frac{d}{2} \right)^2 \tag{23'16}
\]

Throat thickness \( h = \sqrt{\frac{h}{2}} \)

\[
\sum \delta a = \pi d \times \left( \frac{h}{\sqrt{2}} \right)
\]

\[
J = \frac{d^3}{4} \times \frac{\pi dh}{\sqrt{2}} = \frac{\pi dh}{4\sqrt{2}}
\]

Stress

\[
\frac{T_J}{J} = \frac{d}{2} \times \frac{4\sqrt{2}}{\pi dh}
\]

Substituting the values, we get

Maximum stress

\[
\frac{2\sqrt{2}(8 \times 100 \times 1000)}{\pi(120)^2(6)} = 83.4 \text{ N/mm}^2.
\]

Example 23'10. A bracket is subjected to a load of 100 kN as shown in Fig. 23'17. A bracket is welded to a stanchion by means of three lines of weld on three sides as indicated in the figure. Find out size of the welds so that the load is carried safely.

Solution.

If \( x \) is the distance of centroid of weld area from \( AB \),

\[
\bar{y} = \frac{2\times120 \times 60}{2\times120 + 240} = 30 \text{ mm}
\]

\( \therefore \) Eccentricity of the load = \( 100 + 120 - 30 = 190 \text{ mm} \)

\[
I_{xx} = \frac{1}{12} (240)^2 + 2 \times 120 \times (120)^2
\]

\[
= 460.8 \times 10^4 \text{ mm}^4
\]

\[
\text{where } t = \text{ thickness of the weld.}
\]

\[
I_{yy} = 2.120 \times (30)^2 + 2 \times \frac{1}{12} t(120)^2 + 2 \times 120 \times (30)^2
\]

\[
= 72 \times 10^4 \text{ mm}^4
\]

\( \therefore \)

\[
J = I_{xx} + I_{yy} = 460.8 \times 10^4 + 72 \times 10^4 = 532.8 \times 10^4 \text{ mm}^4
\]

Area = \( 2 \times 120 \times t + 240 t = 480 t \text{ mm}^2 \)

The maximum stress due to torsion will occur either at C or D.

Length of radius vector for C or D

\[
= \sqrt{(120)^2 + (90)^2} = 150 \text{ mm}
\]
Maximum stress due to torsion
\[ f_t = \frac{T r}{J} = \frac{10000 \times 190 \times 150}{5328 \times 10^4} \]
\[ = \frac{534.9}{t} \text{ N/mm}^2 \]

Direct stress, \( f_d = \frac{10000}{480} = \frac{208.3}{t} \text{ N/mm}^2 \)

The angle between the stresses is \( \theta \), as shown in Fig. 23.17.
\[ \cos \theta = - \frac{90}{150} = -0.6 \]

Resultant stress
\[ f_r = \sqrt{f_t^2 + f_d^2 - 2f_t f_d \cos \theta} \]
\[ = \sqrt{\left( \frac{534.9}{t} \right)^2 + \left( \frac{208.3}{t} \right)^2 - 2 \left( \frac{534.9}{t} \right) \left( \frac{208.3}{t} \right) \times 0.6} \]
\[ = \frac{680.6}{t} \text{ N/mm}^2 \]

Allowable shear stress = 102.5 N/mm²

\[ t = 6.64 \text{ mm} \]

Size of weld = \( \sqrt{2} \times 9.39 \text{ mm} \).

Hence provided 10 mm size weld.

**PROBLEMS**

1. A 150 mm × 15 plate is welded to other plate by two side welds 12 cm each and end fillet of 100 mm length. Find the safe axial load to which this joint may be subjected if the size of the weld is 7 mm.

2. A 100 mm × 10 plate is welded to other by means of two end fillets and two side fillets of 8 mm as shown in Fig. 23.18. If the plate is loaded to its full strength, find out the required overlap length.

3. An equal angle 75 mm × 75 mm @ 110 kg/m is subjected to a load for 180 kN, whose line of action passes through the centroid of the section, which is 22.2 mm from the heel. This angle is to be welded to a gusset plate. If the size of the weld is to be 8 mm, find the length of the size fillet welds.

4. An I-section is made up of a 200 mm × 10 mm thick web plate welded to two flange plates 120 mm × 10 mm thick by means of fillet welds to size 6 mm. Calculate the maximum shear force which this section can resist.

5. Fig. 23.19 shows a 10 mm angle bracket 100 mm wide welded to the flange of a steel stanchion. It carries a vertical load of 240 kN. The connection consists of continuous 10 mm weld extending along the top and both sides and returned at the bottom of the bracket. Treating the 240 kN load as a vertical shear load (i.e., neglecting bending moment), calculate the depth of bracket, taking 110 N/mm² as the working stress in the transverse weld and 79 N/mm² in the longitudinal weld. (U.L.)

6. An I-section bracket carrying 120 kN load, is connected to by a column as shown in Fig. 23.20 means of two side fillet welds 200 mm deep. The load is eccentric by 70 mm. Calculate the size of the fillet weld.

7. A bracket consisting of a Tee-section 150 mm × 150 mm and 10 mm is connected to a column as shown in Fig. 23.20. The bracket carries 150 kN load at 80 mm eccentricity. If the size of the weld is 6 mm, find out the maximum throat stress.
8. The bracket shown in Fig. 23-21 is welded to a stanchion by side fillet welds on three sides indicated by heavy lines. Calculate the maximum forces per inch of weld metal when the bracket carries the load of 200 kN acting as shown.

![Fig. 23-20](image)

9. A bracket is welded to a stanchion by fillet welds, having a throat thickness of 9 mm and a load of 180 kN is applied in the plane of the bracket as shown in Fig. 23-22. The weld extends round three sides and has the given dimensions. Determine the maximum stress on the throat of the weld.

![Fig. 23-21](image)

10. The dimensions of a plate bracket welded to the face of a stanchion are given in Fig. 23-23. Assuming a maximum weld stress of 6 tons/in² on the throat of the fillet, determine the maximum permissible value of \( W \).

\( (L.S.E.) \)

**ANSWERS**

1. 172'5 kN
2. 72'5 mm
3. 218 mm and 91'5 mm
4. 160 mm
5. 9'5 mm
6. 1100 N/mm²
7. 1800 N/mm²
8. 8'03 ton.
Method of Tension Coefficients

24.1. INTRODUCTION

In Volume 1, we have studied two methods of analysis of plane frames: (i) method of joints, and (ii) method of sections. We can also find out the forces in the members of plane frame, by the graphical method. We now introduce a more general method, called the 'method of tension coefficients' which is equally applicable to both plane frames as well as space frames. The method of tension coefficients, first introduced by Prof. R.V. Southwell, is in effect a neat and systematic presentation of the 'method of joints'. The method is particularly useful to space frames in which other methods prove to be cumbersome and tedious. In this chapter, we shall introduce the method, and illustrate its applications for plane frames. Its application for the space frames has been illustrated in chapter 25.

24.2. TENSION COEFFICIENTS

The tension coefficient for a member of a frame is defined as the pull or tension in that member divided by its length. Thus,

\[ t = \frac{T}{L} \]  \hspace{1cm} (24'1)

where \( t \) is the tension coefficient for the member, \( T \) is the pull in the member and \( L \) is its length.

Consider a member \( AB \) of a pin-jointed perfect frame in equilibrium, under a given system of external forces (or reactions) acting at the joints. Let \( T_{AB} \) be the resulting pull in the member.

Let \( (x_A, y_A) \) be the coordinates of \( A \) and \( (x_B, y_B) \) be the coordinates of joint \( B \), referred to suitable axes of reference.

![Diagram](image-url)

Fig. 24-1

Component of pull \( T_{AB} \) in \( x \)-direction is

\[ T_{AB} \cos \theta = T_{AB} \frac{AC}{AB} = T_{AB} \frac{(x_B - x_A)}{L_{AB}} \]

\[ = t_{AB} (x_B - x_A) \]  \hspace{1cm} (1)

where \( t_{AB} = \) tension coefficient for \( AB = \frac{T_{AB}}{L_{AB}} \).

Similarly, component of \( T_{AB} \) in \( y \)-direction is

\[ T_{AB} \sin \theta = T_{AB} \frac{CB}{AB} = T_{AB} \frac{(y_B - y_A)}{L_{AB}} \]

\[ = t_{AB} (y_B - y_A) \]  \hspace{1cm} (2)

Thus, we observe that the component of forces in the members along the \( x \) and \( y \) directions can very easily be expressed in terms of tension coefficients and the coordinates of the ends of the members with reference to the chosen reference axes.

The length of the member \( AB \) is obviously given by

\[ L_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \]  \hspace{1cm} (24'2)

A member which is in compression will have a negative tension coefficient.
24.3. ANALYSIS OF PLANE FRAMES

Let us now apply the method of tension coefficient for the analysis of plane frame. Let us consider a joint A, where a number of members AB, AC, AD, etc., are meeting. Let \( P_A \) be the external force acting at the joint A, and let \( X_A \) and \( Y_A \) be the components of this force \( P \), in \( x \) and \( y \) directions. Since the joint is in equilibrium under the external force \( P_A \) and the pulls in the members meeting at joint A, the algebraic sum of resolved parts of these forces in any direction must be equal to zero.

Resolving the forces in \( x \) and \( y \) directions, and equating the algebraic sum of these forces in each direction to zero, we get

\[ t_{AB} (x_B - x_A) + t_{AC} (x_C - x_A) + t_{AD} (x_D - x_A) \ldots + X_A = 0 \]

and

\[ t_{AB} (y_B - y_A) + t_{AC} (y_C - y_A) + t_{AD} (y_D - y_A) \ldots + Y_A = 0 \]

\[ \ldots [24'3 (a)] \]

and \[ \ldots [24'3 (b)] \]

The above equations may also be written in compact form as under:

\[ \Sigma x (x_B - x_A) + X_A = 0 \]

and

\[ \Sigma y (y_B - y_A) + Y_A = 0 \]

\[ \ldots [24'4 (a)] \]

and

\[ \ldots [24'4 (b)] \]

where \( (x_B, y_B) \) are the coordinates of the far end of each member and \( (x_A, y_A) \) are the coordinates of the near end of each member meeting at the joint. The end of the member at the joint under consideration is known as the near end.

Thus, we obtain two equations at each joint, in which tension coefficients are the only unknowns. If the frame has \( j \) joints, we will have \( 2j \) such equations, the solution of which will yield tension coefficients for each member. If there are \( n \) members in a perfect frame, we have

\[ n = 2j - 3 \]

There will be \( n \) tension coefficients, for which we require only \( (2j - 3) \) equations, while available equations are \( 2j \). The three surplus equations will be useful either for determining the external reactions acting on the frame or for applying the check.

In order to apply the method, it is assumed that all the members are in tension. In the final solution, a member in compression will then automatically get negative tension coefficient. However, in setting the equations, utmost care must be taken in assigning correct sign to relevant terms. The term will be taken as positive or negative according as they tend to move the joint in the positive or negative directions of the axes of reference. The origin should be so selected that the coordinates of various joints can be written easily.

The method will now be illustrated with the help of few examples.

Example 24'1. A plane frame consists of two members \( AB \) and \( CB \), hinged at \( A \) and \( C \) to the wall, as shown in Fig. 24'2. Determine the forces in the two members due to a vertical force \( P \) applied at joint \( A \).

**Solution.**

![Fig. 24-2](attachment:image.png)

Let us take the origin at joint \( C \), and \( CX \) and \( CY \) be the axes of reference. The coordinates of the three joints are

\[ C(0, 0); B(2, 0); A(0, 1.5) \]

There are only two members and therefore, there will be only two tension coefficients. Let us therefore take joint \( B \) and set two equations at that joint, assuming that every member is in a state of tension, exerting a pull on the joint, though in the present case member \( BA \) will be in tension while member \( BC \) will be in compression. The tension coefficient for \( BC \) will automatically work out to be negative.

Length of tension \( L_{BA} = \sqrt{(0-2)^2 + (1.5-0)^2} = 2.5 \text{ m} \)

Length of compression \( L_{AC} = 2 \text{ m} \) (given)
At the joint $B$, we have the following two equations in $x$ and $y$ directions:

$$t_{BA}(x_B - x_A) + t_{BC}(x_C - x_B) + 0 = 0$$
and

$$t_{BA}(y_B - y_A) + t_{BC}(y_C - y_B) - P = 0$$

(Negative sign has been placed before $P$ since force $P$ acts in the negative $y$-direction)

Substituting the values, we get

$$t_{BA}(0 - 2) + t_{BC}(0 - 2) = 0$$
and

$$t_{BA}(1.5 - 0) + t_{BC}(0 - 0) - P = 0$$

Solving (1) and (2), we get

$$t_{BA} = \frac{P}{1.5}\text{ kN/m}$$
and

$$t_{BC} = -\frac{P}{1.5}\text{ kN/m}$$

Minus sign suggests that member $BC$ will be in compression.

**: Force in member $BA$**

$$= T_{BA} = t_{BA} \cdot L_{BA}$$

$$= \frac{P}{1.5} \times 2.5 = 1.6667P\text{ (tension). Ans.}$$

**: Force in member $BC$**

$$= T_{BC} = t_{BC} \cdot L_{BC} = -\frac{P}{1.5} \times 2$$

$$= -1.3333P = 1.3333P\text{ (comp.).}$$

**Example 24.2.** A truss, shown in Fig. 24.3 is loaded with two point loads of $2P$ and $P\text{ kN at joint B and C. Determine the forces in all the members.**

**Solution.**

**Given:** $L_{AB} = 8\text{ m}; L_{BC} = 4\text{ m}; L_{CD} = 8\text{ m}$ and $L_{AD} = 12\text{ m}.$

Let us keep the origin at $A$, with positive reference directions as shown in Fig. 24.3.

Now $AE = AB \cos 60^\circ = 8 \cos 60^\circ = 4\text{ m} = FD$

$BE = AB \sin 60^\circ = 8 \sin 60^\circ = 6.9282\text{ m}$

$ED = 12 - 4 = 8\text{ m}$

$BD = (BE^2 + ED^2)^{1/2} = 10.583\text{ m}$

The coordinates of various points are as under:

$A, (0, 0); B, (4, 6.9282); C, (8, 6.9282); D, (12, 0).$

To find the reaction $R_D$ at $D,$ take moments about $A.$

$$R_D = \frac{1}{12}[2P \times 4 + P \times 8] = 1.3333P$$

$$R_A = 2P + P - 1.3333P = 1.6667P$$

There are four joints, and hence eight equations will be available, while there are only five unknown tension coefficients. Let us set down the equations for each joint assuming that every member is in state of tension, exerting a pull at the joint, though actually all members except $AD$ are in compression.

**Joint $A$:**

$$t_{AB}(x_B - x_A) + t_{AD}(x_D - x_A) = 0$$
and

$$t_{AB}(y_B - y_A) + t_{AD}(y_D - y_A) + R_A = 0$$

Substituting the values, we get

$$4t_{AB} + 12t_{AD} = 0\quad\ldots(1)$$
and

$$6.928t_{AB} + 1.6667P = 0\quad\ldots(2)$$

Solving, we get $t_{AB} = -0.2406P\text{ kN/m}$
and

$$t_{AD} = \frac{0.0802P\text{ kN/m}}{2}.$$
**Strength of Materials and Theory of Structures**

\[ T_{AB} = -0.2406 \times 8P - 1.9246P \text{ kN} \]  
\[ (i.e. \text{ compression}) \]

\[ T_{AD} = -0.0802P \times 12 = 0.9624P \text{ kN} \]  
\[ (tension) \]

**Joint B**:

\[ f_{BA}(x_A - x_B) + f_{BD}(x_D - x_B) + f_{BC}(x_C - x_B) = 0 \]

and

\[ f_{BA}(y_A - y_B) + f_{BD}(y_D - y_B) + f_{BC}(y_C - y_B) = 2P = 0 \]

Substituting the values, we get

\[ -4f_{BA} + 8f_{BD} + 4f_{BC} = 0 \]  
\[ \text{...(3)} \]

and

\[ -6.9282f_{BA} - 6.9282f_{BD} - 2P = 0 \]  
\[ \text{...(4)} \]

From (4), \( f_{BA} + f_{BD} = -0.2887P \)

But

\[ f_{BA} = -0.2406P \]

\[ \therefore f_{BD} = -0.2887P + 0.2406P = -0.0481P \]

Hence from (3), \(-f_{BA} + 2f_{BD} + f_{BC} = 0\)

or

\[ 0.2406P = 2 \times 0.0481P + f_{BC} = 0 \]

Hence \( f_{BC} = -0.1444P \)

Thus \( T_{BC} = -0.1444P \times 4 = -0.5776P \) (i.e. compression)

and

\[ T_{BC} = -0.0481P \times 10.583 = -0.509P \]  
\[ (i.e. \text{ compression}) \]

**Joint C**:

\[ f_{CB}(x_C - x_B) + f_{CD}(x_D - x_C) = 0 \]

\[ f_{CB}(y_B - y_C) + f_{CD}(y_D - y_C) = P = 0 \]

Substituting the values, we get

\[ -4f_{CB} + 4f_{CD} = 0 \]  
\[ \text{...(5)} \]

and

\[ f_{CD} = -6.9282f_{CD} = 0 \]  
\[ \text{...(6)} \]

Solving, we get \( f_{CD} = -0.1444P \)

and

\[ f_{CB} = f_{CD} = -0.1444P \) (which checks the earlier result).

\[ \therefore T_{CD} = -0.1444P \times 8 = -1.1552P \) (i.e. compression)

Thus the values of forces in all the members have been obtained. Additional two equations available at joint D can be used to check the values of \( T_{BD} \) and \( T_{DA} \) obtained earlier.

**Example 24'3.** Fig. 23'4 shows a Warren type cantilever truss along with the imposed loads. Determine the forces in all the members, using the method of tension coefficients.

**Solution.**

\[ AF = h = \sqrt{8^2 - 4^2} = 6.928 \text{ m} \]
Let us take origin at F. and select x-axis and y-axis along FD and FA respectively. The coordinates of various points are as under:

<table>
<thead>
<tr>
<th>Joint</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>A</td>
<td>0</td>
<td>6'928</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>6'928</td>
</tr>
<tr>
<td>C</td>
<td>12</td>
<td>6'928</td>
</tr>
<tr>
<td>D</td>
<td>16</td>
<td>0</td>
</tr>
<tr>
<td>E</td>
<td>8</td>
<td>0</td>
</tr>
</tbody>
</table>

Joint D: The two equations are

\[ t_{oa}(x_e-x_o) + t_{cd}(x_c-x_d) + X_d = 0 \]  \( \ldots (i) \)

\[ t_{de}(y_e-y_d) + t_{dc}(y_c-y_d) + Y_d = 0 \]  \( \ldots (ii) \)

Substituting the values, we obtain

\[ -8t_{de} - 4t_{dc} = 0 \]  \( \ldots (1) \)

\[ 6'928 t_{de} - 2 = 0 \]  \( \ldots (2) \)

From (2), \( t_{de} = +0.289 \)  \( \ldots (a) \)

From (1), \( t_{de} = -1/4t_{dc} = -0.144 \)  \( \ldots (b) \)

Joint C: The two equations are

\[ t_{cd}(x_o-x_c) + t_{ce}(x_e-x_c) + t_{cb}(x_b-x_c) + X_c = 0 \]  \( \ldots (iii) \)

\[ t_{cd}(y_o-y_c) + t_{ce}(y_e-y_c) + t_{cb}(y_b-y_c) + Y_c = 0 \]  \( \ldots (iv) \)

Substituting the values, we get

\[ 4t_{cd} - 4t_{ce} - 8t_{cb} = 0 \]  \( \ldots (3) \)

\[ -6'928 t_{cd} - 6'928 t_{ce} - 3 = 0 \]  \( \ldots (4) \)

From (4), \( t_{ce} = -t_{cd} = -0.433 = -0.289 - 0.433 = -0.772 \)  \( \ldots (c) \)

From (3), \( t_{cb} = 0.5t_{cd} - 0.5t_{ce} = (0.5 \times 0.289) + (0.5 \times 0.722) \)

\[ = +0.5055 \]  \( \ldots (d) \)

Joint E: The two equations are

\[ t_{ec}(x_c-x_e) + t_{ed}(x_d-x_e) + t_{eb}(x_b-x_e) + t_{ef}(x_f-x_e) + X_e = 0 \]  \( \ldots (v) \)

\[ t_{ec}(y_c-y_e) + t_{ed}(y_d-y_e) + t_{eb}(y_b-y_e) + t_{ef}(y_f-y_e) + Y_e = 0 \]  \( \ldots (vi) \)

Substituting the values, we get

\[ 4t_{ec} + 8t_{eb} - 4t_{ef} = 0 \]  \( \ldots (5) \)

\[ 6'928 t_{ec} + 6'928 t_{eb} - 2 = 0 \]  \( \ldots (6) \)

From (2), \( t_{eb} = 0.2887 - t_{ec} = 0.2887 + 0.722 = +1.011 \)  \( \ldots (e) \)

From (5), \( t_{ef} = 0.5t_{ec} + t_{ed} - 0.5t_{eb} \)

\[ = (-0.5 \times 0.722) / (0.5 \times 1.011) = -0.4323 \]

Joint B: The two equations are

\[ t_{bc}(x_c-x_b) + t_{be}(x_e-x_b) + t_{ba}(x_a-x_b) + X_b = 0 \]

\[ t_{bc}(y_c-y_b) + t_{be}(y_e-y_b) + t_{ba}(y_a-y_b) + Y_b = 0 \]

Substituting the values, we get

\[ 8t_{bc} + 4t_{be} - 4t_{ba} = 0 \]  \( \ldots (7) \)

\[ -6'928 t_{be} - 6'928 t_{bc} - 3 = 0 \]  \( \ldots (8) \)

From (8), \( t_{be} = -t_{bc} = -0.433 = -1.011 - 0.433 = -1.444 \)  \( \ldots (f) \)

From (7), \( t_{ba} = 2t_{bc} + t_{be} - t_{bc} = (2 \times 0.5055) + 1.011 + 1.444 \)

\[ = +1.732 \]

The values of tension coefficients and forces in various members are tabulated below.

(+ for tension; — for compression)

<table>
<thead>
<tr>
<th>Member</th>
<th>Length (m)</th>
<th>T (kN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB</td>
<td>4</td>
<td>+3.466</td>
</tr>
<tr>
<td>BC</td>
<td>8</td>
<td>+0.5055</td>
</tr>
<tr>
<td>CD</td>
<td>8</td>
<td>+0.289</td>
</tr>
<tr>
<td>DE</td>
<td>8</td>
<td>-0.144</td>
</tr>
<tr>
<td>EF</td>
<td>8</td>
<td>-1.011</td>
</tr>
<tr>
<td>FB</td>
<td>8</td>
<td>-1.444</td>
</tr>
<tr>
<td>EC</td>
<td>8</td>
<td>-0.722</td>
</tr>
<tr>
<td>EB</td>
<td>8</td>
<td>+1.011</td>
</tr>
</tbody>
</table>
PROBLEMS

1. Using the method of tension coefficients, find forces in the members of the triangular type cantilever truss shown in Fig. 20'10 (Vol. 1).

2. Using the method of tension coefficients, determine the forces in the members of the crane structure shown in Fig. 20'17 (Vol. 1).

3. Using the method of tension coefficients, determine the forces in the members of the frame shown in Fig. 20'18 (Vol. 1).

4. Using the method of tension coefficients, determine the forces in the members of the frame shown in Fig. 20'19 (Vol. 1).

5. Find the forces in all the members of the frame shown in Fig. 20'23 (Vol. 1). The frame is supported on a pinned support at \( P \) and roller support at \( R \).

25. INTRODUCTION

A space frame or space truss is a three dimensional assemblage of the members, each member being joined at its ends to the foundation or to other members by frictionless ball-and-socket joints. The simplest space frame consists of six members joined to form a tetrahedron [Fig. 25'1 (a)]. By beginning with six members forming a tetrahedron, a stable space frame can be constructed by successive addition of three new members and a joint. One such frame is shown in Fig. 25'1 (b)].

In order to form a stable (or rigid) space frame, a sufficient number of members have to be arranged in a suitable manner explained above, starting with a basic tetrahedron. The original
tetrahedron consists of six members and four joints. Since for each additional joint there are three additional members, the relationship between the number of members \( n \) and number of joints is given by

\[
n - 6 = 3(j - 4) \\
\text{or} \\
\frac{n}{3} = j - 6,
\]

The above equation give the minimum number of members required to build a stable truss or frame. If a truss or frame has less than this number of members, it cannot be stable. If it has more members, then the truss or frame will be termed internally statically indeterminate.

25.2. Method of Tension Coefficients Applied To Space Frames

The method of tension coefficients, explained in the previous chapter, can be extended to space frames by writing the equation of equilibrium of forces at each joint, in each direction. However, in space frame, there are three axes of reference: x-axis, y-axis and z-axis, the first two axes being considered in a horizontal plane and z-axis in a vertical plane. Thus, there will be three equations in place of two equations used for plane frames.

**Fig. 25-2**

Let us consider a member \( AB \) of a space frame, carrying a tensile force \( T_{AB} \). Let \( (x_A, y_A, z_A) \) be coordinates of point \( A \), and \( (x_B, y_B, z_B) \) be the coordinates of point \( B \), referred to the three axes

**SPACE FRAMES**

\( X, Y, Z \) of reference. Let the line \( AB \) be inclined at angles \( \theta_x, \theta_y, \theta_z \) respectively to the three reference axes. Evidently,

\[
L_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} \quad \ldots (25.2)
\]

The direction cosines are:

\[
l_{AB} = \cos \theta_x; \quad m_{AB} = \cos \theta_y \quad \text{and} \quad n_{AB} = \cos \theta_z \quad \ldots (25.2')
\]

As shown in Fig. 25.2, the components of \( AB \) along \( X, Y \) and \( Z \) axes are:

\[
AL = (x_B - x_A); \quad AM = (y_B - y_A); \quad AN = (z_B - z_A)
\]

Hence,

\[
AL = (x_B - x_A) = AB \cos \theta_x = l_{AB} \cdot L_{AB}
\]
\[
AM = (y_B - y_A) = AB \cos \theta_y = m_{AB} \cdot L_{AB}
\]
\[
AN = (z_B - z_A) = AB \cos \theta_z = n_{AB} \cdot L_{AB}
\]

The resolved component of force \( T_{AB} \) along \( x \)-direction is

\[
T_{AB} \cos \theta_x = T_{AB} \frac{(x_B - x_A)}{L_{AB}} = t_{AB}(x_B - x_A)
\]

where \( t_{AB} = \text{tension coefficient for } AB \).

Similarly, the resolved component of force \( T_{AB} \) along \( y \) and \( z \) directions are

\[
T_{AB} \cos \theta_y = T_{AB} \frac{(y_B - y_A)}{L_{AB}} = t_{AB}(y_B - y_A)
\]

and

\[
T_{AB} \cos \theta_z = T_{AB} \frac{(z_B - z_A)}{L_{AB}} = t_{AB}(z_B - z_A)
\]

Let \( P_A \) be the external force at joint \( A \), and let \( X_A, Y_A \) and \( Z_A \) be the resolved component of this force in \( X, Y \) and \( Z \) directions respectively. Let \( AB, AC, AD, \ldots \) be the members meeting at joint \( A \). Since the joint is in equilibrium under the external force \( P_A \) and the pulls in the members meeting at the joint \( A \), the algebraic sum of resolved parts of these forces in any direction must be equal to zero.

Hence resolving the forces in \( X, Y \) and \( Z \) directions and equating the algebraic sum of these forces in each direction to zero, we get the following three equations for joint \( A \):

\[
t_{AB}(x_B - x_A) + t_{AC}(x_C - x_A) + t_{AD}(x_D - x_A) + \ldots + X_A = 0 \quad \ldots (25.5)
\]
\[
t_{AB}(y_B - y_A) + t_{AC}(y_C - y_A) + t_{AD}(y_D - y_A) + \ldots + Y_A = 0 \quad \ldots (25.5')
\]
\[
t_{AB}(z_B - z_A) + t_{AC}(z_C - z_A) + t_{AD}(z_D - z_A) + \ldots + Z_A = 0
\]
The above equations may also be written in compact form as

\( \Sigma(x_f-x_n) + X_A = 0 \)
\( \Sigma(y_f-y_n) + Y_A = 0 \)
\( \Sigma(z_f-z_n) + Z_A = 0 \) ...(25.5)

where \((x_f, y_f, z_f)\) are the coordinates of the far end of each member and \((x_n, y_n, z_n)\) are the coordinates of the near end of each member meeting at the joint. The end of the member at the joint under consideration is known as the near end.

If there are \(j\) joints, there will be \(3j\) such equations, while the number of unknowns (tension coefficients) will be equal to \(n\), where \(n = 3j - 6\). Thus, we will have six surplus equations which can be utilised either for determining the external reactions or for verifying the results.

The method will now be illustrated with the help of few examples.

25.3. ILLUSTRATIVE EXAMPLES

Example 25.1. A pair of shear leg has length of each leg as 5 m, and the distance between their feet is 4 m. The line joining the feet of the legs is 7 m from the foot of the guy rope. If the length of the guy rope is 10 m, find the thrust in each leg and the pull in the guy rope when a load of 100 kN is suspended from the head.

Solution.

Let the origin pass through the foot of the guy rope, and let the \(Y\)-axis pass through the mid-point of the line joining the feet of the two legs, as shown in Fig. 25.3, along with the direction of \(X\) and \(Z\) axes. \(H\) is the head of the shear leg. Let the guy rope make an angle \(\theta\) with the \(Y\)-axis.

Now \(CH^2 = AC^2 = \sqrt{7^2 - 2^2} = \sqrt{49 - 4} = 4.5826\) m

From \(\triangle OCH, \cos \theta = \frac{OH^2 + OC^2 - CH^2}{2OC.OH}\)
\(\cos \theta = \frac{10^2 + 7^2 - 21}{2 \times 7 \times 10} = 0.9143\)
\(\theta = 23.9^\circ; \sin \theta = 0.4051\)

From \(H\), drop a perpendicular \(HD\) on \(Y\)-axis.

\(OD = OH \cos \theta = 10 \times 0.9143 = 9.143\) m
\(HD = OH \sin \theta = 10 \times 0.4051 = 4.051\) m
\(CD = OD - OC = 9.143 - 7 = 2.143\) m

Hence the co-ordinates of various points are as under:

<table>
<thead>
<tr>
<th>Point</th>
<th>Co-ordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>(O)</td>
<td>0 0 0</td>
</tr>
<tr>
<td>(H)</td>
<td>0 9.143 4.051</td>
</tr>
<tr>
<td>(A)</td>
<td>-2 7 0</td>
</tr>
<tr>
<td>(B)</td>
<td>+2 7 0</td>
</tr>
</tbody>
</table>

At the head \(H\), the three equations are as under:

\(t_{HA}(x_A - x_H) + t_{HB}(x_B - x_H) + t_{HO}(x_O - x_H) = 0 \) ...(i)
\(t_{HA}(y_A - y_H) + t_{HB}(y_B - y_H) + t_{HO}(y_O - y_H) = 0 \) ...(ii)
\(t_{HA}(z_A - z_H) + t_{HB}(z_B - z_H) + t_{HO}(z_O - z_H) + Z_H = 0 \) ...(iii)

Substituting the values, we get

\(-2t_{HA} + 2t_{HB} = 0 \) ...(1)
\(-2.143t_{HA} - 2.143t_{HB} - 9.143t_{HO} = 0 \) ...(2)
and

\(-4.051t_{HA} + t_{HB} + t_{HO} = 100 = 0 \) ...(3)

From (1), \(t_{HA} = t_{HB}\) (as expected)

From (2), \(-2.143 + 2.143t_{HA} = 9.143t_{HO}\)

From which \(t_{HA} = -2.1332t_{HO}\)

From (3), \(-4.051 - 2.1332 - 2.1332 + 1 \cdot t_{HO} = 100\)

From which \(t_{HO} = 7.5573\)

\(t_{HA} = t_{HB} = -2.1332 \times 7.5573 = -16.1212\)

Hence pull in guy rope = \(t_{HO} \times L_{HO} = 7.5573 \times 10 = 75.57\) kN
Force in each shear leg = \(-16 \times 1212 \times 5 = -80.61\) kN

\[\because\text{Thrust in each leg} = 80.61\text{ kN}\]

Example 25.2. A space frame consists of six members: AF, BE, vF, FE, EC and FD. The frame is pinned to a vertical wall at ABCD in such a way that ABCD form a square as shown in Fig. 24.4. Also, ABEF is a rectangle in a horizontal plane. Using method of tension coefficients, find forces in each member due to a load of 100 kN applied at E acting towards the joint D. (Based on U.L.)

Solution.

Select the origin at A. Let X-axis be directed along AF, Y-axis be directed along AB and Z-axis be directed vertically through A. Thus, ABEF is in the X-Y plane which is a horizontal plane.

Here, the load of 100 kN is in an inclined direction which does not pass through any of the three axes. Hence it is essential to find its components along X, Y and Z directions.

Length \(FD = \sqrt{3^2 + 4^2} = 5\) m.

Fig. 25.5 (a) shows plane DFEC, in which the load of 100 kN acts along ED. Evidently, \(\theta = \tan^{-1} \frac{3}{4}\). from which \(\theta = 30.964\), \(\sin \theta = 0.5145\) and \(\cos \theta = 0.8575\).
Joint E:
The three equations at the joint E are:

\[
\begin{align*}
\tau_E (x_B - x_E) + \tau_{EC} (x_C - x_E) + \tau_{EB} (x_E - x_E) + P_E &= 0 \\
\tau_E (y_B - y_E) + \tau_{EC} (y_C - y_E) + \tau_{EB} (y_E - y_E) + P_E &= 0 \\
\tau_E (z_B - z_E) + \tau_{EC} (z_C - z_E) + \tau_{EB} (z_E - z_E) + P_E &= 0
\end{align*}
\]

Substituting the values, we get:

\[
\begin{align*}
-4\tau_{EC} - 4\tau_{EB} - 68.6 &= 0 \\
-3\tau_{EC} - 51.45 &= 0 \\
-3\tau_{EC} - 51.45 &= 0
\end{align*}
\]

From (2), \( \tau_{EF} = -17.15 \)
From (3), \( \tau_{EC} = -17.15 \)
From (1), \( \tau_{EB} = 0 \)

\[
\begin{align*}
T_{EF} &= -17.15 \times 3 = -51.45 \text{ kN} \\
T_{EC} &= -17.15 \times 5 = -85.75 \text{ kN} \\
T_{EB} &= 0
\end{align*}
\]

Joint F:
The three equations are

\[
\begin{align*}
\tau_A (x_A - x_F) + \tau_{FA} (x_B - x_F) + \tau_{FD} (x_D - x_F) + \tau_{FE} (x_E - x_F) &= 0 \\
\tau_A (y_A - y_F) + \tau_{FA} (y_B - y_F) + \tau_{FD} (y_D - y_F) + \tau_{FE} (y_E - y_F) &= 0 \\
\tau_A (z_A - z_F) + \tau_{FA} (z_B - z_F) + \tau_{FD} (z_D - z_F) + \tau_{FE} (z_E - z_F) &= 0
\end{align*}
\]

Substituting the values, we get:

\[
\begin{align*}
-4\tau_{FA} - 4\tau_{FD} &= 0 \\
3\tau_A + 3\tau_{FE} &= 0 \\
-3\tau_{FD} &= 0
\end{align*}
\]

From (6), \( \tau_{FD} = 0 \)
From (5), \( \tau_{AB} = -\tau_{FE} = +17.15 \)
From (4), \( -4\tau_{FA} - 4 \times 17.15 = 0 \)

Hence \( T_{FD} = 0 \)
\( T_{FB} = +17.15 \times 5 = +85.75 \text{ kN} \)
\( T_{FA} = -17.15 \times 4 = -68.60 \text{ kN} \)

Example 25.3. The feet of a tripod resting on a smooth ground are tied by horizontal bars forming a triangle BCD, as shown in Fig. 25.6 (a), where E is the mid-point of CD and F is the mid-point of BE. The apex A [Fig. 25.6 (b)] of the tripod is 3 m vertical above point F. Determine the forces in all the members due to a load of 100 kN suspended from apex A.

Fig. 25.6 (a) shows the plan of the base triangle BCD in the horizontal plane (x-y plane). Let B be the origin, and let x-axis be directed along BD; y-axis will be perpendicular to x-axis, while z-axis will be directed vertically upwards.

Drop CC’ perpendicular to BD. Obviously, \( BC = C_1D = 1 \text{ m} \). Also,

\[
CC_1 = \sqrt{(2.5)^2 - (1)^2} = 2.29 \text{ m}
\]

\[
\cos \alpha = \frac{1}{2.25} = 0.4 ; \sin \alpha = \frac{2.29}{2.25} = 0.916
\]

Since F is the mid-point of BE and \( C_1 \) is the mid-point of BD, \( FC_1 \) will be parallel to ED and will be equal to half of ED.

\[
\therefore \quad FC_1 = \frac{1}{2} ED = \frac{1}{2} CD = \frac{1}{2} \times 2.5 = 0.625 \text{ m}
\]

From F, drop perpendiculars \( FF_1 \) and \( FF_2 \) on x and y-axes.

From \( \triangle FF_1C_1, FF_1 = FC_1 \sin \alpha = 0.625 \times 0.916 = 0.5725 \text{ m} \)

\[
F_1C_1 = FC_1 \cos \alpha = 0.625 \times 0.4 = 0.25 \text{ m}
\]

\[
BF_1 = 1 - 0.25 = 0.75
\]

Since point A is vertically above F, the x and y coordinates of point A will be the same as that of point F.

Alternatively, the coordinates of point F may be found as under:

1. For \( E \):
\[
x_E = \frac{x_C + x_D}{2} = \frac{1 + 2}{2} = 1.5
\]
\[
y_E = \frac{y_C + y_D}{2} = \frac{2.29 + 0}{2} = 1.145
\]
\[
z_E = \frac{z_C + z_D}{2} = \frac{0 + 0}{2} = 0
\]
Example 25.4. A space frame shown in Fig. 25.7 is supported at A, B, C and D in a horizontal plane, through ball joints. The member EF is horizontal, and is at a height of 3 m above the base. The loads act on joint E and F, shown in the figure act in a horizontal plane. Find the forces in all the members of the frame.
Solution.

Let the origin be at \( B \), with \( X \) and \( Y \) axes along \( BC \) and \( BA \) respectively, and let \( Z \)-axis be directed vertically. The coordinates of various points are as under:

<table>
<thead>
<tr>
<th>Point</th>
<th>( X )</th>
<th>( Y )</th>
<th>( Z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( B )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( C )</td>
<td>7</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( D )</td>
<td>7</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>( E )</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( F )</td>
<td>5</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

\[
L_{AE} = \sqrt{(2 - 0)^2 + (3 - 0)^2 + (3 - 0)^2} = 4.6904 \text{ m}
\]
\[
L_{BE} = L_{CF} = L_{DF}
\]
\[
L_{AB} = \sqrt{(5 - 0)^2 + (3 - 0)^2 + (3 - 0)^2} = 6.5574 \text{ m}
\]
\[
L_{EF} = 3 \text{ m (given)}
\]

Joint \( E \): The three equations at joint \( E \) as follows:

\[
\begin{align*}
& t_{EA} (x_A - x_E) + t_{EB} (x_B - x_E) + t_{EF} (x_F - x_E) + X_E = 0 \quad \text{(i)} \\
& t_{EA} (y_A - y_E) + t_{EB} (y_B - y_E) + t_{EF} (y_F - y_E) + Y_E = 0 \quad \text{(ii)} \\
& t_{EA} (z_A - z_E) + t_{EB} (z_B - z_E) + t_{EF} (z_F - z_E) + Z_E = 0 \quad \text{(iii)}
\end{align*}
\]

Substituting the values, we obtain

\[
\begin{align*}
& -2 t_{EA} - 2 t_{EB} + 3 t_{EF} + 5 = 0 \quad \text{(1)} \\
& 3 t_{EA} - 3 t_{EB} + 10 = 0 \quad \text{(2)} \\
& -3 t_{EA} - 3 t_{EB} = 0 \quad \text{(3)}
\end{align*}
\]

From (3), \( t_{EA} = -t_{EB} \)

From (2), \( t_{EA} + t_{EB} = -10 \)

or \( t_{EA} = -\frac{5}{3} \) \( \text{...(a)} \)

\[
\begin{align*}
& t_{EB} = +\frac{5}{3} \quad \text{...(b)}
\end{align*}
\]

From (1), \( (2 \times -\frac{5}{3}) - (2 \times -\frac{5}{3}) + 3 t_{EF} + 5 = 0 \)

Substituting the values, we get

\[
\begin{align*}
& 2 t_{FD} + 2 t_{FC} - 3 t_{EF} = 0 \quad \text{(4)} \\
& 3 t_{FD} - 3 t_{FC} = 15 = 0 \quad \text{(5)} \\
& -3 t_{FD} - 3 t_{FC} = 15 = 0 \quad \text{(6)}
\end{align*}
\]

From (4), \( t_{FD} + t_{FC} = 2 5 \) \( t_{EF} = 2 5 \) \( \text{...(d)} \)

From (5), \( t_{FD} = -t_{FC} = -5 \) \( \text{...(e)} \)

From (6), \( t_{FD} + t_{FC} + t_{EF} = 0 \) \( \text{...(f)} \)

From (e) and (f), \( t_{FD} = -2 5 \)

From (d) and (e), \( t_{FB} = +0 7143 \)

From (f), \( t_{FC} = +1 7857 \)

Hence the tension coefficients and forces in the various members will be as shown in the table below.

<table>
<thead>
<tr>
<th>Member</th>
<th>Length (m)</th>
<th>( t )</th>
<th>( T ) (kN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( EA )</td>
<td>4.6904</td>
<td>-1.6667</td>
<td>-7.817</td>
</tr>
<tr>
<td>( EB )</td>
<td>4.6904</td>
<td>+1.6667</td>
<td>+7.817</td>
</tr>
<tr>
<td>( EF )</td>
<td>3.0</td>
<td>-1.6667</td>
<td>+5.0</td>
</tr>
<tr>
<td>( FC )</td>
<td>4.6904</td>
<td>+1.7857</td>
<td>+8.376</td>
</tr>
<tr>
<td>( FD )</td>
<td>4.6904</td>
<td>-2.50</td>
<td>-11.726</td>
</tr>
<tr>
<td>( FB )</td>
<td>6.5574</td>
<td>+0.7143</td>
<td>+4.684</td>
</tr>
</tbody>
</table>
PROBLEMS

1. Find forces in all the members of the space frame shown in Fig. 25'8. Take $AB=4$ m, $AD=5$ m and $AF=6$ m.

2. Fig 25'9 shows the plan of a tripod, the feet $A$, $B$ and $C$ being in the same horizontal plane and the apex $D$ being $3.75$ m above the plane. Horizontal loads of $100$ kN and $150$ kN are applied at $D$ in the directions shown. Find the forces in the members assuming that all joints are pin-joints. *(Based on U.L.)*

3. A frame pedestal shown in Fig. 25'10 is simply supported at $B$ and has two reaction supports at $A$ and $C$. Determine the reaction and forces in all the members due to horizontal force $100$ kN acting at $F$. Take $AB=BC=4$ m.

4. Fig. 25'11 shows two views of a tripod bracket. All connections are pinned. Find the forces in magnitude and nature in the three members due to a vertical load of $100$ kN acting at $O$. *(Based on U.L.)*
ANSWERS

1. $T_{EA} = 0; T_{ID} = +37.5; T_{IE} = 0; T_{FO} = +1874; T_{DA} = -30$
   $T_{GB} = +28.11; T_{OC} = -60; T_{OH} = +46.85; T_{HE} = 0$
   $T_{ID} = +40; T_{HE} = -37.48; T_{ED} = 0$

2. $T_{DA} = +63.7; T_{DC} = +96; T_{DB} = -191.1$

3. $T_{AB} = -75; T_{BC} = -25; T_{CD} = 0; T_{DA} = +25$
   $T_{EA} = +35.36; T_{EB} = -79.06; T_{EP} = -100; T_{EH} = 0$
   $T_{EB} = 0; T_{TC} = 0; T_{FO} = 0; T_{OC} = 0; T_{GD} = 0; T_{HD} = 0$
   $T_{HA} = 0$

4. $T_{OA} = +80.04 \text{kN}; T_{DB} = -8.89 \text{kN}; T_{OC} = -97.5 \text{kN}$

---

**Plastic Theory**

**26.1. INTRODUCTION**

A structure may reach its limit of usefulness through instability, fatigue or excessive deflection. Alternatively, if none of these failure modes occur, then the structure will continue to carry load beyond the elastic limit until it reaches its ultimate load through plastic deformation, and then collapse. Plastic analysis is based on this mode of failure. The concept of ductility of structural steel forms the basis for the plastic theory of bending.

The rigorous analysis of a structure according to the theory of elasticity demands that the stress satisfy two sets of conditions: (1) the equilibrium conditions and (2) compatibility conditions. The first set of conditions must be invariably satisfied in any material. However, the second condition ceases to be valid as soon as plastic yielding occurs. The elastic method of design assumes that a frame will become useless as soon as yield stress is reached. The working stress is, therefore, kept much below the yield stress. The design so produced gives a structure of unknown ultimate strength. The elastic methods of analysis are also very cumbersome, specially for redundant frames. In plastic method of design, the limit load of a system is a statically determinate quantity. The limit load is independent of all imperfections of the structure, such as faulty length of bars, settlement of supports and residual stresses caused by rolling or welding. The plastic method of design gives an economical design. The margin of safety provided in this method is not less than that provided according to the past practice.

The need for the study of plastic behaviour was appreciated by A.E.H. Love in 1892. The possibility of the development of plastic hinge was first suggested by G.V. Kazinczy in 1914. Prof. H. Maier-Leibnitz of Germany carried out load tests on encastre and conti-
262. THE DUCTILITY OF STEEL

The plastic theory is based on the ductility of steel. Through ductility, structural steel has capacity of absorbing large deformation beyond elastic limit without the danger of fracture. However, in the plastic range, the behaviour of steel depends strongly not only on its chemical composition but also on the mechanical and thermal treatments to which it has been subjected.

Fig. 26'1 (a) shows the complete stress-strain curve of mild steel. Fig. 26'1 (b) shows the portion ABC enlarged. It will be seen that the stress-strain relation is linear in the elastic range. The upper yield is reached at point A, and then the stress suddenly drops to lower yield stress at B. The strain then increases up to C at constant stress. This represents the plastic range. Beyond C the strain increases with further increase of stress and the material is said to be in strain hardening range. For ordinary steel the elastic strain is about 1/12 to 1/15 of strain at the beginning of the strain hardening and about 1/200 of maximum strain.

Experience shows that the metal of rolled beams does not usually exhibit any upper yield point and that even when an upper yield point exists, it can be removed by cold working such as straightening. Hence the theory of plastic bending is based on the assumption of a steel without upper yield point. The strain at point C is about 1.5%. In plastic design, at ultimate load the critical strains will not have exceeded about 1.5% elongation. Hence the strain hardening range is neglected in simple theory of plastic bending. This reduces complications in the calculations, and still leaves available a major portion of reserve ductility of steel which can be used as an added margin of safety. Fig. 26'1 (c) shows the idealised stress-strain curve which forms the basis of plastic design.

263. ULTIMATE LOAD CARRYING CAPACITY OF MEMBERS CARRYING AXIAL FORCES

Consider three bars OA, OB and OC, meeting at a common point O and hinged at the other ends A, B and C respectively (Fig. 26'2). Let a vertical load P be applied at the point O. We shall first solve the problem by elastic method.

Let \( P_1 \) be the tensile force in OB and \( P_2 \) be the force in OA and OC. Point O moves vertically to \( O' \) after the application of the load. Let \( \Delta_1 \) and \( \Delta_2 \) be the axial deformations of rods BO and AO (or CO) respectively. The dotted lines show the deformation portion of the structure.

From Statics,

\[ P_1 + 2P_2 \cos \theta = P \]  
...(1)

The structure is statically indeterminate to single degree. The second equation is obtained from the compatibility of deformations:

\[ \Delta_2 = \Delta_1 \cos \theta \]

or

\[ \frac{P_2 L_2}{AE} = \frac{P_1 L_1}{AE} \cos \theta \]  
...(2 a)
PLASTIC THEORY

fails (or becomes useless) when the central bar reaches yield stress \( \sigma_y \). At the yield condition, the force \( P_1 \) in the central bar will be equal to \( A \sigma_y \) while the force \( P_2 \) (from Eq. 2) will be equal to \( A \sigma_y \cos^2 \theta \). Hence the total load, called the elastic limit load is

\[
P_{el} = P_1 + 2P_2 \cos \theta = A \sigma_y + 2A \sigma_y \cos^2 \theta
\]

or

\[
P_{el} = A \sigma_y (1 + 2 \cos^2 \theta)
\]  

Fig. 26-2 (e) shows a plot between the load \( P \) versus the vertical deflection \( \Delta \), given by Eq. 26-2. If \( \theta = 45^\circ \), we get

\[
\Delta = 2 \Delta_1 = \frac{2 \sigma_y L_1}{E}
\]
When the external load \( P_0 \) is increased, the central rod is fully in plastic stage while the end rods are still in elastic stage. This condition is generally called the "\textit{contained plastic flow condition}" as represented by lines BC and AC of Fig. 26.2(b). When all the three rods have become plastic, the condition is known as the "\textit{unrestricted flow condition}".

The load \( P_L \) given by Eq. 26.3 may be considered as the "failure load". The service load may be taken as a certain portion \( \frac{1}{Q} \) of load \( P_0 \), where \( Q \) is a safety factor, usually called a "load factor" in plastic analysis. Thus,

\[
\frac{P}{Q} = P_L \quad \ldots(26.3)
\]

where \( P = \text{service load} \).

The saving achieved by designing according to the plastic theory instead of elastic theory is equal to

\[
\left(1 - \frac{\text{Area by plastic theory}}{\text{Area by elastic theory}}\right) \times 100 = \left(1 - \frac{1 + 2 \cos \theta}{1 + 2 \cos \theta}\right) \times 100
\]

When \( \theta = 45^\circ \), this amount to about 29%.

**Example 26.1.** A rigid beam ABC is kept in horizontal position by three rods as shown. All the three rods are made of the same material and have equal area of cross section of 200 mm\(^2\). The length of the outer rod is 1 m while the length of the central rod is 2 m. Calculate the collapse load for the structure, applied at the centre of the beam. Take the stress at yield equal to 250 N/mm\(^2\) and \( E = 2 \times 10^5 \text{ N/mm}^2\).

**Solution.**

\[
P = 2P_1 + P_2 = (50000 \times 2) + 50000 = 150000 \text{ N}.
\]

This is evidently equal to \( 3\sigma_yA \).

The deflection at this stage is given by

\[
\Delta L = \frac{P_L L^3}{AE} = \frac{50000 \times 2000}{200 \times 2 \times 10^5} = 25 \text{ mm}.
\]

Fig. 26.3(b) shows the complete load-deflection diagram.

**Example 26.2.** A rigid beam ABCD is hinged at A and being supported by two vertical rods attached at B and C, as shown in Fig. 26.4. Determine (i) the load \( P \) when first yield occurs in any of
the bars and (ii) when the whole arrangement collapse. Each rod has an area of section of 200 mm$^2$. Take $\sigma_f = 252$ N/mm$^2$.

Solution.

![Diagram](image)

Fig. 26.24

(a) Load at first yield

From statics, $P_1 + P_2 + R = P$ ... (1)

where $P_1$ and $P_2$ are the forces in rods at $B$ and $C$, and $R$ is the reaction at $A$, assumed vertically upwards.

Taking moments about $A$, we get

$P + 2P_2 = 2.5P$ ... (2)

Fig. 26.3 (b) shows the deformed shape of the arrangement. If $A_1$ and $A_2$ are the vertical extensions of the two bars,

$\frac{A_1}{A_2} = \frac{1}{2}$

or

$2A_1 = A_2$ ...

or

$2 \frac{P_1 \times 1}{AE} = \frac{P_2 (1.5)}{AE}$

or

$P_2 = \frac{2}{1.5} P_1$ ... (3)

Eq. (3) shows that $P_2$ is greater than $P_1$. Since the area of sections of both the bars are equal, it is evident that yield first occurs in bar 2. As the load $P$ is increased, the force $P_2$ will go on increasing till yield occurs in it. Hence, at the first yield,

$P_2 = \sigma_f A = 252 \times 200 = 50400$ N.

(b) Load at complete collapse

When the load $P$ is further increased, the load in rod 1 increases while the load in rod 2 remains constant at a value of 50400 N. When yield is reached in bar 1, the whole structure collapses. The force $P_1$ corresponding to yield in the first rod is evidently equal to $\sigma_f A = 252 \times 200 = 50400$ N. Substituting these values of $P_1$ and $P_2$ (each equal to 50400 N) in Eq. (2), we get

$P_1 = \frac{1}{2.5} [P_1 + 2.5 P_2]$

$= \frac{2.75}{2.5} P_2 = \frac{2.75}{2.5} \times 50400 = 55500$ N.

Thus according to the elastic solution, the load carrying capacity of structure is 55500 N.

Example 26.3. Compute the ultimate load $P$ at the collapse of the structure shown in Fig. 26.5. All the four rods have equal area of cross-section.

Solution.

![Diagram](image)

Fig. 26.5

PLASTIC THEORY

Corresponding value of $P_1$ is

$P_1 = \frac{1}{2} \frac{1}{2} P_2 = 0.75 P_1$

Substituting these in Eq. (2) the load at the first yield is given by

$P_1 = \frac{1}{2.5} [P_1 + 2.5 P_2]$

$= \frac{2.75}{2.5} P_2 = \frac{2.75}{2.5} \times 50400 = 55500$ N.

It should be noted that in plastic analysis, compatibility equation 3 is no longer useful. However, the equilibrium equations (Eqs. 1 and 2) still hold good. This shows the simplicity of plastic analysis.

Example 26.3. Compute the ultimate load $P$ at the collapse of the structure shown in Fig. 26.5. All the four rods have equal area of cross-section.
The structure will collapse when it is turned into a mechanism. There are two possibilities, and both of these should be investigated. In the first possibility, rods 1 and 2 become plastic and the collapse may take place by rotation about point C. In the second possibility, rods 4, 3 and 2 may become plastic and failure may take place by rotation about A. The free body diagrams for both these possibilities are shown in Fig. 26.6 (a) and (b).

The first possibility of collapse is shown in Fig. 26.6 (a). By inspection, rod 1 will yield first, when the force in it is \( P_1 = \sigma_y A \). With the further increase in the external load, \( P_1 \) will remain constant at \( \sigma_y A \) while \( P_2 \) will increase till it also becomes equal to \( \sigma_y A \). At this stage, the structure will turn into mechanism, and collapse will take place by rotation about C. Just before such collapse, we get, by taking moments about C,

\[
\sigma_y A \cdot 2L + \sigma_y A \cdot L = P_{11} \left( \frac{2L}{3} \right)
\]

or

\[
P_{11} = \frac{9}{2} \sigma_y A \tag{1}
\]

Let us now consider the second possibility, when the rod 4, 3 and 2 yield, and collapse takes place by rotation about A. At the yield stage, force carried by each of these rods is equal to \( \sigma_y A \). Hence we get by taking moments about A,

\[
\sigma_y A \cdot L + 2\sigma_y A \cos \theta \cdot \frac{L}{2} = P_{12} \cdot \frac{2L}{3}
\]

or

\[
P_{12} = \frac{3}{4} \sigma_y A (1 + 4 \cos \theta) \tag{2}
\]

It will be seen that \( P_{12} \) is less than \( P_{11} \) for all values of \( \theta \). Hence the collapse load is given by Eq. (2).

---

### 26.4. PLASTIC BENDING OF BEAMS

**PLASTIC THEORY**

![Diagram](image)

**Fig. 26.7.**

**PLASTIC BENDING OF BEAMS**

Let a beam be subjected to an increasing bending moment \( M \) (pure bending). The beam has at least one axis of symmetry so that bending is symmetrical about that axis. When the bending stresses are within the elastic range, the bending stress distribution will be as shown in Fig. 26.7 (b). The neutral axis will pass through the centroid of the section. As the moment is increased, yield stress will appear either in the topmost (or in the bottom most fibre, as the case may be) with the neutral axis still passing through the centroid of the section (Fig. 26.7 (b-2)]. The moment at which the first yield has occurred is called the yield moment (\( M_Y \)). With further increase of \( M \), the yield will also occur in the bottom fibre and it will spread inwards in the top portion (Fig. 26.7 (b-3)). The neutral axis no longer passes through the centroid, its location being determined by the fact that the total compressive force is equal to the total tensile force over the cross-section.

Further increase of bending moment will cause the yield to spread further inwards towards neutral axis. A stage is ultimately
reached when the yield spreads right up to the neutral axis and the section becomes fully plastic [Fig. 26.7 (b-5)]. The corresponding bending moment is called the fully plastic moment and is denoted by \( M_p \). Neglecting strain hardening in the outer fibres, no further increase in the bending moment can be attained. The plastic moment, therefore, represents the limiting strength of the beam in bending. The neutral axis in case of fully plastic section will pass through the equal area axis. In case of sections having two axes of symmetry, the location of neutral axis in elastic and fully plastic conditions remain unchanged. When the fully plastic moment is reached, the section will act as a hinge permitting rotation. With further increase of the load, the yield will spread in longitudinal direction.

Moment curvature relationship

The curvature \( \phi \) is the relative rotation of two sections a unit distance apart. As in the elastic bending, we have, according to first approximation:

\[
\phi = \frac{1}{\rho} \cdot \frac{\varepsilon}{y_0} = \frac{\sigma y}{E y_0},
\]

or

\[
\phi E = \frac{\sigma y}{y_0},
\]

where \( y_0 \) = distance of farthest still-elastic fibre and \( \varepsilon \) = maximum elastic strain.

The curvature at any given stage can thus be obtained from the stress distribution. The curvature of a partially plastic section is controlled by the deformations of still-elastic interior fibres. Fig. 26.7 (d) shows the moment-curvature relationship. This moment-curvature relationship is of great importance in the plastic theory. When an unloaded beam is subjected to increasing bending moment, the curvature first increases linearly with bending moment as represented by \( Oa \). This is elastic range. With the appearance of yield under the section of yield moment \( M_y \), the linear relation no longer holds good. As the bending moment is increased further the curvature will increase at a faster rate which corresponds to the spread of yield in inward direction. As the bending moment approaches its fully plastic value the curvature will tend to infinity. This corresponds to fully plastic section. When, at a particular section, the bending moment reaches the value \( M_p \), the bending moment on either side of it will be lesser than \( M_p \). With the attainment of fully plastic moment at a section, the curvature at this section becomes infinitely large. Thus a finite change of slope can occur over an indefinitely small length of the member at this section. This section will therefore, act as if a hinge has been inserted in the member at this section.

26.5. EVALUATION OF FULLY PLASTIC MOMENT

The moment of resistance developed by a fully plastic section is called the fully plastic moment \( M_p \). The following simplifying assumptions are made for evaluation of fully plastic moment (Baker, 1956):

1. The material obeys Hooke's law until the stress reaches the upper yield value; on further straining the stress drops to the lower yield value and thereafter remains constant.
2. The upper and lower yield stresses and the modulus of elasticity have the same values of compression as in tension.
3. The material is homogeneous and isotropic in both the elastic and plastic states.
4. Plane transverse sections remain plane and normal to the longitudinal axial after bending, the effect of shear being neglected.
5. There is no resultant axial force on the beam.
6. The cross-section of beam is symmetrical about an axis through its centroid parallel to the plane of bending.
7. Every layer of the material is free to expand and contract longitudinally and laterally under stress as if separated from the other layers.

![Fig. 26.8. Evaluation of fully plastic moment.](image-url)
Consider a cross-section of a beam subjected to a fully plastic moment $M_p$ of sagging nature. Under the action of this fully plastic moment, every fibre of the cross-section will be stressed to the yield stress $\sigma_Y$ and the stress distribution will be rectangular as shown in Fig. 26.8 (c). The stress in the fibres above the neutral axis will be of compressive nature while the stress of fibres located below the neutral axis will be of tensile nature. Let $A_1$ be the area of the portion of the section situated above the neutral axis, its C.G. $(g_1)$ being at $y_1$ from the N.A. Similarly, let $A_2$ be the tensile area, with its C.G. $(g_2)$ situated at $y_2$ below the neutral axis.

The compressive force acting over the cross-section = $A_1 \times \sigma_Y$

Total tensile force acting over the cross-section = $A_2 \times \sigma_Y$

\[ A_1 \sigma_Y = A_2 \sigma_Y \]  \hspace{1cm} \text{...(2)}

or

\[ A_1 = A_2 \]

But $A = A_1 + A_2$ \hspace{0.5cm} \text{(total area)}

\[ A_1 = A_2 \frac{A}{2}. \]

Thus, the neutral axis divides the section into two equal parts, i.e., it passes through the equal area axis.

Again these two forces should form a couple such that its magnitude is equal to the externally applied moment $M_p$. Hence

\[ A_1 \sigma_Y y_1 + A_2 \sigma_Y y_2 = M_p \]  \hspace{1cm} \text{...(2)}

But $A_1 = A_2 = \frac{A}{2}$

\[ M_p = \sigma_Y \frac{A}{2} (y_1 + y_2) \]  \hspace{1cm} \text{...(26.6)}

or

\[ M_p = \sigma_Y Z_p \]  \hspace{1cm} \text{...(26.7)}

where $Z_p = \frac{A}{2} (y_1 + y_2)$  \hspace{1cm} \text{...(26.8)}

$Z_p$ is the first moment of area about neutral axis termed as \textit{plastic section modulus}. It should be noted that the fully plastic moment $M_p$ is constant for a particular cross-section of a given material.

Also, yield moment $M_Y$ \textit{i.e.,} the moment at which the first yield occurs, section still being elastic is given by

\[ M_Y = \sigma_Y Z \]  \hspace{1cm} \text{...(26.9)}

where $Z$ = elastic section modulus.

\[ S = \frac{M_p}{M_Y} = \frac{\sigma_Y Z_p}{\sigma_Y Z} = \frac{Z_p}{Z} \]  \hspace{1cm} \text{...(26.10)}

\textbf{EVALUATION OF SHAPE FACTOR}

The shape factor is the property of a section and depends solely on the shape of the cross-section. We shall evaluate the shape factor of some of the standard sections.

(a) Rectangular Section [Fig. 26.9 (a)]

Elastic section modulus, $Z_p = \frac{bd^3}{6}$

Plastic section modulus, $Z_p = \frac{A}{2} (y_1 + y_2)$

\[ = \frac{b \times d}{2} \left[ \frac{d}{4} + \frac{d}{4} \right] = \frac{bd^3}{4} \]  \hspace{1cm} \text{...(26.11)}

\[ S = \frac{Z_p}{Z} = \frac{bd^3}{4} \div \frac{bd^3}{6} = 1.5 \]

(b) Triangular Section [Fig. 26.9 (b)]

\[ I = \frac{bh^3}{12} \]

The distance of extreme fibre from the elastic neutral axis

\[ = \frac{h}{2} \]

\[ Z = \frac{bh^3}{12} \times \frac{3}{h} = \frac{bh^3}{12} \]
For locating the equal area axis, equate the area on either side.

Let the equal area axis be at distance $h_1$ from the apex.

\[
\frac{bh_1}{2} = \frac{1}{2} bh
\]

But

\[
\frac{h_1}{h} = \frac{b}{2} \quad \text{or} \quad b_1 = \frac{bh_1}{h}
\]

\[
\frac{bh_1}{h} = \frac{1}{2} \frac{bh}{2}
\]

or

\[
h_1 = \frac{h}{\sqrt{2}}. \quad \text{Similarly} \quad b_1 = \frac{bh}{\sqrt{2}}
\]

Now

\[
y_1 = \frac{h_1}{3} = \frac{h}{3\sqrt{2}} = 0.235h
\]

and

\[
y_2 = \frac{h - h_1}{3} = \frac{b_1 + 2b}{b_1 + b} = \frac{h - h/\sqrt{2}}{3} \cdot \frac{2b + b/\sqrt{2}}{b + b/\sqrt{2}}
\]

\[
= \frac{8 - 5\sqrt{2}}{6} \quad h = 0.155h
\]

\[
Z_p = A \left( y_1 + y_2 \right) = \frac{bh}{4} \left\{ 0.235h + 0.155h \right\}
\]

\[
= 0.098 bh^2
\]

\[
S = \frac{Z_p}{Z} = 0.098 bh^2 \times \frac{24}{bh^2} = 2.34.
\]

(c) Circular Section [Fig. 26.9 (c)]

\[
Z = \frac{\pi}{32} d^3
\]

\[
Z_p = A \left( y_1 + y_2 \right) = \frac{\pi}{4} d^2
\]

\[
y_1 + y_2 = \text{distance of C.G. of semi-circle from N.A.}
\]

\[
= \frac{2d}{3\pi}
\]

\[
Z = \frac{1}{2} \times \frac{\pi}{4} d^3 \left( \frac{2d}{3\pi} + \frac{2d}{3\pi} \right) = \frac{d^3}{6}
\]

\[
S = \frac{Z_p}{Z} = \frac{d^3}{6} \times \frac{\pi}{32} d^2 = 1.7.
\]

(b) Hollow Circular Section [Fig. 26.9 (d)]

\[
d = \text{internal diameter}
\]

\[
D = \text{external diameter}
\]

For a circular section, $Z_p = \frac{d^3}{6}$ in general.

Hence for a hollow circular section,

\[
Z = \frac{D^3}{6} - \frac{d^3}{6} = \frac{D^3 - (kD)^3}{6}
\]

\[
= \frac{\pi}{64} D^5 \left( 1 - k^3 \right)
\]

\[
S = \frac{Z_p}{Z} = \frac{D^3}{6} \left[ 1 - k^3 \right] = \frac{\pi}{32} D^3 \left[ 1 - k^3 \right]
\]

\[
= 1.7 \left( \frac{1 - k^3}{1 - k^3} \right)
\]

26.6. PLASTIC HINGE

A plastic hinge is a zone of yielding due to flexure in a structural member. At those sections where plastic hinges are located, the member acts as if it were hinged, except with a constant restraining moment $M_p$. Just like any ordinary hinge, the plastic hinge allows the rotation of members on its two sides without change in curvature of members. The plastic hinge is capable of resisting rotation until fully plastic moment is developed and then permitting rotation of any magnitude while the bending moment remains constant at $M_p$.

The hinge extends over a length of member that is dependent on loading and the geometry. The hinge length $\Delta L$ is the length of the beam over which the bending moment is greater than the yield moment $M_Y$. However, in all of its length $\Delta L$ the section are not plastic to its full depth. To illustrate this we shall consider the case of a simply supported beam loaded with a central point load $W$, the section of the beam being rectangular.

Let the yielded portion (i.e., the plastic hinge) extreme points distant $x$ from either end. The moment at these extreme points is $M_Y$ and the moment at other points beyond these is less than $M_Y$. 

PLASTIC THEORY

Let

\[
\frac{d}{D} = k
\]

\[
Z = \frac{\pi}{64} \left[ \frac{D^3 - d^3}{D^3} \right] = \frac{\pi}{64} \left[ \frac{D^3 - (kD)^3}{D^3} \right]
\]

\[
= \frac{\pi}{64} D^5 \left( 1 - k^3 \right)
\]

For a circular section, $Z_p = \frac{d^3}{6}$ in general.
From Fig. 26.10 (b), we get

\[ \frac{M_p}{M_y} = \frac{L/2}{x} \]

But \[ \frac{M_p}{M_y} = S = \frac{3}{2} \] for a rectangular section

\[ \frac{3}{2} = \frac{L}{2x} \]

\[ x = \frac{L}{3} \]

Hence \( \triangle L = L - 2x = L - \frac{2L}{3} = \frac{L}{3} \).

Similarly, it can be shown that if the beam is of I-section, the length \( \triangle L \) is about \( \frac{L}{8} \). The length \( \triangle L \), in fact, represents the length of elasto-plastic zone.

Because of the shape of the moment curvature diagram [Fig. 26.7 (d)], the curvature remains very small near ends \( C \) and \( D \) of the plastic region, while in the neighbourhood of point \( E \), the curvature is extremely high as shown in Fig. 26.10 (c). The beam, therefore, deforms very nearly as if it consisted of two rigid portions connected by a hinge in \( E \) [Fig. 26.10 (d)]. In most of the analytical work, it is assumed that all plastic rotations occur at a point, i.e. length of the hinge approaches zero.

26.7. LOAD FACTOR

The load factor is the ratio of the collapse load to the working load:

\[ Q = \frac{W_c}{W} \quad \text{or} \quad \frac{W_c}{W} = Q \]

where \( Q \) = Load factor

\( W_c = \) Collapse load or limit load

\( W = \) Working load.

The value of load factor depends upon type of loading, the end conditions of the supports and the cross-section of the member.

Let \( M_{\text{max}} = \) maximum bending moment corresponding to working load \( W \)

\( M_p = \) fully plastic moment corresponding to collapse load \( M_c \).

Since bending moment at a given section is directly proportional to load, we have

\[ M \propto W \quad \text{or} \quad M = aW \]  

\( (\text{For simply supported beam } M = \frac{WL}{4} \text{ and hence } a = \frac{L}{4}) \)

Similarly, \( M_p \propto W_p \)

or \( M_p = aW_c = a.QW \)  

\[ \frac{M_p}{M_{\text{max}}} = Q \]  

Now elastic section modulus required \( \Rightarrow Z_e = \frac{M_{\text{max}}}{f_t} \)  

Plastic section modulus required \( \Rightarrow Z_P = \frac{M_p}{\sigma_Y} \)  

\[ \frac{Z_P}{Z_e} = \frac{M_p}{f_t} \frac{1}{\sigma_Y} \]

But \( \frac{M_p}{M_{\text{max}}} = Q \) ; \( \frac{Z_P}{Z_e} = S \)

and \( \frac{\sigma_Y}{f_t} = F \) = factor of safety elastic method.

Substituting these in (11), we get

\[ S = \frac{Q}{F} \]

or \( Q = S \times F \)  

\[ (26.16) \]
26.8. METHODS OF LIMIT ANALYSIS : BASIC THEOREMS

In the elastic method of analysis, three conditions must be satisfied: (1) continuity condition, (2) equilibrium condition, and (3) limiting stress condition. Thus, an elastic analysis requires that the deformations must be compatible, the structure should be in equilibrium and the bending moments anywhere in the structure should be less than \( M_f \) (or the stress should be less than \( \sigma_f \)).

Compared to this, an analysis according to the plastic method must satisfy the following fundamental conditions:

1. **Mechanism condition.** The ultimate load or collapse load is reached when a mechanism is formed. There must, however, be just enough plastic hinges that a mechanism is formed.

2. **Equilibrium condition.** The summation of the forces and moments acting on a structure must be equal to zero.

3. **Plastic moment condition.** The bending moment anywhere must not exceed the fully plastic moment.

It should, however, be noted that all the three conditions cannot be satisfied in one operation. Two theorems have been evolved which must be satisfied to ensure that all the conditions are fulfilled. The general method of limit analysis and design are based on the two fundamental theorems evolved by Greenberg and Prager. The first theorem, called the **static or lower bound theorem**, furnishes a lower boundary for the limit load, while second theorem, called the **kinematic or upper bound theorem** gives an upper boundary for the limit load.

**Basic Theorems**

1. **Static theorem or lower bound theorem**

   The static theorem states that for a given frame and loading, if there exists any distribution of bending moment throughout the frame which is both safe and statically admissible, with a set of loads \( W \), the value of \( W \) must be less than or equal to the collapse load \( W_c \).

   The distribution of bending moment, such that it satisfies all the conditions of equilibrium is called **statically admissible distribution**. If the distribution of bending moment is such that the fully plastic moment is not exceeded anywhere in frame, it is called **safe distribution**.

2. **Kinematic or upper bound theorem**

   The upper bound theorem states that for a given frame subjected to a set of loads \( W \), the value of \( W \) which is found to correspond to any assumed mechanism will always be greater than or equal to the actual collapse load \( W_c \). This theorem satisfies the equilibrium condition as well as mechanism condition, and provides the upper bound or limit of collapse load. If the values of \( W \) corresponding to a number of mechanisms for a given frame under given set of loading are found, the collapse load \( W_c \) will be the smallest of all these found.

**Methods of Analysis**

Based on the above two theorems, there are two basic methods of limit analysis: (1) static method, and (2) kinematic method.

1. **Static method.** Static method is based on the static or lower bound theorem according to which a load computed on the basis of an assumed equilibrium moment diagram in which the moments not greater than \( M_f \), is less than or at best equal to the true ultimate load. In this method, a moment diagram is sketched in such a way that the conditions of equilibrium are satisfied. The moments must either be less than or equal to \( M_f \). If a mechanism is formed, then the solution of equilibrium equation will give true collapse load. If the mechanism is not formed, the moment at some of the sections will have to be increased so as to obtain a mechanism, i.e. the existing load will have to be increased. The load will become equal to the collapse load when a mechanism is formed. The procedure for application of static theorem is as follows:

   1. Convert the structure into statically determine structure by removing the redundant forces.
   2. Draw free bending moment diagram for the structure.
   3. Draw the bending moment diagram for the redundant forces.
   4. Draw the composite bending moment diagram in such a way that a mechanism is obtained.
   5. Find out the value of collapse load by solving equilibrium equations.
   6. Check the moments to ensure that \( M < M_f \). If it is so, correct value of collapse load is obtained.
The method is suitable only for simple structures. For complicated frames, the method becomes very difficult and, therefore, kinematic method is preferred.

2. Kinematic or Mechanism method. Kinematic method is based on the kinematic or upper bound theorem according to which a load computed on the basis of an assumed mechanism will always be greater than or at best equal to the true ultimate load. For the application of this method, it is very essential to know the possible types and number of mechanisms. There are four types of independent mechanisms (Fig. 26.11): (i) beam mechanism, (ii) panel mechanism, (iii) gable mechanism, and (iv) joint mechanism. Various combination of the independent mechanisms may be made to obtain a certain number of composite mechanisms.

(a) BEAM MECHANISMS

(b) PANEL MECHANISM (c) GABLE MECHANISM (d) JOINT MECHANISM

Fig. 26.11.
Types of independent mechanisms.

For a particular structure with a loading, the number of independent mechanisms is given by

\[ N = n - T \]  

...(26.17)

where \( N \) = number of independent mechanisms
\( n \) = number of possible hinges
\( T \) = number of redundancies.

A number of possible collapse mechanisms may be obtained by the combination of independent mechanisms. The correct mechanism will be the one which results in the lowest possible load (upper bound theorem) and for which the moment does not exceed the plastic moment at any section of the structure (lower bound theorem). The procedure of application of the kinematic theorem is as follows:

1. Determine the location of possible plastic hinges.
2. Select possible independent and composite mechanisms.
3. Solve equilibrium equation by virtual displacements method for the lowest load.
4. Check that \( M < M_p \).

Principle of virtual work. It is stated as follows:

"If a deformable structure in equilibrium under the action of a system of external forces is subjected to a virtual deformation compatible with its conditions of support, the work done by these forces on the displacements associated with the virtual deformation is equal to the work done by the internal stresses on the strains associated with this deformation." This principle has wide utility for the structure at collapse. During collapse there is no change in the elastic strain energy stored in the beam since the bending moment and, therefore, the curvature remains the same. So the work done during small motion of collapse mechanism is equal to the work absorbed by the plastic hinge. The work absorbed in the hinges is always positive irrespective of the sign of B.M.

26.9. DETERMINATION OF COLLAPSE LOAD FOR SOME STANDARD CASES OF BEAMS

1. Simply supported beam carrying a concentrated load \( W \)

Let the beam section have a plastic moment of resistance \( M_p \).

We shall solve the problem by both the methods.

(a) Static Method:

The maximum bending moment of \( \frac{Wab}{L} \) evidently occurs under the load. When the load is increased to the collapse load \( W_C \), the maximum bending moment will be equal to \( \frac{W_Cab}{L} \), as shown in Fig. 26.12 (b). This should evidently be equal to the plastic moment of resistance \( M_p \).

\[ \frac{W_Cab}{L} = M_p \]

or

\[ W_C = \frac{M_pL}{ab} \]
Since the bending moment nowhere in the beam exceeds $M_P$, the load given by the above expression is the true collapse load.

(b) Kinematic Method:

The collapse mechanism (beam mechanism) is shown in Fig. 26'12 (c). Collapse will occur when a hinge is formed under the load.

Let $\theta$ angle of rotation of the left portion of the beam.

Hence angle of rotation of the right portion of the beam

$$\theta_1 = \frac{a \theta}{b}$$

Rotation of the hinge under the action of plastic moment

$$\theta + \theta_1 = \theta + \frac{a \theta}{b} = \theta \cdot \frac{L}{b}$$

The work absorbed by the hinge $= M_P \cdot \frac{\theta L}{b}$

The work done by the load $= W_C a \theta$

Equating the two, we get

$$W_C a \theta = M_P \cdot \frac{\theta \cdot L}{b}$$

or

$$W_C = \frac{M_P \cdot L}{ab}$$

The value of B.M. anywhere does not exceed $M_P$, and hence above value of collapse load is correct.

PLASTIC THEORY

If the load is acting at the centre of the beam, $a=b=L/2$ and hence

$$W_C = \frac{M_P \cdot L}{2} \cdot \frac{L}{2} = \frac{4M_P}{L}$$

2. Simply supported beam carrying uniformly distributed load

Let $W=\text{Total U.D.L.}$

$W_C=\text{full plastic moment of resistance of the beam section}$.

(a) Static method:

The maximum bending moment of $\frac{WL}{8}$ will occur at the centre of the beam. When the load $W$ is increased to the collapse load $W_C$, the maximum bending moment at the centre will be equal to $\frac{W_C \cdot L}{8}$, and a hinge will form there to get a collapse mechanism.

For equilibrium, the maximum moment must be equal to the plastic moment of resistance $M_P$.

$$W_C \cdot \frac{L}{8} = M_P$$

or

$$W_C = \frac{8M_P}{L}$$

(b) Kinematic method:

The collapse mechanism is shown in Fig. 26'13 (c). The central hinge will rotate through an angle $2\theta$, while the beam will deflect by $\frac{L}{2} \cdot \theta$ vertically downwards at its centre.
Average vertical moment of the U.D.L. = \( \frac{1}{2} \cdot \frac{L}{2} \theta = -\frac{L\theta}{4} \)

\[
\therefore \quad \text{Work done by the load} = W_C \cdot \frac{L\theta}{4}
\]

\[
\text{Work absorbed by the hinge} = M_P \cdot 2\theta
\]

\[
\therefore \quad W_C \cdot \frac{L\theta}{4} = M_P \cdot 2\theta
\]

or

\[
W_C = \frac{8M_P}{L}, \text{ as before.}
\]

The bending moment nowhere exceeds \( M_P \) and hence the assumed mechanism and the collapse load is correct.

3. Propped cantilever with eccentric concentrated load

\[
\text{Fig. 26.14}
\]

The shape of the bending moment diagram during elastic stage will be the same as that shown in Fig. 26.13 (b). Since the static B.M. at \( C \) is greater than at \( A \), the plastic hinge will first develop at \( C \) and then at \( A \). The structure will then be converted into a mechanism and it will ultimately collapse. During collapse, the moments at both \( A \) and \( C \) will be \( M_P \).

(i) Static method:

From Fig. 26.14 (b), the equilibrium equation is

\[
\frac{W_C \cdot ab}{L} = M_P + M_P \cdot \frac{b}{L} = M_P \left( \frac{L+b}{L} \right)
\]

or

\[
M_C = M_P \left( \frac{L+b}{ab} \right)
\]

The moment anywhere is not more than \( M_P \). Hence the above expression gives correct value of the collapse load.

(ii) Kinematic method:

The collapse mechanism is shown in Fig. 26.14 (c). During collapse, one hinge will be formed at the fixed end \( A \) and the other hinge will be formed at \( C \). Let the rotation of the left portion be \( \theta \).

The rotation of the right portion will be \( \theta_1 = \theta - \frac{a}{b} \). Thus, the hinge at \( A \) will rotate through angle \( \theta \) while the hinge at \( C \) will rotate through \( \theta + \theta_1 \). The collapse load will move down by \( ab \).

The equilibrium equation, therefore:

\[
W_C \cdot a\theta = M_P \cdot \theta + M_P \left( \theta + \frac{a\theta}{b} \right)
\]

\[
W_C \cdot a\theta = M_P \theta \left( \frac{b+b+a}{b} \right) = M_P \cdot \frac{\theta}{b} (L+b)
\]

or

\[
W_C = M_P \cdot \frac{(L+b)}{ab}, \text{ as before.}
\]

If the load acts at the centre of the cantilever, \( a=b=\frac{L}{2} \)

\[
W_C = M_P \cdot \frac{(L+L)}{\frac{L}{2} \cdot \frac{L}{2}} = \frac{6M_P}{L}
\]

4. Propped cantilever carrying U.D.L.

Fig. 26.15 (b) shows the bending moment diagram at collapse. One plastic hinge will form at the fixed end \( A \) and the other at \( C \). The exact location of \( C \) is to be determined. Let \( M_C \) be the simply supported bending moment at \( C \), distant \( x \) from \( B \).

Then

\[
M_C = -\left( \frac{W_C \cdot x}{L} - \frac{W_C}{L} \cdot \frac{x^2}{2} \right) = -\left( M_P + M_P \cdot \frac{x}{L} \right)
\]

or

\[
M_P \left( 1 + \frac{x}{L} \right) = \frac{W_C \cdot x}{L} \left( \frac{L}{2} - \frac{x}{2} \right)
\]

or

\[
M_P (L+x) = W_C \cdot \frac{x}{2} \cdot (L-x)
\]

or

\[
M_P = \frac{W_C}{2} \cdot \frac{x(L-x)}{L+x} = \frac{W_C}{2} \cdot \frac{Lx-x^2}{L+x}
\]

For maxima,

\[
\frac{\partial M_P}{\partial x} = 0 = x^2 + 2Lx - L^2
\]

This gives

\[
x = L(\sqrt{2}-1) = 0.414 \, L
\]
(a) Static method:

The equilibrium equation is

\[ M_p = \frac{W_c}{2} \cdot \frac{x(L-x)}{L+x} \text{, from (1)} \]

or

\[ M_c = \frac{2M_p}{L(3-2\sqrt{2})} = \frac{2M_p}{L}(3+2\sqrt{2}) = 11'656 \frac{M_p}{L} \]

(b) Kinematic method:

The rotation of left portion is \( \theta \) while that of right portion is \( \frac{L-x}{x} \cdot \theta \). The hinge at C will, therefore, rotate through \( \theta + \frac{L-x}{x} \cdot \theta \).

The downward movement of the load is equal to \( (L-x) \cdot \theta \), where \( v = 4'414 \frac{L}{x} \).

Work done by the load:

\[ \frac{W_c(L-x) \cdot \theta}{2} = \frac{W_c(2-\sqrt{2}) \cdot \theta}{2} \]

Work absorbed by the hinges:

\[ = M_p \cdot \theta + M_p \left[ \theta + \frac{L-x}{x} \cdot \theta \right] = M_p \cdot \theta \left[ 1 + \frac{L}{x} \right] = M_p \cdot \theta \cdot \frac{\sqrt{2L}}{L(\sqrt{2}-1)} \]

Hence the equilibrium equation is

\[ \frac{W_c(L-2\sqrt{2}) \cdot \theta}{2} = M_p \cdot \theta \cdot \frac{\sqrt{2L}}{L(\sqrt{2}-1)} \]

5. Fixed beam carrying an eccentric point load

Fig. 26.16 (b) shows the bending moment diagram for elastic stage. The bending moment at B is the greatest. As the load is increased the plastic hinge will first form at B, then at C and finally at A. At this stage, the beam will be converted into a mechanism and ultimately it will collapse.

(1) Static method:

From Fig. 26.16 (c), the equilibrium equation is

\[ \frac{W_c \cdot L}{ab} = M_p + M_r = 2M_p \]

\[ W_c = \frac{2M_p \cdot L}{ab} \]

(2) Kinematic method:

The rotation of the hinge at \( A = \theta \). The load \( W_c \) will, therefore, move downward by \( a\theta \). The rotation of hinge at \( B \) is \( \theta_1 = \frac{a\theta}{b} \). The rotation of hinge at \( C = \theta + \theta_1 = \theta + \frac{a\theta}{b} \)
Work done by the load = $W_c a \theta$

Work absorbed by the hinges = $M_F \theta + M_P \theta \left( 1 + \frac{L}{b} + \frac{a}{b} \right) = \frac{2L}{b} M_P \theta$

$$W_c a \theta = \frac{2L}{b} M_P \theta$$

$$W_c = \frac{2M_P L}{ab}$$

If, however, the load is placed at the middle of the beam

$$a = b = \frac{L}{2}$$

$$W_c = \frac{2M_P L}{L} = \frac{8M_P}{2}$$

If the beam carries uniformly distributed load, it can be shown that the collapse load,

$$W_c = \frac{16M_P}{L}$$

6. Three span continuous beam with U.D.L.

Let the total uniformly distributed load on each span be $W_c$. A continuous beam will collapse in the same manner as fixed beam by the formation of three plastic hinges, two at the supports and one between the supports of any span. The failure of one span will result in the failure of the whole structure.

![Fig. 26-17](a)

![Fig. 26-17](b)

During the elastic stage the ordinates of bending moment diagram will be $\frac{W_L}{10}$ at the inner supports and $\frac{W_L}{8}$ at the mid span. When the collapse load is applied, the plastic hinges will form at $E$, $B$, $C$, and $F$, and the beams $AB$ and $CD$ will collapse. The beam $BC$ can still take more load, but for all practical purposes the continuous beam has been rendered useless. The spans $AB$ and $CD$ may be looked upon as propped cantilevers with uniformly distributed loading. The collapse load is, therefore, equal to $11\,656 \cdot \frac{M_P}{L}$ and the hinges in the end span form at $0'414L$ from the outer supports. The plastic bending moment diagram can now be drawn as shown in Fig. 26'17 (b).

**Example 26'4.** Calculate the plastic section modulus, shape factor and plastic moment of the following sections:

(a) ISMB 200 [Fig. 26.18 (a)] having the following properties:

$$I_{xx} = 2235'4 \, cm^4; \, Z_{xx} = 223'5 \, cm^3; \, A = 32'33 \, cm^3;$$

Thickness of web = 5'7 mm; Thickness of flange = 10'8 mm.

(b) ISHT 150 [Fig. 26.18 (b)] having the following properties:

$$I_{xx} = 573'7 \, cm^4; \, A = 37'42 \, sq. \, cm \, \text{and} \, \text{distance of C.G. from the top is 26'6 \, mm}.$$ 

Take the yield stress for mild steel as 253 N/mm².

**Solution.**

(a) I-section: Given:

$$I_{xx} = 2235'4 \times 10^4 \, mm^4; \, Z_{xx} = 223'5 \times 10^3 \, mm^3 \, \text{and} \, A = 3233 \, mm^2$$

$$Z_P = \frac{A}{2} (y_1 + y_2)$$

Since the equal area axis coincides with the centroidal axis, $y_1$ and $y_2$ are equal. To find $y_1$ of the upper half area, we have

$$y_1 = \frac{100 \times 10'8 (100 - 5'4) + 5'7 (100 - 10'8) (100 - 10'8)}{(100 \times 10'8) + (5'7) (100 - 10'8)}$$

$$= \frac{102168 + 22576}{1080 + 508'4} = 78'6 \, mm$$

![Fig. 26-18](a)
\[ Z_p = \frac{A}{2} (y_1 + y_2) = 3233 \times 78.6 = 254106 \text{ mm}^3 \]
\[ Z = 223.5 \times 10^3 \text{ mm}^3 = 223500 \text{ mm}^3 \]
\[ S = \frac{Z_p}{Z} = 1.14 \]
\[ M_p = Z_p.\sigma_Y = 254106 \times 253 = 6429 \times 10^4 \text{ N-mm} \]
\[ = 6429 \text{ kN-m}. \]

(b) Tee Section

Given \( I_s = 573.7 \times 10^4 \text{ mm}^4 \); \( A = 3742 \text{ mm}^2 \)

Elastic section modulus \( Z = \frac{573.7 \times 10^4}{150 - 26.6} = 46491 \text{ mm}^3 \)

Let the equal area axis pass through the flange at distance \( x \) below the top fibre.
\[ 250x = 250(10.6 - x) + (150 - 10.6) \times 7.6 \]
\[ \Rightarrow 500x = 3709 \]
From which \( x = 7.42 \text{ mm} \)

\( y_1 = \) Distance of C.G. of the top area from equal area axis
\[ = \frac{7.42}{2} = 3.71 \text{ mm} \]

Distance of bottom of flange from equal area axis
\[ = 10.6 - 7.42 = 3.18 \text{ mm} \]

\( y_2 = \) Distance of C.G. of bottom area from the equal area axis
\[ = 250 \times 3.18 \times (3.18)^4 + (150 - 10.6) \times 7.6((150 - 10.6)^{4} + 3.18) \]
\[ = 41.94 \text{ mm} \]

\[ Z_p = \frac{A}{2} (y_1 + y_2) = \frac{3742}{2}[3.71 + 41.94] = 85417 \text{ mm}^3 \]
\[ S = \frac{Z_p}{Z} = \frac{85417}{46491} = 1.84 \]
\[ M_p = Z_p.\sigma_Y = 85417 \times 253 = 21.61 \times 10^6 \text{ N-mm} \]
\[ = 21.61 \text{ kN-m}. \]

Example 26.5. A beam of rectangular cross-section \( b \times d \) is subjected to a bending moment \( 0.9 \) \( M_p \). Find out the depth of the elastic core.

Solution.

Let the total depth of the elastic core = 2\( x \). Therefore, the depth of the plastic zone = \( \frac{d}{2} - x \), on either side.

Example 26.6. A fixed beam of span \( L \) carries a uniformly distributed load \( W \) on the left half portion. Determine the value of \( W \) at collapse. The elastic moment of resistance of the beam is \( M_p \).
Solution.

Let the maximum free bending moment occur at C, distant x from A.

\[ M_r = -\frac{3Wx}{4} + \frac{2Wx^2}{L} \]

For maxima,

\[ \frac{dM_r}{dx} = 0 = -\frac{3W}{4} + \frac{2Wx}{L} \]

\[ x = \frac{3}{8} L \]

\[ M_r = -\frac{3W}{4} \times \frac{3}{8} L + \frac{W}{L} \left( \frac{3}{8} L \right)^2 = -\frac{9}{64} WL \]

At collapse, it becomes equal to \( \frac{9}{64} WcL \) (Numerically)

Hence as per static method, the equilibrium equation is

\[ \frac{9}{64} WcL = M_r + M_p \]

\[ W_c = \frac{2M_p}{L} \times \frac{64}{9} = \frac{128 M_p}{9} \] (Answer).

Example 26.7. A beam ABC of span L is fixed at the ends A and C, and carries a point load at a distance \( \frac{L}{4} \) from the left end. Find the value of the load at collapse if the left half of the beam has a plastic moment of resistance \( 2M_p \) and the right half has a plastic moment \( M_p \).

Solution.

There are two possible collapse mechanisms. In the first mechanism [Fig. 26.21 (b)], hinges may form at A, D and C. The equilibrium equation is

\[ W_c \times \frac{3}{4} L \theta = 2M_p.3\theta + 2M_p.4\theta + M_p\theta \]

\[ W_c = \frac{20M_p}{L} \]

In the second mechanism [Fig. 26.21 (c)], hinges may form at A, B and C. The hinge at B will work corresponding to the least moment of resistance at B, i.e. \( M_r \). The equilibrium equation is

\[ W_c \times \frac{L}{4} \theta = 2M_p. \theta + M_p. 2\theta \]

\[ W_c = \frac{20M_p}{L} \]

The mechanism is the one which gives the minimum of the collapse load (upper bound theorem). However, in the present case, both mechanisms give equal collapse load of \( \frac{20 M_p}{L} \).

Example 26.8. A simply supported beam consists of a steel rod of diameter 30 mm. The span of the beam is 2 m. The steel rod is bored for a length of 0.5 m at each end. Find the diameter of the bore so that plastic hinges may form simultaneously at A, B and C, as shown in Fig. 26.21. If \( a_v = 253 \text{ N/mm}^2 \), find the collapse load.

Solution.

When the plastic hinge is formed, the bending moment at the centre = \( \frac{W_c \times 2}{2} = \frac{W_c}{4} \) = plastic moment of unbored rod = \( M_p \).

Bending moment at B and C

\[ = \frac{W_c}{2} \times \frac{1}{2} - \frac{W_c}{2} \times \frac{1}{2} \times \frac{1}{4} = \frac{3}{16} W_c = M_p \].
For the plastic hinges to form at B and C, the fully plastic moment of the bored steel rod must be $\frac{3}{16} W_C$.

Now $M_F : M_{P1} = \frac{W_C}{4} : \frac{3}{16} W_C = 1 : 0.75$.

For unbored steel rod, $Z_P = \frac{d_o^3}{6} - \frac{(30)^3}{6} = 4500 \text{ mm}^3$.

Let $d_i$ be the inner diameter and $d$ the outer diameter of the bored tube. Plastic section modulus of bored tube is

$$Z_{P1} = \frac{d^3 - d_i^3}{6} = \frac{30^3 - d_i^3}{6} = 27000 - d_i^3$$

Since the ratio of $M_F$ and $M_{P1}$ is $1 : 0.75$, the ratio of $Z_P$ and $Z_{P1}$ must also be $1 : 0.75$.

$$Z_{P1} = 0.75 Z_P$$

or

$$\frac{27000 - d_i^3}{6} = 0.75 \times 4500$$

From which $d_i = 18.9 \text{ mm}$

Now $M_F = Z_P \times \sigma_F = 4500 \times 253 = 1.1385 \times 10^6 \text{ N-mm}$

At collapse, $M_F = \frac{W_C}{4} \text{ kN-m}$

$$W_C = 1.1385 \times 4 = 4.55 \text{ kN}.$$
or
\[ W_c = 4.5 \frac{M_p}{L} \]  \hspace{1cm} (3)

(IV) Let the span CD collapse first

The equilibrium equation is
\[ 2W_c \cdot \frac{L}{3} = M_p \cdot 20 + M_p \cdot 30 \]

\[ W_c = \frac{15}{4} \frac{M_p}{L} \]  \hspace{1cm} (4)

The actual collapse load will be the least of these four. Hence
\[ W_c = \frac{15}{4} \frac{M_p}{L} \]  \hspace{1cm} (Answer)

The bending moment diagram, corresponding to this value of \( W_c \) is shown. Since the B.M. nowhere exceeds \( M_p \), the above value of \( W_c \) is correct.

2616. PORTAL FRAMES

In the case of a portal frame, at least four hinges (natural + plastic) are necessary to convert it into a mechanism. In general, one hinge more than the number of redundancies will be required. In the case of a portal frame hinged at both the legs, where redundancy is one, two plastic hinges will be required for the mechanism to form. Similarly, for a portal frame fixed at the base, four plastic hinges are necessary. The degree of redundancy can be found by the expression:
\[ T = 3a + R - 3, \]
where
\[ T = \text{number of redundancies} \]
\[ a = \text{number of areas completely enclosed by the members} \]
\[ R = \text{total number of reaction components} \]

In the case of a simple single span portal frame, \( a = 0 \). If the legs are fixed, \( R = 3 + 3 = 6 \). Hence \( T = 6 - 3 = 3 \). Thus a fixed portal frame has 3 redundancies. For finding out the number of redundancies of other cases, reader is advised to go through chapter 6. The number of independent mechanisms is given by Eq. 26'17. In addition to these, a number of combined mechanisms may be possible, and all of these should be tried to determine the minimum collapse load.

Example 2610. Determine the value of \( W \) at collapse for the portal frame loaded as shown in Fig. 26'24. All the members have the same plastic moment of resistance \( M_p \).

\[ W_c = \frac{2L}{3} \theta + W_c \frac{L}{3} \theta = M_p 2\theta + M_p 3\theta + M_p \theta \]

\[ \therefore \quad W_c = \frac{6M_p}{L} \]

\[ 2W_c \cdot \frac{L}{2} \theta = M_p 2\theta + M_p \theta \]

\[ \therefore \quad W_c = \frac{6M_p}{L} \]
Example 26.11. Determine the value of $W$ at collapse for the portal frame shown in Fig. 26.26. All the members have the same plastic moment of resistance.

Solution.

\[ T = 3a + R - 3 \]
\[ a = 0; R = 3 + 3 = 6; \]
\[ a = 0; R = 3 + 3 = 6; \]
\[ T = 6 - 3 = 3. \]

Thus the frame is statically indeterminate to third degree. The total number of independent mechanisms are given by

\[ N = n - T \]

where $n =$ number of possible hinges

\[ = 5 \text{ (one each at points A, B, C, D and E)} \]
\[ N = 5 - 3 = 2. \]

Thus there are two independent mechanisms: (1) beam mechanism, and (2) panel mechanism. In addition to these, a combined mechanism, consisting of beam and panel mechanism is possible. All the three mechanisms are shown in Fig. 26.17 (a), (b) and (c) respectively.
(a) Beam mechanism [Fig. 26'27 (a)]
\[ 2W_C \cdot \frac{L}{2} = M_P \theta + M_P 2\theta + M_P \theta \]
\[ W_C = \frac{4M_P}{L} \]

(b) Panel mechanism [Fig. 26'27 (b)]
\[ W_C \cdot \frac{L}{2} = M_P \theta + M_P \theta + M_P \theta + M_P \theta \]
\[ W_C = \frac{8M_P}{L} \]

(c) Combined mechanism [Fig. 26'27 (c)]
\[ 2W_C \cdot \frac{L}{2} + W_C \cdot \frac{L}{2} = M_P \theta + M_P 2\theta + M_P 2\theta + M_P \theta \]
\[ W_C = \frac{4M_P}{L} \]

Actual collapse load = \( \frac{4W_P}{L} \). But since this load occurs in two mechanisms, the collapse mechanism will be combination of these two mechanisms, as shown in Fig. 26'27 (d).

In Fig. 26'27 (d), the equilibrium equation is
\[ 2W_P \cdot \frac{L}{2} (\theta + \phi) + W_C \cdot \frac{L}{2} = M_P \theta + M_P \phi + M_P 2(\theta + \phi) + M_P (2\theta + \phi) + M_P \theta \]

or
\[ W_C \cdot \frac{L}{2} (3\theta + 2\phi) = 2M_P (3\theta + 2\phi) \]

or
\[ W_C = \frac{4M_P}{L} \] which is the same as before.

PROBLEMS

1. A beam of rectangular cross-section \( b \times d \) is subjected to a bending moment 0'75 \( M_P \). Find out the depth of the elastic core.

2. A beam \( ABC \), 2.5 m long is suspended at \( B \) and \( C \) in horizontal position by means of two wires of 2 cm diameter. The bar is hinged at \( A \). A load \( P \) is applied as shown in Fig. 26'28. Find out the value of \( P \) when the yield appears in any of the bars, (b) when the whole arrangement collapses. Take \( \sigma_Y = 253 \) N/mm².

3. A load \( P \) is supported by three rods as shown in Fig. 26'29. Find the value of \( P \) at collapse. All the bars are of the same area of cross-section, \( A \). Show that this load is \( \sqrt{2} \) times the elastic load.

4. For the structure shown in Fig. 26'30, compute the value of \( P \) (i) when the first yield occurs, (ii) when the whole arrangement collapses. The vertical bars have the same area of cross-section \( A \). The horizontal beam is rigid.

5. (a) A simply supported beam carries uniformly distributed load. Prove that \( \frac{W_C}{W_P} = S \).

(b) Show that the fully-plastic moment of a beam of rectangular cross-section is 50% greater than the bending moment at which the beam reaches the limit of elasticity.

6. A beam fixed at both the ends carries uniformly distributed load. Prove that \( \frac{W_C}{W_P} = \frac{4}{3} S \).

7. A simply supported beam of span \( L \) carries a central point load. Calculate the value of the load at collapse in terms of the plastic moment of resistance \( M_P \).

8. If a propped cantilever, with a constant \( M_P \), carries a central point load, determine its value at collapse.

9. A beam of span \( L \) and constant \( M_P \) is fixed at its end. It carries a load \( W \). Determine its value at collapse if (a) \( W \) is concen-
10. A $ISWB$ 600 @ 145'1 kg/m is supported over a length of 5 m. Its one end is fixed and the other is hinged. It is loaded with a uniformly distributed load $W$ and a concentrated load $0.5W$ at 2 m from the fixed end. Find the value of $W$ for collapse to take place. If the load factor is 2, find out the value of safe working load. Given $Z_{xx}=385.42$ cm$^3$, shape factor $=1.14$ and $\sigma_v=25.30$ kg/cm$^2$.

11. A fixed beam of span 6 m carries a uniformly distributed load $W$ on the left half portion. If the fully plastic moment of the beam is 100 kN-m, find the value of the collapse load.

12. A uniform beam of rectangular cross-section is built in at each end and carries a vertical load at the mid-length. Determine the plastic zones and their extent at the collapse load of the beam.

13. A uniform beam of length $L$ is built-in at one end and simply supported at the other. A load $W$ is applied to the beam at a distance $aL$ from the built-in end. If the fully plastic moment of the beam is $M_F$, find the value of $W$ for collapse, and find the value of $a$ for which the collapse load is a minimum.

14. A uniform beam of length $3L$ is built-up at each end and carries vertical loads $W$ and $2W$ at the third points. If the plastic moment of the beam is $M_F$, estimate the value of $W$ for complete collapse.

15. Find the value of $W$, for the beam shown in Fig. 26'31 so that collapse may take place. The plastic moment of the beam section is $M_F$.

16. Find the collapse load $W_c$ for the continuous beam shown in Fig. 26'32. The beam has constant plastic moment $M_F$.

17. A portal frame shown in Fig. 26'33 is hinged at the ends and has fully plastic moment $M_F$. It is loaded with a vertical load $W$ and a horizontal load $\frac{W}{2}$ as shown in Fig. 26'33. Find the value of $W$ at collapse.

18. A portal frame shown in Fig. 26'34 has uniform section throughout. Determine the value of the plastic moment of resistance in terms of the load parameter $W_c$ at collapse.

19. A portal frame of height $L$ and span $L$ is hinged at the base and is of uniform plastic moment $M_F$. It carries a single central vertical load. Find the value of $W$ at collapse.

20. A fixed rectangular portal frame of height $L$ and span $2L$, is of uniform section with fully plastic moment $M_F$. A horizontal load $W$ is applied at the top of the column and another load $W$ is applied vertically at the centre of the beam. Find the value of $W$ at collapse.

**ANSWERS**

1. 0.868 d.
2. (a) 115'2 kN  \hspace{1cm} (b) 139 kN.
3. $P_{pl} = \left(1 + \frac{1}{\sqrt{2}}\right)A \sigma_y$; $P_L = A \sigma_y (1 + \sqrt{2})$.
4. (i) $\frac{5}{3} \sigma_y A$  \hspace{1cm} (ii) $2 \sigma_y A$.
5. $\frac{6M_F}{L}$.
6. $\frac{4M_F}{L}$.
9. \( \frac{8M_p}{L} \); \( \frac{16M_p}{L} \).
10. \( W_c = 146 \text{ t} \); Safe, \( W = 73 \text{ t} \).
11. \( W_c = 235.6 \text{ kN} \).
12. \( \frac{L}{12} \) at either ends and \( \frac{L}{6} \) at the middle.
13. \( W_c = \frac{M_p}{L} \left[ \frac{2 - a}{a(1 - a)} \right] ; a = 0.586 \).
14. \( W_c = \frac{1.2M_p}{L} \).
15. \( W_c = 3.74 \).
16. \( W_c = \frac{4}{3} M_p \).
17. \( \frac{8}{3} \frac{M_p}{L} \).
18. \( M_p = W_c \).
19. \( \frac{8M_p}{L} \).
20. \( \frac{3M_p}{L} \).

### Building Frames

#### 27.1. INTRODUCTION

A building frame may contain a number of bays and may have several storeys. A multi-storeyed, multi-panelled frame is a complicated statically indeterminate structure. It consists of a number of beams and columns built monolithically, forming a network. The doors and walls are supported on beams which transmit the loads to the columns. A building frame is subjected to both the vertical as well as horizontal loads. The vertical loads consist of the dead weight of structural components such as beams, slabs, columns etc. and live load. The horizontal loads consist of the wind forces and earthquake forces. The ability of multi-storey building to resist the wind and other lateral forces depends upon the rigidity of connections between the beams and columns. When the connections of beams and columns are fully rigid, the structure as a whole is capable of resisting the lateral forces acting on the structure.

In ordinary reinforced concrete skeleton buildings, a continuous beam is rigidly connected with columns. Due to this, the moments in the beam depend not only upon the number and length of spans composing the beam itself, but also upon the rigidity of columns, with which it is connected. The bending moment at the end of any one span of the continuous beam cannot be transferred to the beam in the next span without subjecting the columns to bending. Instead of transmitting the bending moment in full to the beam in the next span, part of the moment is transferred to the columns above and below the beam, and the balance to the beam. Due to this, the effect of loading on one span upon the other spans is much lower than in ordinary continuous beams which are not connected to the columns.
272. SUBSTITUTE FRAME

The analysis of a multi-storeyed multi-panelled building frame is very cumbersome, since the frame contains a number of continuous beams and columns. As stated in the previous article, the effect of loading on the span upon other spans is much smaller. The moments from floor to floor, through columns, are very small in comparison to beam moments. In other words, the moments in one floor have negligible effect on the moments of the floor above and below it. Therefore, a substitute frame consists of one floor, connected above and below with their four ends either hinged or fixed or restrained.

Fig. 271 (a) shows a building frame consisting of five storeys and three bays. Fig. 271 (b) shows the substitute frame of determining bending moment in the second floor. Generally, it is sufficient to consider two adjacent spans on each side of joint considered. The substitute frame gives the results which are safe for all practical purposes.

Types of substitute frames

Under ordinary conditions, the following four types of substitute frames are considered sufficient:

(a) Three span structure with two storey columns.
(b) Substitute frame for wall columns.
(c) Substitute frame for two panel wide building.
(d) Substitute frame for one panel wide building.

Fig. 272 (a) shows the most general substitute frame consisting of three span, two-storey substitute structure with irregular spacing of columns. Fig. 272 (b) shows the substitute frame for finding the bending moments in wall columns. This consists of three spans and three two-storey columns, one of which is the wall column. Fig. 272 (c) shows the substitute frame for structures with two panels wide. Fig. 272 (d) shows the substitute frame for one span multi-storey frame.

End conditions for substitute frames

The restraining effect of any one member, upon other members forming a joint depends also upon the condition existing on the other end of the restraining member. The other end may have three conditions: (i) free to turn (i.e. hinged), (ii) partially restrained, or
The restraining effect is largest for the rigidly fixed conditions of the end and smallest for free end. It should be noted that the restraining effect of a fixed member is one-third larger than the restraining effect if it were free to turn. The rigidity of any member is expressed by the ratio \( \frac{I}{L} \) where \( I \) is its moment of inertia and \( L \) is its length (for beam) or height (for column). If the loaded member has rigidity \( \frac{I_1}{L_1} \) and the restraining member has rigidity \( \frac{I}{L} \), then this restraining member is considered as fixed at the other end if \( \frac{I_1}{L_1} \geq \frac{I}{L} \) is equal to or greater than 10. The end of a member is considered as a partly restrained when it runs into another joint composed of several members, a condition which is often found in concrete skeleton structure. No restraint exists if \( \frac{I_1}{L_1} \div \frac{I}{L} = 0 \). The other ends of the member of the substitute frame are sometimes taken as hinged (except for columns fixed in the ground). This gives severest condition for a particular reaction under investigation. The moments obtained by assuming the ends hinged gives the moments nearest to the value obtained from full frame analysis and compensates to some extent for the error caused due to neglecting loads on members of distant span.

27.3. ANALYSIS FOR VERTICAL LOADS

A building frame is a three dimensional structure consisting of a number of bays in two directions at right angles to each other. A building structure may be assumed to be consisting of two sets of plane frames crossing each other at right angles. The vertical members (i.e. columns) are common to both these sets of frame. Each set of frames is analysed separately. Since moments in the vertical members occur in two planes, the stresses in columns should be found for moments acting in two planes simultaneously and the corresponding vertical loads.

(a) Maximum bending moments in beams

The magnitude of bending moments in beams and columns respectively depend upon their relative rigidity. Generally, the beams are made of the same dimensions in all the floors, while the dimensions of columns vary from story to story. Columns have smallest dimensions at the top and largest dimensions at the bottom. Due to this reason, the ratio of the rigidity of the beam to that of the column is larger in the upper floors than in the lower floors. The positive bending moments in the beams increase with decrease of the rigidity of the columns, while the negative B.M. in them increase with the increase in the rigidity of the columns. Due to this, the positive B.M. are the largest in the upper storeys where the columns are least rigid and the negative bending moments are maximum in the lower storeys where the columns are rigid.
In order to find the maximum moment in a given span of the beam, the substitute frame is so selected that span under investigation forms the centre span. This substitute frame may be moved from floor to floor. However, since the beams in all floors are made of the same dimensions and provided with same amount of steel, only one substitute frame may be sufficient when placed in a position in the structure for which the bending moments are the largest. The beams should be loaded with live loads as follows for maximum effects:

(i) For maximum positive B.M. At the mid-point C of a span AB, the loads should be placed on the span and on alternative spans, as shown in Fig. 27'3 (a).

(ii) For maximum negative B.M. At the mid-point C of a span AB, the span AB should be unloaded while load should be placed on spans adjacent to the span under consideration, as shown in Fig. 27'3 (a).

(iii) For maximum negative B.M. At the support A, loads should be placed on the two spans adjacent to the support as shown in Fig. 27'3 (c).

When the spans of the beams are not equal, substitute frames should be selected in which the largest span forms the centre span, and also frames in which the smallest span forms the centre span. Several trial computations may be necessary to get the frame for which the bending moments are maximum. To get the bending moment in the wall columns and wall beams, substitute frame shown in Fig. 27'2 (b) should be used.

The bending moments due to dead loads are found separately. The bending moments for dead and live loads are then added, and the beam is designed.

(b) Maximum bending moment in columns

The bending moments in columns increase with increase in their rigidity. Hence they are largest in the lower storeys, and smallest in the upper storey. The maximum compressive stresses in columns is found by combining maximum vertical loads with the maximum bending moments. The maximum tensile stresses in columns is found by combining the maximum bending moment with minimum vertical loads. Though the bending moment is smallest in the upper floors, its effect is much larger since the dimensions of the columns are the smallest there and also the vertical loads are much smaller than in lower storeys. Also the possibility of tensile stresses in columns is much larger in upper storeys than in lower storeys.

The maximum moments in columns occur when alternative spans are loaded as shown in Fig. 27'4 (a), (b). The corresponding axial loads are found. The column is designed to resist the stresses provided by every combination of axial load and the corresponding moment.
274. METHODS OF COMPUTING B.M.

The bending moments in the beams and columns of a substitute frame may be computed by the following methods:

1. Slope deflection method.

The slope-deflection method results in too many equations to be solved simultaneously. Hence moment distribution method, using two cycles is used. Taylor, Thomson and Smulski recommended the use of building frame formulae which they have developed using slope deflection equations.

Example 27.1. Fig. 27.5 shows an intermediate frame of a multi-storeyed building. The frames are spaced at 4 metres centre to centre. Analyse the frame taking live load of 4000 N/m² and dead load as 3000 N/m², 3250 N/m² and 2750 N/m² for panels AB, BC and CD respectively. The self weight of the beams may be taken as under:

Beams of 7 m span : 5000 N/m.
Beams of 5 m span : 3500 N/m.
Beams of 3.5 m span : 2500 N/m.

The relative stiffness of the members are marked on the figure itself.

Solution.

1. Substitute frame

Let us analyse the second floor ABCD. The substitute frame is shown in Fig. 27.6, assuming the far ends of the columns fixed. Other floors can be analysed in a similar manner.

2. Loading and fixed end moments

Since the frames are spaced @ 4 m c/c, the live loads transferred from the floors will be

4000 x 4 = 16000 N/m.
The total dead load on a beam will be equal to dead load from the floors plus the dead load due to the self weight of the beam.

Thus, the total dead load on the beam $AB$, per metre run = $(3000 \times 4) + 5000 = 17000$ N/m. Dead loads for other beams are tabulated in Table 27.1.

The fixed end moment is calculated from the following expressions

$$M_f = \pm \frac{wL^2}{12}$$

where $w$ is the U.D.L. and $L$ is the span of the beam. Clockwise moments are taken as positive and anticlockwise moments are taken as negative. The fixed end moments due to dead load, and due to dead and live load combined are tabulated in Table 27.1.

**Table 27.1**

<table>
<thead>
<tr>
<th>Member</th>
<th>Dead load (N/m)</th>
<th>Live load (N/m)</th>
<th>F.E.M. due to dead load (N-m)</th>
<th>F.E.M. due to dead and live load combined (N-m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AB$</td>
<td>17000</td>
<td>16000</td>
<td>69420</td>
<td>134750</td>
</tr>
<tr>
<td>$BC$</td>
<td>15500</td>
<td>16000</td>
<td>15820</td>
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<tr>
<td>$CD$</td>
<td>14500</td>
<td>16000</td>
<td>30210</td>
<td>63540</td>
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</table>

3. Distribution factor

The distribution factors at a joint depends upon the relative stiffnesses of the members meeting at the joints. These are tabulated in Table 27.2 on next page.

### Table 27.2

<table>
<thead>
<tr>
<th>Joint</th>
<th>Members</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>Distribution factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$AE$</td>
<td>$2.5K$</td>
<td></td>
<td>0.263</td>
</tr>
<tr>
<td></td>
<td>$AI$</td>
<td>$2.5K$</td>
<td>$9.5K$</td>
<td>0.263</td>
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<tr>
<td></td>
<td>$AB$</td>
<td>$4.5K$</td>
<td></td>
<td>0.474</td>
</tr>
<tr>
<td>$B$</td>
<td>$BA$</td>
<td>$4.5K$</td>
<td></td>
<td>0.392</td>
</tr>
<tr>
<td></td>
<td>$BF$</td>
<td>$2.5K$</td>
<td></td>
<td>0.217</td>
</tr>
<tr>
<td></td>
<td>$BC$</td>
<td>$2.5K$</td>
<td>$11.5K$</td>
<td>0.174</td>
</tr>
<tr>
<td></td>
<td>$BJ$</td>
<td>$2.5K$</td>
<td></td>
<td>0.217</td>
</tr>
<tr>
<td>$C$</td>
<td>$CB$</td>
<td>$2K$</td>
<td></td>
<td>0.20</td>
</tr>
<tr>
<td></td>
<td>$CG$</td>
<td>$2.5K$</td>
<td></td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>$CD$</td>
<td>$3K$</td>
<td></td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>$CK$</td>
<td>$2.5K$</td>
<td></td>
<td>0.25</td>
</tr>
<tr>
<td>$D$</td>
<td>$DC$</td>
<td>$3K$</td>
<td>$10K$</td>
<td>0.375</td>
</tr>
<tr>
<td></td>
<td>$DH$</td>
<td>$2.5K$</td>
<td>$8K$</td>
<td>0.3125</td>
</tr>
<tr>
<td></td>
<td>$DL$</td>
<td>$2.5K$</td>
<td></td>
<td>0.3125</td>
</tr>
</tbody>
</table>

(A) **MAXIMUM NEGATIVE B.M. AT SUPPORTS**

4. Maximum Negative B.M. at Joint A

The condition of loading to obtain maximum negative B.M. at a joint $A$ is as follows: Live load on $AB$ only, while the dead load is on $AB$ and $CD$. The effect of load on other spans is neglected. The moment distribution is carried out in Table 27.3. The distribution is done at Joint $B$ and the carry over effect (i.e. half the moment) is transferred to joint $A$. After adding the total moment at $A$, distribution is done at $A$. The distribution at joint $B$ has not been recorded in Table 27.3.
Table 27.3

Moment Distribution for $-ve$ B.M. at $A$

<table>
<thead>
<tr>
<th>Joint</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Member</td>
<td>$AB$</td>
<td>$BA$</td>
<td>$BC$</td>
<td>$CB$</td>
</tr>
<tr>
<td>D.F.</td>
<td>0.474</td>
<td>0.392</td>
<td>0.174</td>
<td>0.20</td>
</tr>
</tbody>
</table>

1. F.E.M. due to D.L.
2. F.E.M. due to total load
3. Distribution at $B$ and carry over to $A$
4. Add (2) and (3)
5. Distribution
6. Total [sum of (4) and (5)]

5. Maximum Negative B.M. at Joint $B$

The loading conditions are: Live load on $AB$ and $BC$, while dead load on whole of $ABCD$. The moment distribution is carried out in Table 27.4. In the first cycle, joints $A$ and $C$ are balanced and half the moments are carried over to joint $B$ for beams $BA$ and $BC$ respectively. In the second cycle, joint $B$ is balanced and final moments are found. Thus, in the first cycle, unbalanced moment at $C$ is $+1950$, the distributed moment to $CB$ will be $-1950 \times 0.2 = -390$, and the carry over moment at $B=+200/2=+100$. Similarly, the unbalanced moment at $A$ is $-134750$, the distributed moment for $AB=+134750 \times 0.474=+63880$, the carried over moment to $B=+31940$. The total moments at $BA$ and $BC$ will be $+166690$ and $-32260$, leaving an unbalanced moment of $+134430$. The distributed moments to $BA$ and $BC$ will be $-134430 \times 0.392=-52700$ and $-134430 \times 0.174=-23390$ respectively.

6. Maximum Negative B.M. at $C$

The conditions of loadings are: Live load on $BC$ and $CD$, and dead load on $ABCD$. In the first cycle, joints $B$ and $D$ are balanced and effects are carried over to $C$. In the second cycle, joint $C$ is balanced, as shown in Table 27.5.

7. Maximum Negative B.M. at $D$

The conditions for loadings are: Live load on $CD$ and dead load on $ABCD$. In first cycle, joint $C$ is balanced and its effect is carried over to $D$. In the second cycle, joint $D$ is balanced as shown in Table 27.5.
### Table 27.5
**Moment Distribution for —ve B.M. at C**

<table>
<thead>
<tr>
<th>Joint</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Member</td>
<td>AB</td>
<td>BA</td>
<td>BC</td>
<td>CB</td>
</tr>
<tr>
<td>D.F.</td>
<td>0.474</td>
<td>0.392</td>
<td>0.174</td>
<td>0.20</td>
</tr>
<tr>
<td>1. F.E.M. due to D.L.</td>
<td>-69420</td>
<td>+69420</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. F.E.M. due to total load</td>
<td></td>
<td></td>
<td>-32160</td>
<td>+32160</td>
</tr>
<tr>
<td>3. Distribution at B and D and carry over to C</td>
<td></td>
<td></td>
<td>-3240</td>
<td>-11910</td>
</tr>
<tr>
<td>4. Add (2) and (3)</td>
<td></td>
<td></td>
<td>+28920</td>
<td>-75450</td>
</tr>
<tr>
<td>5. Distribution</td>
<td></td>
<td></td>
<td>+9310</td>
<td>+13960</td>
</tr>
<tr>
<td>6. Total (sum of 4 and 5)</td>
<td></td>
<td></td>
<td>+38230</td>
<td>-61490</td>
</tr>
</tbody>
</table>

### Table 27.6
**Moment Distribution for +ve B.M. at C**

<table>
<thead>
<tr>
<th>Joint</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Member</td>
<td>AB</td>
<td>BA</td>
<td>CB</td>
<td>CB</td>
</tr>
<tr>
<td>D.F.</td>
<td>0.474</td>
<td>0.392</td>
<td>0.174</td>
<td>0.20</td>
</tr>
<tr>
<td>1. F.E.M. due to D.L.</td>
<td>-15820</td>
<td>+15820</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. F.E.M. due to total load</td>
<td></td>
<td></td>
<td>-63540</td>
<td>+63540</td>
</tr>
<tr>
<td>3. Distribution at C and carry over to D</td>
<td></td>
<td></td>
<td>+7160</td>
<td></td>
</tr>
<tr>
<td>4. Add (2) and (3)</td>
<td></td>
<td></td>
<td>+9310</td>
<td>+13960</td>
</tr>
<tr>
<td>5. Distribution</td>
<td></td>
<td></td>
<td>+38230</td>
<td>-61490</td>
</tr>
</tbody>
</table>

### Building Frames

- **(B) Maximum Positive B.M. at Mid-SPANS**

The conditions of loadings are: Live load on $AB$ and $CD$ and Dead load on $ABCD$. In the first cycle, distribution is performed at joints $A$, $B$, and $C$. Half of these distributed moments are carried over to the opposite ends, i.e., from $A$ to $B$ and $B$ to $A$, and from $C$ to $B$. In the second cycle, distribution is performed at $A$ and $B$, as illustrated in Table 27.7. Thus the end moments at $A$ and $B$ for beam $AB$ are $-83140$ and $+105680$ respectively. The free B.M. at mid-span of $AB$ is:

$$AB = \frac{wL^2}{8}$$

$$= \frac{(17000 + 16000)(7)^2}{8} = 202120 \text{ N-m}$$

Therefore, Net B.M. at centre of $AB = 202120 - \frac{(83140 + 105680)}{2} = 107710 \text{ N-m}$

*Note: The table for +ve B.M. at mid-span of $AB$ is not fully visible.*
9. **Maximum +ve B.M. in mid-span of BC**

The conditions for loadings are: Live load on BC and dead load on ABCD. In the first cycle, moments are distributed at A, B, C and D. These distributed moments are carried over from A to B, from B to C, from C to D and from D to C. Finally, the moment and distributed at joints B and C, as shown in Table 27.8.

<table>
<thead>
<tr>
<th>Table 27.8</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Moment Distribution for +ve B.M. at mid-span of BC</strong></td>
</tr>
<tr>
<td>Joint</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>Member</td>
</tr>
<tr>
<td>D.F.</td>
</tr>
<tr>
<td>1. F.E.M. due to D.L.</td>
</tr>
<tr>
<td>2. F.E.M. due to total load.</td>
</tr>
<tr>
<td>3. Distribution at A, B, C and D</td>
</tr>
<tr>
<td>4. Carry over</td>
</tr>
<tr>
<td>5. Distribution at A and B</td>
</tr>
<tr>
<td>6. Total moments</td>
</tr>
</tbody>
</table>

Thus, the end moments at B and C are -41670 and +30310 respectively. The free B.M. at the centre of span BC is

\[
\text{Net B.M. at the Centre of BC} = \frac{wL^2}{8} = \frac{(15500+16000) \times 3.5^2}{8} = 48230 \text{ N-m}
\]

10. **Maximum +ve B.M. in mid-span of CD**

Conditions of loadings are: Live load on CD and AB and dead load on ABCD. In the first cycle, the moment distribution is done at joints B, C and D, and half the distributed moments are carried over to the opposite ends, i.e. from D to C and C to D, and from B to C. In the second cycle, distribution is performed at C and D, as illustrated in Table 27.9.

<table>
<thead>
<tr>
<th>Table 27.9</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Moment Distribution for the B.M. at Mid-span of CD</strong></td>
</tr>
<tr>
<td>Joint</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>Member</td>
</tr>
<tr>
<td>D.F.</td>
</tr>
<tr>
<td>1. F.E.M. due to D.L.</td>
</tr>
<tr>
<td>2. F.E.M. due to total load.</td>
</tr>
<tr>
<td>3. Distribution at A, B, C and D</td>
</tr>
<tr>
<td>4. Carry over</td>
</tr>
<tr>
<td>5. Distribution at B and C</td>
</tr>
<tr>
<td>Thus, the end moments at C and D are -5445 and +4420 respectively. The free B.M. at the centre of span CD is</td>
</tr>
</tbody>
</table>

\[
\text{Net B.M. at the Centre of CD} = \frac{wL^2}{8} = \frac{(14500+16000) \times 3.5^2}{8} = 48230 \text{ N-m}
\]

11. **Maximum Negative B.M. at Centre of Spans BC**

The condition for loadings are: Live loads on AB and CD, and dead load on ABCD. In the first cycle, moment distribution is carried out at all the four joints A, B, C and D. These moments are then carried over to joints B and C and from joints A and B, as well as between themselves. The second distribution is carried out at joints B and C, as shown in Table 27.16.
Table 27'10
Moment Distribution for Max. –ve B.M. at centre of Span BC

<table>
<thead>
<tr>
<th>Joint</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Member</td>
<td>AB</td>
<td>BA</td>
<td>BC</td>
<td>CB</td>
</tr>
<tr>
<td>D.F.</td>
<td>0.474</td>
<td>0.392</td>
<td>0.174</td>
<td>0.20</td>
</tr>
<tr>
<td>1. F.E.M. due to D.L.</td>
<td>-134750</td>
<td>+134750</td>
<td>-15820</td>
<td>+15820</td>
</tr>
<tr>
<td>2. F.E.M. due to total load</td>
<td>-63540</td>
<td>+63540</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Distribution at A, B, C and D</td>
<td>+46620</td>
<td>-20700</td>
<td>-10350</td>
<td>-11910</td>
</tr>
<tr>
<td>4. Carry over to B and C</td>
<td>+31940</td>
<td>-10350</td>
<td>-11910</td>
<td></td>
</tr>
<tr>
<td>5. Distribution at B and C</td>
<td>-10350</td>
<td>-11910</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Final moments</td>
<td>-38130</td>
<td>+19460</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus the end moments at B and C are -38130 and +19460 respectively. Free B.M. at the centre of span BC

\[
= \text{Dead load intensity} \times \frac{L^2}{8} = \frac{15500(3.5)^2}{8} = 23730
\]

Net B.M. at centre of BC

\[
= 23730 - \frac{38130 + 19460}{8} = -5065
\]

12. Maximum Negative B.M. at Centre of Span AB and CD

Since spans AC and CD are large, free B.M. at their mid-span will be large. It will be seen that the net B.M. at the centre of these spans will either be positive, or will be negative but of negligible small magnitude. Due to this reason, these spans are not being investigated for maximum negative B.M. However, conditions for maximum negative B.M. at the centre of span AB will be when live load is on BC and dead load is on ABCD. Similarly, the loading condition for maximum negative B.M. at centre of CD will be when span BC is loaded with live load, and dead load is on ABCD.

(D) BENDING MOMENTS IN COLUMNS

For maximum B.M. in columns, alternate spans should be loaded with live load, while the whole floor is loaded with dead load. The two possible load conditions are shown in Fig. 27'4. For the present case, the loadings will be as shown in para (13) and (14) below. See Tables 27'11 and 27'12.

Table 27'12
Bending Moment in Columns

<table>
<thead>
<tr>
<th>Joint</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column D.F.</td>
<td>0.263</td>
<td>0.217</td>
<td>0.25</td>
<td>0.3125</td>
</tr>
<tr>
<td>(a) Just above floor</td>
<td>0.263</td>
<td>0.217</td>
<td>0.25</td>
<td>0.3125</td>
</tr>
<tr>
<td>(b) Just below floor</td>
<td>0.263</td>
<td>0.217</td>
<td>0.25</td>
<td>0.3125</td>
</tr>
<tr>
<td>Horizontal Members</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(i) Just above floor</td>
<td>0.263</td>
<td>0.217</td>
<td>0.25</td>
<td>0.3125</td>
</tr>
<tr>
<td>(ii) Just below floor</td>
<td>0.263</td>
<td>0.217</td>
<td>0.25</td>
<td>0.3125</td>
</tr>
</tbody>
</table>

13. Max. B.M. in Columns

The loading conditions are: Live load on AB and CD, and dead load on ABCD. The moment distribution is carried out as illustrated in Table 27'11. In the first cycle, distribution is done at
all the four joints A, B, C and D. The carry over moments are then transferred to the appropriate points. These carried over moments are added to the original F.E.M. to get new moments at each joint. These new moments are distributed to the columns meeting at the joints.


The loading conditions are: Live load on BC and dead load on A B C D. The moment distribution is carried out as illustrated in Table 27.12. In the first cycle, distribution is done at all the four joints. The carry over moments are then transferred to appropriate points. These carried over moments are added to the original F.E.M. to get new moments at each joint. The new moments are distributed to the columns meeting at the joints.

Table 27.12
Bending Moments in Columns

<table>
<thead>
<tr>
<th>Joint</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column D.F.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a) Just above floor</td>
<td>0.263</td>
<td>0.217</td>
<td>0.25</td>
<td>0.3125</td>
</tr>
<tr>
<td>(b) Just below floor</td>
<td>0.263</td>
<td>0.217</td>
<td>0.25</td>
<td>0.3125</td>
</tr>
<tr>
<td>Horizontal Members</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D.F.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. F.E.M. due to D.L.</td>
<td>-69420</td>
<td>+69420</td>
<td>-30210</td>
<td>+30210</td>
</tr>
<tr>
<td>2. F.E.M. due to total load</td>
<td>-32160</td>
<td>+32160</td>
<td>-14600</td>
<td>+14600</td>
</tr>
<tr>
<td>3. Distribution</td>
<td>+32900</td>
<td>-14600</td>
<td>-6480</td>
<td>+400</td>
</tr>
<tr>
<td>4. Carry over</td>
<td>-7300</td>
<td>+16450</td>
<td>-200</td>
<td>-2400</td>
</tr>
<tr>
<td>5. New moments (sum of 1, 2, 4)</td>
<td>-76720</td>
<td>+85870</td>
<td>-32360</td>
<td>+28920</td>
</tr>
<tr>
<td>6. Distribution to columns</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a) Just above floor</td>
<td>+20180</td>
<td>-11610</td>
<td>+1740</td>
<td>-9350</td>
</tr>
<tr>
<td>(b) Just below floor</td>
<td>+20180</td>
<td>-11610</td>
<td>+1740</td>
<td>-9350</td>
</tr>
</tbody>
</table>

Due to this reason, suitable assumptions are made so that the frame subjected to horizontal forces can be analysed by using simple principles of mechanics. Following approximate methods are commonly used for the analysis of building frames subjected to lateral forces:

1. Portal method
2. Cantilever method.
27.6. PORTAL METHOD

For the purposes of analysis, it is assumed that the horizontal forces are acting on the joints. The portal method is based on the following two important assumptions:

(i) the points of contraflexure in all the members lie at their mid-points, and
(ii) the horizontal shear taken by each interior column is double the horizontal shear taken by each of exterior column.

Similarly, consider the second storey, where the exterior columns $A_2-A_3$ and $D_2-D_3$ have shear $Q$. The value of shear $Q$ is found by

\[ P_1 + P_2 = Q + 2Q + 2Q + Q \]

\[ Q = \frac{1}{6} (P_1 + P_2) \]

Similarly, for the bottom storey, the shear $R$ is given by

\[ P_1 + P_2 + P_3 = R + 2R + 2R + R \]

\[ R = \frac{1}{6} (P_1 + P_2 + P_3) \]

Knowing the horizontal shears at the point of contraflexure, the bending moment in the column can be easily found.

Let us consider the floor $A_2B_2C_2D_2$ between third and second storey. The shear acting at the point of contraflexure are as shown in Fig. 27.9. The joint $A_2$ is subjected to clockwise moment of $Ph/2$ at $A_2$ in column $A_2A_3$, and to a clockwise moment equal to $Qh/2$ at $A_2$ in column $A_2A_4$. The beam $A_2B_2$ is thus required to resist a clockwise moment of $m = (P + Q)h/2$ at $A_2$. Similarly, at joint $B_2$ there will be a clockwise moment equal to $(2P + 2Q)h/2$. But there are two beams to resist this. Hence clockwise moment in each beam will be $(P + Q)h/2$. Thus the ends of each beam receive the same clockwise moment of $(P + Q)h/2$, with the result that points of contraflexure will lie in the middle of the beams.

The moment $m$ acting at each end of the beam $A_2B_2$, $B_2C_2$, $C_2D_2$ give rise to vertical reactions in columns. If $L$ is the span of these beams, each beam will impose an upward pull of $\frac{2mL}{L}$ on wind-
ward column and a push of \( \frac{2m}{L} \) on Leeward column connected to the beam, for each span. The vertical reactions will neutralize for any intermediate column, provided span of beams on either side are equal. Only the end columns will experience vertical reactions. The windward column will have an upward pull of \( \frac{2m}{L} \) and the Leeward column will have a downward push of \( \frac{2m}{L} \).

The method of analysis is illustrated in Example 27.2.

**277. CANTILEVER METHOD**

The cantilever method is based on the following assumptions:

(i) Points of contraflexure in each member lies at its mid-span or mid-height.

(ii) The direct stresses (axial stresses) in the columns, due to horizontal forces, are directly proportional to their distance from the centroidal vertical axis of the frame.

![Fig. 27-10 (a)](image)

Fig. 27-10 (a) shows a building frame subjected to horizontal forces. Fig. 27-10 (b) shows the top storey, up to the points of contraflexure of the columns. The reactions at the points of contraflexure will be direct and shear forces only. Let \( V_1, V_2, V_3 \) and \( V_4 \) be the axial forces in the columns \( AE, BF, CG \) and \( DH \), having areas of cross-sections \( a_1, a_2, a_3 \) and \( a_4 \) respectively.

From statics, we have

\[
P = H_1 + H_2 + H_3 + H_4
\]

...(i)

From assumption 2, we have

\[
\frac{V_1}{a_1} = \frac{V_2}{a_2} = \frac{V_3}{a_3} = \frac{V_4}{a_4}
\]

...(ii)

where \( x_1, x_2, x_3 \) and \( x_4 \) are the centroidal distances of the columns from vertical centroidal axis of the frame.

By taking moments about the point of intersection of the vertical centroidal axis and top beam, we get

\[
\frac{H_1 + H_2 + H_3 + H_4}{2} = V_1 x_1 + V_2 x_2 + V_3 x_3 + V_4 x_4
\]

or

\[
V_1 x_1 + V_2 x_2 + V_3 x_3 + V_4 x_4 = \frac{Ph}{2}
\]

...(iii)

From (ii) and (iii), axial forces \( V_1, V_2, V_3 \) and \( V_4 \) can be determined.

![Fig. 27-10 (b)](image)

![Fig. 27-11](image)
In order to determine \( H_1 \), take moments about the point of contraflexure \( M_1 \) in beam \( AB \) [Fig. 27'11 (b)]:

\[
H_1 \cdot \frac{h}{2} = V_1 \cdot \frac{L_1}{2}
\]

\[
\therefore H_1 = \frac{V_1 L_1}{h} \quad \text{(a)}
\]

Similarly, taking moments about point of contraflexure \( M_2 \) in beam \( BC \),

\[
H_1 \cdot \frac{h}{2} + H_2 \cdot \frac{h}{2} = V_1 \left( L_1 + \frac{L_2}{2} \right) + V_2 \cdot \frac{L_2}{2}
\]

\[
\therefore (H_1 + H_2) = \frac{2 \left( V_1 L_1 + (V_1 + V_2) \frac{L_2}{2} \right)}{h} \quad \text{(b)}
\]

Since \( h_1 \) is known from (a), \( H_2 \) can be determined. In a similar manner, \( H_2 \) and \( H_3 \) can be determined.

**Example 27'2.** Analyse the building frame, subjected to horizontal forces, as shown in Fig. 27'12. Use portal method.

**Solution.**

1. **Horizontal shear**

Let the horizontal shears in the exterior columns be \( P \) and in the interior columns be \( 2P \) for the top storey. Similarly, for the bottom storey, let the shears be \( R \) and \( 2R \) for the exterior and interior columns.

For the top storey, we have

\[ P + 2P + 2P + P = 120 \]

For the bottom storey, we have

\[ R + 2R + 2R + R = 120 + 180 \]

\[ \therefore R = \frac{300}{6} = 50 \text{ kN}. \]

2. **Moments at the ends of columns**

For the top storey,

\[ M_{EA} = M_{BA} = M_{DB} = M_{BC} = P \times \frac{h}{2} = 20 \times \frac{3.5}{2} = 35 \text{ kN-m} \]

\[ M_{FB} = M_{BF} = M_{DC} = M_{CD} = 2P \times \frac{h}{2} = 20 \times 3.5 = 70 \text{ kN-m}. \]

For the bottom storey,

\[ M_{HE} = M_{EH} = M_{DL} = M_{HL} = R \times \frac{h}{2} = 50 \times \frac{3.5}{2} = 87.5 \text{ kN-m} \]

\[ M_{HH} = M_{EH} = M_{KG} = M_{Kg} = 2R \times \frac{h}{2} = 50 \times 3.5 = 175 \text{ kN-m}. \]

3. **Moments at the ends of the beams**

**First floor beams**

\[ m_{EB} = M_{EA} + M_{SE} = 35 + 87.5 = 122.5 \text{ kN-m} \]

Similarly, \( m_{EF} = M_{FG} = M_{BH} = 122.5 \),

since the point of contraflexure lies at the middle of each span.

In general, \( m = (P + R) \times \frac{h}{2} = (20 + 50) \times \frac{3.5}{2} = 122.5 \)

**Roof beams**

\[ m_{AB} = m_{BA} = m_{BC} = m_{CB} = m_{CD} = m_{DC} = P \times \frac{h}{2} \]

\[ = 20 \times \frac{3.5}{2} = 35 \text{ kN-m}. \]

4. **Shear in beams**

Since no external vertical force is acting on the beam, shear \( F \) is given by

\[ F = \frac{m_1 + m_2}{L} \]

where \( m_1 \) and \( m_2 \) are the moments at ends of the beam of span \( L \).

Thus,

\[ F_E = \frac{122.5 + 122.5}{7} = 35 \text{ kN} \uparrow \]

\[ F_K = 35 \text{ kN} \downarrow \]
STRENGTH OF MATERIALS AND THEORY OF STRUCTURES

\[ F_{BG} = F_{GB} = \frac{122.5 + 122.5}{3.5} = 70 \text{ kN} \]
\[ F_{OH} = F_{HO} = \frac{122.5 + 122.5}{5} = 49 \text{ kN} \]
\[ F_{AB} = F_{BA} = \frac{35 + 35}{7} = 10 \text{ kN} \]
\[ F_{BC} = F_{CB} = \frac{35 + 35}{3.5} = 20 \text{ kN} \]
\[ F_{CD} = F_{DC} = \frac{35 + 35}{5} = 14 \text{ kN} \]

5. Axial force in columns

The axial forces in the columns will be as under:

* Column $AE$ = shear in beam $AB = 10 \text{ kN}$↑

* Column $EF$ = axial force in $AE$ + shear in $EF$
  \[= 10 + 35 = 45 \text{ kN} \]

* Column $DH$ = shear in beam $DC = 14 \text{ kN}$↓

* Column $HL$ = axial force in $DH$ + shear in $HG$
  \[= 14 + 49 = 63 \text{ kN} \]

Since the spans are not equal, interior columns will also have axial forces.

* Column $BF = F_{BA} - F_{BC} = 10 - 20 = -10 \text{ kN}$ (i.e. ↑)

* Column $FJ = (-10) + (F_{BE} - F_{EG})$
  \[= (-10) + (35 - 70) = -45 \text{ kN} \]

Alternatively, axial force in $BF$

\[ = \frac{2m'}{L_1} - \frac{2m'}{L_2} = \frac{2 \times 35}{7} - \frac{2 \times 35}{3.5} = -10 \text{ kN} \]

and axial force in column

\[ F_{J} = (-1) + \left( \frac{2m'}{L_1} - \frac{2m'}{L_2} \right) \]
\[ = (-10) + \left( \frac{2 \times 122.5}{7} - \frac{2 \times 122.5}{3.5} \right) \]
\[ = -45 \text{ kN} \]

Axial force in $CG = \frac{3m'}{L_2} - \frac{2m'}{L_3}$
\[ = \frac{2 \times 35}{3.5} - \frac{2 \times 35}{5} = 6 (↑) \]

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Axial force in column $GK$

\[ = 6 + \left( \frac{2m}{L_3} - \frac{2m}{L_4} \right) \]
\[ = 6 + \left( \frac{2 \times 122.5}{3.5} - \frac{2 \times 122.5}{5} \right) = 21 \]

Check: Total axial force at the base

\[ = -45 (↑) - 45 (↑) + 27 (↑) + 63 (↑) \]
\[ = 0 \]

Example 27.3. Re-analyse the frame of example 27.2 by cantilever method, assuming that all the columns have the same area of cross-section.

![Diagram](image)

Solution.

1. Location of centroidal axis of the columns

Let the centroidal axis be at a distance $x$ from the windward column $AEF$. Taking moment of areas of the columns about $AEF$, we get

\[ x = \frac{(2 \times 0) + (2 \times 7) + (2 \times 10.5) + (2 \times 15.5)}{8} = 8.25 \text{ m} \]

\[ x_1 = 8.25 \text{ m} \]

\[ x_2 = 8.25 - 7 = 1.25 \text{ m} \]

\[ x_3 = 3.5 - 1.25 = 2.25 \text{ m} \]

\[ x_4 = 7 + 3.5 + 5 = 8.25 \text{ m} \]

2. Axial forces in columns of first storey

Let the axial force in column $EF = V_1 = V$.

Since the areas are equal, the axial forces in other columns will be in proportion to their distances from the centroidal axis.
Since there is point of contraflexure at the middle of column, \( AE, \)
\[
M_{EA} = 47.6 \text{ kN-m}
\]
\[
M_{BF} = M_{BA} + M_{BC} = 47.6 + 27.4 = 75 \text{ kN-m}
\]
\[
M_{FB} = 75 \text{ kN-m}
\]
\[
M_{CO} = 27.4 + 29.9 = 57.3
\]
\[
M_{OC} = 57.3
\]
\[
M_{DH} = M_{DC} = 29.9 \text{ kN-m}
\]
\[
M_{HD} = 29.9 \text{ kN-m}
\]
(b) Bottom storey
\[
M_{EI} + M_{EA} = M_{EF}
\]
\[
M_{EI} = M_{EF} - M_{EA} = 166.8 - 47.6 = 119.2 \text{ kN-m}
\]
\[
M_{Br} = 119.2 \text{ kN-m}
\]
\[
M_{Fe} + M_{Fn} = M_{Fe} + M_{Go}
\]
\[
M_{Ft} = 166.8 + 96 - 75 = 187.8 \text{ kN-m}
\]
Hence
\[
M_{Br} = 187.8 \text{ kN-m}
\]
\[
M_{Gr} = 143.2 \text{ kN-m}
\]
\[
M_{El} + M_{Dh} = M_{Oh}
\]
\[
M_{El} = 104.7 - 29.9 = 74.8
\]
\[
M_{Lh} = 74.8
\]
Alternatively, the moment at the column ends can be found by first determining horizontal shears \((H)\) at the point of contraflexure and multiplying there by half the height of the column.

Thus,
\[
M_{AE} = H_1' \times \frac{h}{2} ; M_{BF} = H_1' \times \frac{h}{2} \text { etc.}
\]
Similarly,
\[
M_{EI} = H_4' \times \frac{h}{2} ; M_{Br} = H_4' \times \frac{h}{2} \text { etc.}
\]

The method of determining horizontal shears have been explained in § 27.7.

For example,
\[\frac{V_1' L_4}{h} \times \frac{7}{35} = 27.23\]
\[
H_1' = \frac{2 \left( V_1' L_4 + (V_4' + V_3') \right) L_4}{h} - H_1'
\]

The factor method is more accurate than either the portal method or the cantilever method, and is more useful when the moments of inertia of various members (of columns and beams) are different. Both cantilever method as well as portal method assume uniform moments of inertia of members. These methods, therefore, depend on some stress assumptions, thus limiting the analysis to be based on equations of statics only. The factor method is based on assumptions regarding the elastic action of the structure. For analysis by factor method the relative stiffness \( K' (=h/L) \) for each member of the frame or structure should be known. The procedure consists of the following steps:

1. Calculate the girder factor \( g' \) for each joint from the following expression
\[
g = \frac{\sum K_g}{\sum K} \quad \text{...(27.1)}
\]
where \( \sum K_g \) = Sum of relative stiffnesses of all column members at the joint considered
\( \sum K \) = Sum of relative stiffnesses of all the members at the joint considered.

These values of girder factor \( g' \) are entered in a tabular form as shown in Table 27.13. The values are entered at the end of each girder meeting at that joint.

2. Calculate the column factor \( c' \) for all joints from the following expression:
\[
c = 1 - g \quad \text{...(27.2)}
\]
where \( g' \) = girder factor of the joint.

The values of column factors are entered in Table 27.13 at the end of each column at the joint.

For columns which are fixed at the base, the column factor \( c' \) is taken as 1.00.
Fig. 27'16 shows a simple frame with two storeys and two bays, used for illustration purpose. The relative stiffnesses \( k_1, k_2, k_3 \ldots \ldots k_{10} \) of all the ten members are entered on/near each member. Table 27'13 is used for computation of column factor (c), girder factor (g), column moment factor (C) and girder moment factor (G).

![Diagram of a simple frame](image)

3. As shown in Table 27'13, in the first column of the table, the name of all the joints are entered. The 2nd column contains all the members at each joint. In 3rd column, the corresponding girder or column factors are entered against each girder or column. In column 4 of the table, half the values of the column factor/girder factor of opposite end of the members are entered. For example, if \( c_1 = \) column factor of member DG, it is entered in column 3 opposite DG, while column factor \( c_2 \) of member GD is entered in column 3 opposite member GD. Hence in column 4, half the column factor of opposite end, i.e. \( c_1/2 \) is entered opposite member DG. Similarly, for member GD, a value of \( c_1/2 \) is entered opposite it for the same reason. So in this way column no. 4 is entered. The values in column no. (3) and (4) of Table 27'13 for each member are added and entered in column no. 5. In column no. 6, the relative stiffness values \( K = I/L \) for each member is entered.

4. The sum of columns (3) and (4), which is entered in column no. '5' is multiplied by relative stiffness of respective members (which are entered in column no. 6). This product is termed as column moment factor 'C' for columns and girder moment factor 'G' for girders. This is entered in column no. '7' of Table 27'13.
The column moment factor 'C', gives the relative values of moments at the ends of columns for each storey in which the column occurs. The sum of column end moments is equal to the horizontal shear on that storey multiplied by the storey height. Hence the column moment factors (C) are converted into end moments for columns by direct proportion for each storey.

Similarly, the girder moment factor G, gives the relative values of moment at ends of each girder for the joint. The sum of girder end moment at each joint is equal to the sum of end moments in the columns at the joint. Hence the girder moment factors are converted into end moments for girders by direct proportion for each storey.

(5) Calculation of column moments:

(a) Total column moments (\(A\)) for each storey is found by the relation

\[
A = \frac{H \times h}{\Sigma C}
\]

...(27.3)

where \(A\)=Total column moment for each storey

\(H\)=Total horizontal force above the storey considered

\(h\)=height of the storey considered

\(\Sigma C\)=Sum of column end moment factors of that storey

Thus, for each storey, different column moments \(A_1, A_2, \ldots\) etc. are calculated.

(b) The column moment factor 'C' of each member is multiplied by the total column moment (\(A\)) of that storey in which the column occurs. For example in Fig. 27.16 if we want to find the column moment \(M_{GD}\) of column GD, we have

\[
M_{GD} = A_1 \times C_{GD}
\]

...(27.4)

where \(A_1\)=Total column moments of first storey (Eq. 27.3) and

\(C_{GD}\)=column moment factor for column GD.

Similarly, moment \(M_{HE}\) in column HE of first storey is

\[
M_{HE} = A_1 \times C_{HE}
\]

Also for column DA of ground storey,

\[
M_{DA} = A_0 \times C_{DA}
\]

where \(A_0\)=total column moments of ground storey.

(6) Calculations of Girder/beam moments

(a) For calculation of girder/beam moments, a constant 'B' is found for each joint.

\[
B = \frac{\text{Sum of column moments at the joint}}{\text{Sum of the girder moment factors at that joint}}
\]

(b) This constant 'B' is multiplied by the girder moment factor (G) to obtain the girder moments.

For example (Fig. 27.16),

\[
M_{DB} = B_0 \times G_{DB}
\]

\[
M_{SH} = B_h \times G_{SH}
\]

\[
M_{H} = B_i \times G_{HI}
\]

and so on.

Here, \(B_0, B_h, B_i\) are the constants for joints D, H and I respectively.

The factor method of analysing the building frame has been illustrated in Example 27.4.

Example 27.4. Analyse the frame as shown in 27.17 by factor method. Sketch the B.M.D. The relative 'K' value are written on the members.

---

**Building Frames**

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Fig. 27.17
Step 1. Calculate the girder factor \( g \) at all joints by Eq. 27'1.

\[
g = \frac{\sum K_C}{\sum K}
\]

where \( \sum K \) = Sum of relative Stiffness of columns at that joint

\( \sum K \) = Sum of relative Stiffness of all the members at that joint

Joint Q \( g_Q = \frac{K_{QM} + K_{QR}}{K_{QR}} = \frac{2}{2+2} = 0.5 \)

Joint R \( g_R = \frac{K_{RV} + K_{RS} + K_{RS}}{2+1+1} = 0.4 \)

Joint S \( g_S = \frac{K_{SO} + K_{SR} + K_{ST}}{2+1+2} = 0.4 \)

Joint T \( g_T = \frac{K_{TP}}{K_{TP} + K_{TS}} = 2+2 = 0.5 \)

Joint M \( g_M = \frac{K_{MM} + K_{MQ}}{K_{MM} + K_{MQ} + K_{MN}} = \frac{2+2}{2+2+3} = 0.57 \)

Joint N \( g_N = \frac{K_{NM} + K_{NR}}{K_{NH} + K_{NR} + K_{NM} + K_{NO}} = \frac{2+2}{2+2+3+2} = 0.44 \)

Joint O \( g_O = \frac{K_{OK} + K_{OS}}{K_{OK} + K_{OS} + K_{ON} + K_{OT}} = \frac{2+2}{2+2+2+2} = 0.44 \)

Joint P \( g_P = \frac{K_{PL} + K_{PT} + K_{PS}}{2+2+3} = 0.57 \)

Joint I \( g_I = \frac{K_{IE} + K_{IM}}{K_{IE} + K_{IM} + K_{IL}} = \frac{2+2}{2+2+3} = 0.57 \)

Joint J \( g_J = \frac{K_{IF} + K_{IN}}{K_{IF} + K_{IN} + K_{II} + K_{Jk}} = \frac{2+2}{2+2+3+2} = 0.57 \)

Joint K \( g_K = \frac{K_{KO} + K_{KG}}{K_{KO} + K_{KG} + K_{KL}} = \frac{2+2}{2+2+3} = 0.57 \)

Joint L \( g_L = \frac{K_{LM} + K_{LP}}{K_{LM} + K_{LP} + K_{KL}} = \frac{2+2}{2+2+3} = 0.57 \)

Joint E \( g_E = \frac{K_{EA} + K_{EI} + K_{EF}}{3+2+3} = 0.57 \)

Joint F \( g_F = \frac{K_{FB} + K_{FS} + K_{FS}}{2+2+2} = 0.57 \)

Joint G \( g_G = \frac{K_{GC} + K_{GK} + K_{GF} + K_{GK}}{2+2+3+2} = 0.44 \)

Joint H \( g_H = \frac{K_{HD} + K_{HL}}{3+2+3} = 0.57 \)

These values of girder factors are written at the ends of girders beams meeting at each joint as shown in col. 3, Table 27'14.

Step 2. Calculate the column factor \( c \) at the joints by the relation,

\[
c = 1 - g = \text{girder factor at the joint}
\]

Joint Q \( c_Q = 1 - g_Q = 1 - 0.5 = 0.5 \)

Joint R \( c_R = 1 - g_R = 1 - 0.4 = 0.6 \)

Joint S \( c_S = 1 - g_S = 1 - 0.4 = 0.6 \)

Joint T \( c_T = 1 - g_T = 1 - 0.5 = 0.5 \)

Joint M \( c_M = 1 - g_M = 1 - 0.57 = 0.43 \)

Joint N \( c_N = 1 - g_N = 1 - 0.44 = 0.56 \)

Joint O \( c_O = 1 - g_O = 1 - 0.44 = 0.56 \)

Joint P \( c_P = 1 - g_P = 1 - 0.57 = 0.43 \)

Joint I \( c_I = 1 - g_I = 1 - 0.57 = 0.43 \)

Joint J \( c_J = 1 - g_J = 1 - 0.57 = 0.43 \)

Joint K \( c_K = 1 - g_K = 1 - 0.57 = 0.43 \)

Joint L \( c_L = 1 - g_L = 1 - 0.57 = 0.43 \)

Joint E \( c_E = 1 - g_E = 1 - 0.57 = 0.43 \)

Joint F \( c_F = 1 - g_F = 1 - 0.44 = 0.56 \)

Joint G \( c_G = 1 - g_G = 1 - 0.44 = 0.56 \)

Joint H \( c_H = 1 - g_H = 1 - 0.57 = 0.43 \)

Joint A \( c_A = 100 \)

Joint B \( c_B = 100 \) For columns fixed at the base, column

Joint C \( c_C = 100 \) factor is taken as 100

Joint D \( c_D = 100 \)

These values of column factors are written at the end of columns meeting at the joint, and have been entered in column 3 of Table 27'14.

Step 3. Half the values of column/girder factors of the opposite ends are entered in column 4.

The values in col. 3 and col. 4 are added and entered in col. '5' of Table 27'14.

Step 4. Enter the values of relative stiffnesses of all members in col. 6. These values of stiffnesses are multiplied by the values of column 5, to get the values of girders/moment factors (G or C), and are entered in col. 7 of Table 27'14.
### Table 27.14

<table>
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<tr>
<th>Joints</th>
<th>Members</th>
<th>Girder/Column factor (c or g)</th>
<th>Half values of the factors from opposite end</th>
<th>(3)+(4)</th>
<th>Relative stiffness ((K=I/L))</th>
<th>Girder Column Moment ((C,or,G)) factor (=(3)\times(6))</th>
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### Building Frames

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</tr>
</tbody>
</table>
### Building Frame

Thus Table 2714 is completed.

**Step 5. Calculation of Column Moments**

Total column moments for each storey

\[ A = \frac{H \times h}{\Sigma C} \]

where \( H \)=Total horizontal force above the storey considered

\( h \)=height of the storey considered

\( \Sigma C \)=sum of column end moment factors of that storey.

Let

- \( A_s \)=Total column moments of third/top storey
- \( A_2 \)=Total column moments of second storey
- \( A_1 \)=Total column moments of first storey
- \( A_0 \)=Total column moments of ground storey.

\[ A_s = \frac{5 \times 3}{1'42 + 1'36 + 1'76 + 1'72 + 1'76 + 1'72 + 1'42 + 13'6} \]

\[ = 1'2 \text{kN-m} \]

\[ A_2 = \frac{(10 + 5) \times 3}{C_{OM} + C_{MQ} + C_{MR} + C_{MT} + C_{TP} + C_{TT}} \]

\[ = 1'28 + 1'78 + 1'68 + 1'68 + 1'68 + 1'68 + 1'28 + 1'28 \]

\[ = 3'80 \text{kN-m} \]

\[ A_1 = \frac{(10 + 10 + 5) \times 3}{C_{TL} + C_{ML} + C_{TH} + C_{MT} + C_{MR} + C_{MT} + C_{MR} + C_{MR}} \]

\[ = 1'22 + 1'16 + 1'68 + 1'68 + 1'68 + 1'68 + 1'42 + 1'16 \]

\[ = 6'22 \text{kN-m} \]

\[ A_0 = \frac{(10 + 10 + 10 + 5) \times 4}{C_{LE} + C_{AE} + C_{FR} + C_{ER} + C_{TG} + C_{CG} + C_{CG} + C_{CG} + C_{DR}} \]

\[ = 2'61 + 3'54 + 2'12 + 2'56 + 2'12 + 2'56 + 2'61 + 3'54 \]

\[ = 6'46 \text{kN-m} \]

**Column Moments:**

**TOP STOREY:** \( A_0 = 1'2 \text{kN-m} \)

\[ M_{OM} = A_0 \times C_{OM} = 1'2 \times 1'42 = 1'704 \text{kN-m} \]

\[ M_{MQ} = A_0 \times C_{MQ} = 1'2 \times 1'36 = 1'632 \text{kN-m} \]

\[ M_{MN} = A_0 \times C_{MN} = 1'2 \times 1'76 = 2'112 \text{kN-m} \]

\[ M_{NR} = A_0 \times C_{NR} = 1'2 \times 1'72 = 2'064 \text{kN-m} \]

---

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STRENGTH OF MATERIALS AND THEORY OF STRUCTURES

Second Storey: $A_2 = 3.80 \text{kN-m}$

$M_{Si} = 3.80 \times 1.28 = 4.864 \text{kN-m}$
$M_{SM} = 3.80 \times 1.28 = 4.864 \text{kN-m}$
$M_{SN} = 3.80 \times 1.68 = 6.384 \text{kN-m}$
$M_{SK} = 3.80 \times 1.68 = 6.384 \text{kN-m}$
$M_{SP} = 3.80 \times 1.28 = 4.864 \text{kN-m}$
$M_{SL} = 3.80 \times 1.28 = 4.864 \text{kN-m}$

First Storey: $A_1 = 6.422 \text{kN-m}$

$M_{SI} = 6.422 \times 1.22 = 7.834 \text{kN-m}$
$M_{SL} = 6.422 \times 1.16 = 7.449 \text{kN-m}$
$M_{SF} = 6.422 \times 1.68 = 10.789 \text{kN-m}$
$M_{FJ} = 6.422 \times 1.68 = 10.789 \text{kN-m}$
$M_{SK} = 6.422 \times 1.68 = 10.789 \text{kN-m}$
$M_{SH} = 6.422 \times 1.22 = 7.834 \text{kN-m}$
$M_{SL} = 6.422 \times 1.16 = 7.449 \text{kN-m}$

Ground Storey: $A_2 = 6.464 \text{kN-m}$

$M_{SE} = 6.464 \times 2.61 = 16.871 \text{kN-m}$
$M_{AE} = 6.464 \times 3.54 = 22.882 \text{kN-m}$
$M_{PB} = 6.464 \times 2.12 = 13.703 \text{kN-m}$
$M_{BF} = 6.464 \times 2.56 = 16.547 \text{kN-m}$
$M_{GC} = 6.464 \times 2.12 = 13.703 \text{kN-m}$
$M_{CG} = 6.464 \times 2.56 = 16.547 \text{kN-m}$
$M_{HD} = 6.464 \times 2.61 = 16.871 \text{kN-m}$
$M_{DH} = 6.464 \times 3.54 = 22.882 \text{kN-m}$

Step 6. Calculation of beam moments

(a) constant $'B' = \frac{\text{Sum of column moments at the joint}}{\text{Sum of girder moment factors at that joint}}$

Joint Q: $B_0 = \frac{M_{SO}}{G_{QR}} = \frac{1.704}{1} = 1.704$

Joint R: $B_0 = \frac{M_{SK}}{G_{QR} + G_{RS}} = \frac{2.112}{1.30 + 0.60} = 1.111$

(b) BEAM MOMENTS

Beam moments = $B \times $Girder Moment Factor.

$M_{QR} = B_0 \times G_{QR} = 1.217 \times 1.4 = 1.704 \text{kN-m}$
$M_{RB} = B_0 \times G_{RB} = 1.111 \times 1.30 = 1.444 \text{kN-m}$
$M_{MS} = B_0 \times G_{MS} = 1.111 \times 0.60 = 0.667 \text{kN-m}$
$M_{SR} = B_0 \times G_{SR} = 1.111 \times 0.60 = 0.667 \text{kN-m}$
$M_{SF} = B_0 \times G_{SF} = 1.111 \times 1.30 = 1.444 \text{kN-m}$
$M_{TS} = B_0 \times G_{TS} = 1.217 \times 1.4 = 1.704 \text{kN-m}$
$M_{MN} = B_0 \times G_{MN} = 2.741 \times 2.37 = 6.496 \text{kN-m}$
$M_{NS} = B_0 \times G_{NS} = 2.427 \times 2.16 = 5.242 \text{kN-m}$
$M_{SO} = B_0 \times G_{SO} = 2.427 \times 1.32 = 3.203 \text{kN-m}$
$M_{AO} = B_0 \times G_{AO} = 2.427 \times 1.32 = 3.203 \text{kN-m}$
Thus the moments in all the columns and girder are found. Fig. 27.18 shows the B.M. diagram for girder/beams while Fig. 27.19 shows the B.M. diagram for columns.

PROBLEMS

1. What do you understand by a substitute frame? How do you select it? Discuss in brief the method of analysis.

2. Explain the portal method for analysing a building frame subjected to horizontal forces.

3. Explain the cantilever method for analysing a building frame subjected to horizontal forces.

4. A two-span intermediate frame of a multi-storeyed building is shown in Fig. 27.20. The frames are spaced at 5 m intervals. The dead load and live load per metre run of the beam may be taken as 15 kN/m and 20 kN/m respectively. Analyse the frame using two cycle method of moment distribution.
5. If wind loads of 15 kN, 30 kN and 30 kN are acting at joint A, B and C respectively, analyse the frame. (Fig. 27-20) by (a) portal method, (b) cantilever method. Assume that all the columns have equal area of cross-section for the purpose of analysis.

28.1. INTRODUCTION

In Chapter 9, we have discussed, the slope deflection method presented by G.A. Maney (1915), wherein the rotations and displacements of the joints are treated as unknowns. The simultaneous solution of the various slope deflection equations, along with the equilibrium equations gives the values of these unknowns. However, in many cases, especially in the cases of frames, the solutions become clumsy, though the solutions of these sets of linear simultaneous equations can be obtained by methods such as relaxation technique, Gauss-Seidell iteration, Cramer’s rule etc. In chapter 10, we have discussed the ‘moment distribution method’, given by Prof. Hardy Cross (1930). It should be noted that Hardy Cross method of moment distribution is a technique of solving the above-mentioned equations numerically, without explicitly writing them down, by using Gauss Seidell iteration. The quantities iterated in the ‘moment distribution method’ are the increments to the member end moments, instead of end-moment themselves.

We now introduce Kani’s method, given by Dr. Gasper Kani (1947). The Kani’s method is similar to the moment distribution method in that both these methods use Gauss-Seidell iteration procedure to solve the slope deflection equations, without explicitly writing them down. However, the difference between the Kani’s method and the moment distribution method is that Kani’s method iterates the member end moments themselves rather than iterating their increments. Kani’s method essentially consists of a simple numerical operation, performed repeatedly at the joints of a structure, in a chosen sequence.

Let us now develop the method for the following two cases: (i) Continuous beams and Frames without joint translation. (ii) Frames with sway.
28.2. CONTINUOUS BEAMS AND FRAMES WITHOUT JOINT TRANSLATION

Fig. 28.1 shows the final deflected shape of a span \( AB \) of a continuous beam, under imposed loads. Let \( M_{AB} \) and \( M_{BA} \) be the moments developed, and \( \theta_A \) and \( \theta_B \) be the corresponding rotations at \( A \) and \( B \), due to imposed loads. The slope deflection equation for span \( AB \), at joint \( A \) is:

\[
M_A = M_{FAB} + 2EK_{AB} (\theta_A + \theta_B) \quad \text{(28.1)}
\]

where

\[
K_{AB} = \frac{L_{AB}}{L}
\]

\[
M_A = M_{FAB} + 4EK_{AB} \theta_A + 2EK_{AB} \theta_B
\]

or

\[
M_A = M_{FAB} + 2m_{AB} + m_{BA} \quad \text{(28.2)}
\]

where, by definition, 

\[
m_{AB} = 2EK_{AB} \theta_A
\]

and

\[
m_{BA} = 2EK_{AB} \theta_B
\]

Here, \( m_{AB} \) is called the rotational contribution of end \( A \) to \( M_{AB} \), while \( m_{BA} \) is called the rotational contribution of end \( B \) to \( M_{AB} \).

Now, in general, there may be many members meeting at end \( A \), so that \( B \) is the common designation for far ends. For the equilibrium at the joint \( A \), the algebraic sum of end moments of all the members meeting there must be zero.

Hence

\[
\sum_M = 0
\]

where the symbol \( \sum \) represents the sum taken over all the adjacent joints \( B \).

Kani's Method

Now let \( M_{FA} \) be the resultant restraint moment at \( A \), equal to the algebraic sum of all fixed end moments at \( A \), given by

\[
M_{FA} = \sum M_{FAB} \quad \text{(28.5)}
\]

Hence we get from Eq. 28.4.

\[
\sum m_{AB} = -\frac{1}{2} (M_{FA} + \sum m_{BA}) \quad \text{(28.6)}
\]

In fact, Eq. 28.6 gives the algebraic sum of rotational contribution of all members meeting at \( A \). Now, we know that \( m_{AB} \) for any member is proportional to its \( K \) value. The individual share of the members in the total rotational contribution can be found by distributing the total rotational contribution in proportion to their respective \( K \) values.

Thus, for member \( AB \),

\[
m_{AB} = \frac{K_{AB}}{\sum K_{AB}} \sum m_{AB} \quad \text{(28.7)}
\]

or

\[
m_{AB} = -\frac{1}{2} \left( \frac{K_{AB}}{\sum K_{AB}} \right) (M_{FA} + \sum m_{BA}) \quad \text{(28.8)}
\]

Where \( R_{AB} \) is known as rotational factor for \( AB \) given by

\[
R_{AB} = -\frac{1}{2} \left( \frac{K_{AB}}{\sum K_{AB}} \right) \quad \text{(28.9)}
\]

It is to be noted that the rotational factor \( R \) is equal to \(-\frac{1}{2}\) times the distribution factors used in moment distribution.

Eqs (I) and (II) form the basis of Kani's method of solution wherein the support moment \( M_{AB} \) can be determined. Obviously, both \( m_{AB} \) and \( m_{BA} \) must be determined before \( M_{AB} \) can be found. This is accomplished by Gauss-Seidel iteration procedure outlined below. However, sometimes it is preferable to use symbol \( i \) for near end \( A \) and \( j \) for the far ends \( B \). In that case, the various equations take the form listed below:

\[
M_i = M_{Fi} + 2E K_i (\theta_i + \theta_j) \quad \text{(28.1)}
\]

where \( K_i = L_i / L \)

\[
M_j = M_{Fj} + 2m_{ij} + m_{ji} \quad \text{(28.2)}
\]

where

\[
\sum M_{Fi} + \sum_i (2m_{ij} + m_{ji}) = 0 \quad \text{(28.4)}
\]
STRENGTH OF MATERIALS AND THEORY OF STRUCTURES

\[ M_R = \sum M_{R_j} \quad \text{(28.5)} \]

\[ \sum m_i = -\frac{1}{2} \left( M_R + \sum m_j \right) \quad \text{(28.6)} \]

\[ m_i = R_y \left( M_{R_i} + \sum m_j \right) \quad \text{(II)} \text{(28.8)} \]

where

\[ R_y = -\frac{1}{2} \left( \frac{K_y}{\sum K_j} \right) \quad \text{(28.9)} \]

Procedure for Kani's method

Step 1. Calculate fixed end moments \((M_{Fj})\) in all the members of the structure. Find the resultant restraint moment at each joint using Eq. 28.9. Enter these values of resultant restraint moment within the square or circle made at each joint (Fig. 28.2).

Step 2. Calculate the \(K\) values and rotation factors \(R_y\) for all members meeting at each joint. These values are entered outside the first square (or circle) but inside the second square (or circle), towards each member as shown in Fig. 28.2.

Step 3. Compute rotational contribution \((m_{\theta_j})\) of the two ends of all the members by Gauss-Seidell iteration performed on Eq. 28.8 (Eq.11), taking \(m_\theta = 0\) at all joints starting with the approximation that \(\theta = 0\). Continue the iteration through several cycles till practically the same values of \(m_\theta\) are obtained in two successive cycles. Each cycle gives improved approximation for the rotational contribution. All these values of \(m_\theta\) are entered as shown in Fig. 28.2.

KANI'S METHOD

Step 4. Using Eq. 28.2 (Eq.1), determine member end moments \(M_{ij}\).

Example 28.1 illustrates the complete procedure.

Members with far ends hinged

If the end \(j\) of the member \(ij\) is hinged, then its modified stiffness coefficient is \(3E K_i\) as compared to the normal value of \(4E K_i\). Hence for calculating the factor \(R\) in Eq. 28.9, 0.75 times its actual \(K\) value should be used. Eq. 28.8 can then be used without any modification. The member end moment at the hinged end will be zero. However, half the initial F.E.M. at the joint \(j\) will have to be carried to joint \(i\), with negative sign, and added to the F.E.M. of \(i\), to get the modified value of F.E.M. at \(i\). Thus, if \(M_{Fj}\) and \(M_{Fj}\) are the F.E.M.'s at \(i\) and \(j\), computed by taking \(j\) to be fixed, then the modified fixed end moment at \(i\) will be as follows:

Modified \(M_{Fj} = M_{Fj} + \frac{1}{2} (-M_{Fj})\)

This modified fixed end moment at joint \(i\) becomes the starting F.E.M. at \(i\) before beginning the Kani's cycles.

Member with far end fixed: It should be noted that no iteration is performed at the fixed end. The rotational components of fixed joints at far ends are zero.

Example 28.1. A continuous beam ABCD consists of three spans, and is loaded as shown in Fig. 28.3(a). Ends A and D are fixed. Determine the bending moments at the supports, using Kani's method. Also, plot the bending moment diagram and the deflected shape of the beam.

Solution.

Step 1. Computation of fixed end moments (kN-m units)

\[ M_{FAB} = -\frac{2 \times 6^2}{12} = -6.0; \quad M_{FBA} = +\frac{2 \times 6^2}{12} = +6 \]

\[ M_{FBC} = -\frac{5 \times 3 \times 2^2}{8^2} = -24; \quad M_{FBC} = +\frac{5 \times 2 \times 3^2}{8^2} = +36 \]

\[ M_{FCD} = -\frac{8 \times 5}{8} = -5.0; \quad M_{FDC} = +\frac{8 \times 5}{8} = +5.0 \]

Step 2. Rotation factors

As pointed out earlier, rotation factor is equal to \(-0.5\) times the distribution factor used in moment distribution. The relative stiffness, distribution factors and rotation factors are calculated in Table 28.1.
Table 28.1.

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<th>Member</th>
<th>Relative</th>
<th>Sum</th>
<th>Distribution</th>
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<td>$5/17$</td>
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<tr>
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<td>BC</td>
<td>$2/5$</td>
<td></td>
<td>$12/17$</td>
<td>-0.353</td>
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<tr>
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<tr>
<td></td>
<td>CD</td>
<td>$1/5$</td>
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<td></td>
<td>-0.167</td>
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</table>

It is to be noted that the sum of rotation factors at a joint is equal to $-0.5$.

Thus $R_{BA} + R_{BC} = -(0.147 + 0.353) = -0.5$

Step 3. Resultant restraint moments

Compute resultant restraint moment at each joint by Eq. 28.5:

$$M_{fi} = \sum M_{fi}$$

Thus

$$M_{FB} = M_{FBA} + M_{FBC} = 6 - 2.4 = 3.6 \text{ kN-m}$$
$$M_{FC} = M_{FCB} + M_{FCD} = 3.6 - 5 = -1.4 \text{ kN-m}$$

Enter these values within the small square [(Fig. 28.3 (d)]

Step 4. Kani's Iteration cycles

Cycle 1: Kani's iteration procedure can now be commenced, assuming all rotational components ($m_i$) to be zero at all joints which will indirectly mean that $\theta_i = 0$. Note that $m_{AB}$ and $m_{BC}$ are permanently zero since ends A and D are fixed.

Applying Eq. 28.8 at joint C and assuming $m_{BC} = 0$,

we get

$$m_{CB} = R_{CB} (M_{FC}) = -0.333 (-1.4) = +0.466$$
and
$$m_{CD} = R_{CB} (M_{FC}) = -0.167 (-1.4) = +0.234$$

These values are now used in Eq. 28.8 for computing rotational components ($m_{ij}$) at joint B. Thus

$$m_{BC} = R_{BC} (M_{FB} + m_{CB}) = -0.353 (+3.6+0.466) = -1.435$$
$$m_{BA} = R_{BA} (M_{FB} + m_{CB}) = -0.147 (+3.6+0.466) = -0.598$$

Cycle 2: The values of the four rotational components found in cycle 1 will now be used to get better approximations for the rotational components at joint C. Applying Eq. 28.8 at joint C,

$$m_{CB} = R_{CB} (M_{FC} + m_{BC}) = -0.333 (-1.4 -1.435) = +0.944$$
$$m_{CD} = R_{CD} (M_{FC} + m_{BC}) = -0.167 (-1.4 -1.435) = +0.473$$

Similarly, applying Eq. 28.8 at joint B,

$$m_{BC} = R_{BC} (M_{FB} + m_{CB}) = -0.353 (+3.6+0.944) = -1.604$$
$$m_{BA} = R_{BA} (M_{FB} + m_{CB}) = -0.147 (+3.6+0.944) = -0.668$$

Cycle 3: Applying Eq. 28.8 at joint C,
\( m_{CB} = -0.333 \times (-1.4 - 1.604) = +1.000 \)
\( m_{CD} = -0.167 \times (-1.4 - 1.604) = +0.502 \)

At joint B, \( m_{BC} = -0.353 \times (+3.6 + 1.000) = -1.624 \)
\( m_{BA} = -0.147 \times (+3.6 + 1.000) = -0.676 \)

**Cycle 4.**

At C, \( m_{CB} = -0.333 \times (-1.4 - 1.624) = +1.007 \)
\( m_{CD} = -0.167 \times (-1.4 - 1.624) = +0.505 \)

At B, \( m_{BC} = -0.353 \times (+3.6 + 1.007) = -1.626 \)
\( m_{BA} = -0.147 \times (+3.6 + 1.007) = -0.677 \)

**Cycle 5.**

At C, \( m_{CB} = -0.333 \times (-1.4 - 1.626) = +1.008 \)
\( m_{CD} = -0.167 \times (-1.4 - 1.626) = +0.505 \)

At B, \( m_{BC} = -0.353 \times (+3.6 + 1.008) = -1.627 \)
\( m_{BA} = -0.147 \times (+3.6 + 1.008) = -0.677 \)

The iteration is terminated at the end of 5th cycle as there is no change in the values of \( m_{ij} \) as compared to the corresponding values of 4th cycle.

Hence the final values of the rotational components are \( m_{BA} = -0.677; m_{BC} = -1.627; m_{CB} = +1.008 \) and \( m_{CD} = +0.505 \)

It should be clearly noted that we have actually solved the *displacement equations* in the above iteration, since from Eq. 28.3, we observe that

\[ \theta_B = \frac{m_{BA}}{2E K_{BA}} = \frac{-0.677}{2E/6} = -2.031 \]

and

\[ \theta_C = \frac{m_{CB}}{2E K_{CB}} = \frac{1.008}{2E/2175} = +1.26 \]

Note that we obtained both these values of \( \theta_B \) and \( \theta_C \) by slope deflection method in Example 9.3 for the present beam.

Thus it is concluded that Kani's method indirectly solves the displacement equations. *This makes the solution self correcting.*

**Step 5. Computation of final moments at joint**

The final moments (\( M_{ij} \)) are computed from Eq. 28.2. The computations have been arranged in Table 28.2.

**KANI'S METHOD**

**Table 28.2.** Final moments

<table>
<thead>
<tr>
<th>( M_{ij} )</th>
<th>( M_{Fj} )</th>
<th>( 2m_{ij} )</th>
<th>( m_{ij} )</th>
<th>Sum (kn-m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_{AB} )</td>
<td>-6.0</td>
<td>0</td>
<td>-0.677</td>
<td>-6.677</td>
</tr>
<tr>
<td>( M_{BA} )</td>
<td>+6.0</td>
<td>-1.354</td>
<td>0</td>
<td>+4.646</td>
</tr>
<tr>
<td>( M_{BC} )</td>
<td>-2.4</td>
<td>-3.254</td>
<td>+1.008</td>
<td>-4.464</td>
</tr>
<tr>
<td>( M_{CB} )</td>
<td>+3.6</td>
<td>+2.016</td>
<td>-1.627</td>
<td>+3.989</td>
</tr>
<tr>
<td>( M_{CD} )</td>
<td>-5.0</td>
<td>+1.010</td>
<td>0</td>
<td>-3.990</td>
</tr>
<tr>
<td>( M_{DC} )</td>
<td>+5.0</td>
<td>0</td>
<td>+0.505</td>
<td>+5.505</td>
</tr>
</tbody>
</table>

It will be noted that these values are practically the same as obtained by the slope deflection method. The B.M.D. and deflected shape of the beam are shown in Fig. 28.3(b) and (c) respectively.

**Example 28.2** Solve example 28.1 if the ends A and D are hinged (or simply supported).

**Step 1. Computation of fixed end moments**

Considering ends of A and D as fixed, the fixed end moments at various joints will be as under (as calculated in the previous example).

\( M_{FAB} = -6.0 \); \( M_{FBA} = +6.0 \)
\( M_{FBC} = -2.4 \); \( M_{FCB} = +3.6 \)
\( M_{FCD} = -5.0 \); \( M_{DFC} = +5.0 \)

Actually, ends A and D are free. Hence releasing ends A and D, the *modified moments* at A and D will be

\( m_{FBA} = +6.0 + \frac{1}{2}(+6.0) = +9.0 \) kN-m.
\( m_{FCD} = -5.0 + \frac{1}{2}(-5.0) = -7.5 \) kN-m.

**Step 2. Rotation factors**

Since end A is hinged, the stiffness of \( AB \) will be 3/4 times its actual \( K \). Similarly, since end D is hinged, the stiffness of \( DC \) will be 3/4 times its actual \( K \). Based on this, the relative stiffness, distribution factors and rotation factors are calculated in Table 28.3.
Step 3. Resultant restraint moment

Compute resultant restraint moment at each joint by Eq. 28.5

\[ M_R = \sum M_{Rj} \]

\[ M_{FB} = M_{FB} + M_{FBC} = 9.0 - 2.4 = +6.6 \text{ kN-m.} \]

\[ M_{FC} = M_{FCA} + M_{FCD} = 3.6 - 7.5 = -3.9 \text{ kN-m.} \]

Enter these values within the small square (Fig. 28.4)

Step 4. Kani's Iteration cycles

Cycle 1: Kani's iteration procedure can now be commenced assuming all rotational components \((m_{ij})\) to be zero at all joints, which will indirectly mean that \(\theta_i = 0\). Note that \(m_{AB}\) and \(m_{DC}\) are zero.

Applying 28.8 at joint C and assuming \(m_{BC} = 0\) we get

\[ m_{CB} = R_{BC} (M_{FC}) = -0.364 (-3.9) = +1.420 \]

\[ m_{CD} = R_{CD} (M_{FC}) = -0.136 (-3.9) = +0.530 \]

These values are now used in Eq. 28.8 for computing rotational components \((m_{ij})\) at joint B. Thus

\[ m_{BC} = R_{BC} (M_{FB} + m_{CB}) = -0.381 (6.6 + 1.420) = -3.056 \]

\[ m_{BA} = R_{BA} (M_{FB} + m_{CB}) = -0.119 (6.6 + 1.420) = -0.954 \]

Cycle 2

At C,

\[ m_{CB} = -0.364 - 3.9 - 3.056 \]

\[ m_{CD} = -0.136 (-3.9 - 3.056) = +0.946 \]
Cycle 4  
At C,  \(m_{CB} = -0.364( -3.9 - 3.538) = + 2.707\)
\(m_{CD} = -0.136( -3.9 - 3.538) = + 1.012\)
At B,  \(m_{BC} = -0.381( +6.6 + 2.707) = -3.546\)
\(m_{BA} = -0.119( +6.6 + 2.707) = -1.107\)

Cycle 5  
At C,  \(m_{CB} = -0.364( -3.9 - 3.546) = + 2.710\)
\(m_{CD} = -0.136( -3.9 - 3.546) = + 1.013\)
At B,  \(m_{BC} = -0.381( +6.6 + 2.710) = -3.547\)
\(m_{BA} = -0.119( +6.6 + 2.710) = -1.108\)

The iteration may be terminated at the end of 5th cycle, as there is very little difference between the values of \(m_{ij}\) of 5th cycle and those of 4th cycle.

**Step 5. Computation of final moments**

Using Eq.28.2, the final moments at various joints can be computed, as shown in Table 28.4.

<table>
<thead>
<tr>
<th>(M_{BA})</th>
<th>(M_{BC})</th>
<th>(2m_{i})</th>
<th>(m_{j})</th>
<th>Sum (kN-m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>+ 9.0</td>
<td>- 2.216</td>
<td>0</td>
<td></td>
<td>+ 6.784</td>
</tr>
<tr>
<td>- 2.4</td>
<td>- 7.094</td>
<td>+ 2.710</td>
<td></td>
<td>- 6.784</td>
</tr>
<tr>
<td>+ 3.6</td>
<td>+ 5.420</td>
<td>- 3.547</td>
<td></td>
<td>+ 5.473</td>
</tr>
<tr>
<td>- 7.5</td>
<td>+ 2.026</td>
<td>0</td>
<td></td>
<td>- 5.474</td>
</tr>
</tbody>
</table>

The B.M.D. and the deflected shape of the beam are shown in Fig. 28.4(b) and (c) respectively.

**Example 28.3** Solve example 28.1 if there is no support at end D.

**Solution**

**Step 1 : Fixed end moments**

Take Ends B and C as clamped(fixed) so that \(AB\) and \(BC\) are considered as fixed beam. The overhanging portion \(CD\) becomes a cantilever fixed at C. Hence:
\(M_{FAB} = -6.00\) kN-m;  \(M_{FBA} = +6.00\) kN-m
\(M_{FBC} = -2.40\) kN-m;  \(M_{FCB} = +3.6\) kN-m
The rotation factors are computed in Table 28.5:

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>Distribution factor (DF)</th>
<th>Rotation factor R = -0.5 DF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BA</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{5}{17} )</td>
<td>- 0.1471</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>BC</td>
<td>( \frac{21}{5} )</td>
<td>( \frac{12}{17} )</td>
<td>- 0.3529</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>CB</td>
<td>( \frac{21}{5} )</td>
<td>1</td>
<td>- 0.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CD</td>
<td>0</td>
<td>0</td>
<td>0.0</td>
<td></td>
</tr>
</tbody>
</table>

Step 3. Resultant Restraint moment

\[ \sum M_{FB} = +6.0 - 2.40 = +3.6 \text{ kN} \cdot \text{m} \]

\[ \sum M_{FC} = +3.6 - 20 = -16.4 \text{ kN} \cdot \text{m} \]

Step 4. Kani's Iteration Cycles

**Cycle 1**

The iteration cycles are to be performed at joints \( B \) and \( C \) only since \( m_{AB} = 0 \) (fixed end) and \( m_{CD} = 0 \) (fixed end of cantilever).

To start with, let us assume \( m_{CB} = 0 \)

\[ \therefore \text{At joint } B : \]

\[ m_{BC} = R_{BC} (M_{FB} + m_{CB}) = -0.3529 (+3.6+0) = -1.270 \]

\[ m_{BA} = R_{BA} (M_{FB} + m_{CB}) = -0.1471 (+3.6+0) = -0.530 \]

At joint \( C \),

\[ m_{CB} = R_{CB} (M_{FC} + m_{BC}) = -0.5 (-16.4 - 1270) = +8.835 \]

**Cycle 2**

At joint \( B \), \( m_{BC} = -0.3529 (+3.6+8.835) = -4.388 \)

\[ m_{BA} = -0.1471 (+3.6+8.835) = -1.829 \]

At joint \( C \), \( m_{CB} = -0.5 (-16.4 - 4.338) = +10.369 \)

**Cycle 3.**

At joint \( B \), \( m_{BC} = -0.3529 (+3.6+10.369) = -4.930 \)

At joint \( C \), \( m_{CB} = -0.5 (-16.4 - 4.930) = +10.665 \)

**Cycle 4.**

At joint \( B \), \( m_{BC} = -0.3529 (+3.6+10.665) = -5.034 \)

\[ m_{BA} = -0.1471 (+3.6+10.665) = -2.098 \]

At joint \( C \), \( m_{CB} = -0.5 (-16.4 - 5.034) = +10.707 \)

**Cycle 5.**

At joint \( B \), \( m_{BC} = -0.3529 (+3.6+10.707) = -5.049 \)

\[ m_{BA} = -0.1471 (+3.6+10.707) = -2.098 \]

At joint \( C \), \( m_{CB} = -0.5 (-16.4 - 5.049) = +10.724 \)

**Cycle 6.**

At joint \( B \), \( m_{BC} = -0.3529 (+3.6+10.724) = -5.055 \)

\[ m_{BA} = -0.1471 (+3.6+10.724) = -2.107 \]

At joint \( C \), \( m_{CB} = -0.5 (-16.4 - 5.055) = +10.728 \)

At joint \( B \), \( m_{CB} = -0.3529 (+3.6+10.728) = -5.056 \)

\[ m_{BA} = -0.1471 (+3.6+10.728) = -2.108 \]

At joint \( C \), \( m_{CB} = -0.5 (-16.4 - 5.056) = +10.728 \)

The iteration is terminated at the end of 7th cycle.

**Step 5. Computation of Final moments** (Table 28.6)

<table>
<thead>
<tr>
<th>( M_i )</th>
<th>( M_{ij} )</th>
<th>( 2m_{ij} )</th>
<th>( m_{ij} )</th>
<th>Sum.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_{AB} )</td>
<td>- 6.0</td>
<td>0</td>
<td>- 2.108</td>
<td>- 8.108</td>
</tr>
<tr>
<td>( M_{BA} )</td>
<td>+ 6.0</td>
<td>- 4.216</td>
<td>0</td>
<td>+ 1.784</td>
</tr>
<tr>
<td>( M_{BC} )</td>
<td>- 2.4</td>
<td>- 10.112</td>
<td>+ 10.728</td>
<td>- 1.784</td>
</tr>
<tr>
<td>( M_{CB} )</td>
<td>+ 3.6</td>
<td>+ 21.456</td>
<td>- 5.056</td>
<td>+ 20.000</td>
</tr>
<tr>
<td>( M_{CD} )</td>
<td>- 20.0</td>
<td>0</td>
<td>0</td>
<td>- 20.000</td>
</tr>
</tbody>
</table>

The B.M.D. and the deflected shape of the beam are shown in Fig. 28.5(b) and (c) respectively.

**Example 28.4** Analyse the portal frame shown in Fig. 28.6 by Kani's method. Draw the B.M.D. and sketch the deflected shape of the frame. Take EI constant for all the members.
Solution:

Fig. 28.6

The frame is symmetrical, and is symmetrically loaded. Hence it will not sway.

Step 1. Fixed End moments

\[ M_{FBC} = -\frac{6 \times 4^2}{12} = -8 \text{ kN-m} \]

\[ M_{FCB} = +\frac{6 \times 4^2}{12} = +8 \text{ kN-m} \]

\[ M_{FAB} = 0; \quad M_{FDC} = 0. \]

Step 2. Rotation Factors

(TABLE 28.7)

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative Stiffness</th>
<th>Sum</th>
<th>( D_f )</th>
<th>( R = -0.5 \times D_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( BA ) (or ( CD ))</td>
<td>( I = \frac{2}{4} )</td>
<td>1/1</td>
<td>2/3</td>
<td>-0.3333</td>
<td></td>
</tr>
<tr>
<td>( BC ) (or ( CB ))</td>
<td>( I = \frac{3}{4} )</td>
<td>3/4</td>
<td>1/3</td>
<td>-0.1667</td>
<td></td>
</tr>
</tbody>
</table>

Cycle 1: To start with, let \( m_{CB} = 0 \). Hence at joint \( B \),

\[ m_{BC} = R_{BC} (M_{FB} + m_{CB} + m_{AB}) = -0.1667 (-8.0 + 0 + 0) = +1.33 \]

\[ m_{BA} = R_{BA} (M_{FB} + m_{CD} + m_{AB}) = -0.3333 (-8.0 + 0 + 0) = +2.67 \]

Hence at \( C \),

\[ m_{CA} = R_{CA} (M_{FC} + m_{CB} + m_{DC}) = -0.1667 (-8.0 + 1.33 + 0) = -1.56 \]

\[ m_{CD} = R_{CD} (M_{FC} + m_{BC} + m_{DC}) = -0.3333 (-8.0 + 1.33 + 0) = -3.11 \]

Kani's Method

Step 3. Resultant Restraint Moments

\[ M_{FB} = -8; \quad M_{FC} = +8. \]


In the case of frames there are horizontal members as well as vertical members. For horizontal members (i.e., beams) the rotational components \( (m_i) \) are written below the beams, while for vertical members (i.e., columns) the rotational components are written along the columns, as illustrated in Fig. 28.7. Ends \( A \) and \( D \) are fixed. Hence \( m_{AB} = 0 \) and \( m_{DC} = 0 \). We will do the computations up to the accuracy of second decimal place.

Fig. 28.7.
Example 28.5. A continuous beam shown in Fig. 28.9 has rigidly fixed ends C and D, is pinned at E and has rigid joints at A and B. The members are of uniform section and material throughout. Sketch the bending moment diagram for the frame, showing all important values. Also, find the values of the horizontal and vertical reactions at D and E. Use Kani’s method.

Solution

Step 1. Fixed End moments

\[ M_{FAB} = - \frac{12 \times 4 \times 2}{3^2} - \frac{12 \times 2 \times 1^2}{3^2} = -8 \text{ kN-m} \]

\[ M_{FBA} = + \frac{12 \times 2 \times 1^2}{3^2} + \frac{12 \times 1 \times 2^2}{3^2} = 8 \text{ kN-m} \]

\[ M_{FBC} = -\frac{4 \times 4^2}{12} = -5.33 \text{ kN-m} \]

\[ M_{FCD} = + \frac{4 \times 4^2}{12} = +5.33 \text{ kN-m} \]
**Step 2. Rotation Factors.**

**Table 28.9**

<table>
<thead>
<tr>
<th>Joint</th>
<th>Member</th>
<th>Relative stiffness</th>
<th>Sum</th>
<th>( D_F )</th>
<th>Rotation Factor ( R = -0.5 ) ( D_F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( AD )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{21}{3} )</td>
<td>0.5</td>
<td>-0.25</td>
</tr>
<tr>
<td>B</td>
<td>( BA )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{8}{3} )</td>
<td>0.4</td>
<td>-0.2</td>
</tr>
</tbody>
</table>

| B     | \( BE \) | \( \frac{3}{4} \) | \( \frac{10}{3} \) | 0.3 | -0.15 |
|       | \( BC \) | \( \frac{1}{4} \) | \( \frac{1}{2} \) | 0.3 | -0.15 |

**Step 3. Resultant Restraint moments**

- \( M_{FA} = -8 \ \text{kN} \cdot \text{m} \)
- \( M_{FB} = +8 - 5.33 = +2.67 \ \text{kN} \cdot \text{m} \)
- \( M_{FC} = +5.33 \)

**Step 4. Kani's Iteration cycles.**

Ends \( D \) and \( E \) are fixed. Hence \( m_{DA} = 0 \) and \( m_{CB} = 0 \)

Similarly, \( m_{EB} = 0 \)

**Cycle I:** Let us assume that at all joints \( m_y = 0 \) in the beginning.

At joint \( A \),

- \( m_{AB} = R_{AB} (M_{FA} + m_{DA} + m_{BA}) = -0.25 (-8 + 0 + 0) = +2.0 \)
- \( m_{AD} = R_{AD} (M_{FA} + m_{DA} + m_{BA}) = -0.25 (-8 + 0 + 0) = +2.0 \)

At joint \( B \),

- \( m_{BA} = R_{BA} (M_{FB} + m_{AB} + m_{CB} + m_{EB}) = 0.20 (+2.67 +2.0 + 0 + 0) = -0.93 \)
- \( m_{BE} = R_{BE} (M_{FB} + m_{AB} + m_{CB} + m_{EB}) = -0.15 (+2.67 +2.0 + 0 + 0) = -0.70 \)
- \( m_{BC} = R_{BC} (M_{FB} + m_{AB} + m_{CB} + m_{EB}) = 0.15 (+2.67 +2.0 + 0 + 0) = -0.70 \)

**Cycle II**

At \( A \),

- \( m_{AB} = -0.25 (-8 - 0.93) = +2.23 \)
- \( m_{AD} = -0.25 (-8 + 0.93) = +2.23 \)

At \( B \),

- \( m_{BA} = -0.20 (+2.67 + 2.23) = -0.98 \)

**Cycle III**

At \( A \),

- \( m_{AB} = -0.25 (-8 - 0.98) = +2.24 \)
- \( m_{AD} = -0.25 (-8 - 0.98) = +2.24 \)

At \( B \),

- \( m_{BA} = -0.20 (+2.67 + 2.24) = -0.98 \)
- \( m_{BE} = -0.15 (+2.67 + 2.24) = -0.73 \)
- \( m_{BC} = -0.15 (+2.67 + 2.24) = -0.73 \)

**Cycle IV**

At \( A \),

- \( m_{AB} = -0.25 (-8 - 0.98) = +2.24 \)
- \( m_{AD} = -0.25 (-8 - 0.98) = +2.24 \)
At B, 
\[ m_{BA} = -0.20 \times (2.67 + 2.24) = -0.98 \]
\[ m_{BE} = -0.15 \times (2.67 + 2.24) = -0.74 \]
\[ m_{BC} = -0.15 \times (2.67 + 2.24) = -0.74 \]

Step 5. Computation of final moments (Table 28.10)

<table>
<thead>
<tr>
<th>Table 28.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_y )</td>
</tr>
<tr>
<td>-------------</td>
</tr>
<tr>
<td>DA</td>
</tr>
<tr>
<td>AD</td>
</tr>
<tr>
<td>AB</td>
</tr>
<tr>
<td>BA</td>
</tr>
<tr>
<td>BC</td>
</tr>
<tr>
<td>BE</td>
</tr>
<tr>
<td>CB</td>
</tr>
</tbody>
</table>

The B.M.D. for the frame is shown in Fig. 28.11

Step 6. Computation of reactions

Considering the equilibrium of \( AD \) and taking moments about \( A \),
\[ H_D = \frac{M_{DA} + M_{AD}}{3} = \frac{2.24 + 4.49}{3} = 2.243 \text{ kN} \]

Similarly, taking moments about \( B \), of all forces below \( B \), we get
\[ -1.48 + 3H_E = 0 \]
\[ \therefore H_E = \frac{1.48}{3} = 0.493 \text{ kN} \]

Taking moments about \( B \), of all forces to the right to \( B \),
\[ -6.81 + 4.59 + 4V_C + 4 \times 4 \times 2 = 0 \]
\[ \therefore V_C = 7.42 \text{ kN} \]

Taking moments about \( B \), of all forces to the left to \( B \),
\[ 8.28 + 2.24 + 3V_D - (3 \times 2.24) - (12 \times 1) - (12 \times 2) = 0 \]
\[ \therefore V_D = 10.73 \text{ kN} \]

Considering the vertical equilibrium of the whole frame :
\[ V_E + 7.42 + 10.73 - 12 - 12 - (4 \times 4) = 0 \]
\[ V_E = 21.85 \text{ kN} \]

28.3. SYMMETRICAL FRAMES

Symmetrical frames are those which are not only symmetrical about a vertical line through the mid-points of central beams, but have symmetric load system. Such a frame can be easily analysed by considering only half the frame and by taking modified \( K \)-values for half the symmetric beams. This is illustrated below.
\[ \theta = \frac{ML}{2EI} \]

Again, Fig. 28.12(b) shows another beam, \( A'C' \) of span \( L/2 \) and sectional moment of inertia \( I/4 \), such that \( K_{A'C'} = \frac{1}{2} K_{AB} \).

The beam is fixed at \( C' \) and free to rotate at \( A' \). If a moment \( M \) is applied at \( A' \), we have
\[ \theta' = \frac{M (L/2)}{4EI} = \frac{ML}{EI} \]

\[ \theta = \theta' \]

Hence for those frames which are symmetric about mid-points of the horizontal beams, the following procedure may be adopted.

(i) Find fixed end moments at various joints, due to imposed loading.

(ii) Replace the horizontal beams, about which the frame is symmetrical by fictitious beams which are of half the length and are fixed at the ends. For such fictitious beams, the relative stiffness is taken half the value of the actual beam.

(iii) Carry out the computations for half the frame only.

As an illustration, let us re-analyse the portal frame of example 28.4. The frame is symmetrical about mid-point \( F \) of the beam \( BC \). Hence replace \( BC \) by half the beam \( BF \), fixed at \( F \), but having relative stiffness equal to \( \frac{1}{2} \left( \frac{I}{4} \right) = \frac{I}{8} \). However, fixed end moments will remain the same as found earlier.

\[
\begin{align*}
R_{BC} &= -0.5 \left[ \frac{1}{8} \right] = -0.1 \\
R_{BA} &= -0.5 \left[ \frac{1/2}{8 + \frac{1}{2}} \right] = -0.4 \\
M_{FBC} &= -6 \times \frac{4}{12} = -8 \text{ kN-m as before.}
\end{align*}
\]

Also, \( m_{EB} = 0 \) and \( m_{AB} = 0 \) since \( A \) and \( F \) are fixed.

Hence the distribution factors are as follows:
\[
R_{BC} = -0.5 \left[ \frac{1/8}{8 + \frac{1}{2}} \right] = -0.1
\]
\[
R_{BA} = -0.5 \left[ \frac{1/2}{8 + \frac{1}{2}} \right] = -0.4
\]
\[
M_{FBC} = -6 \times \frac{4}{12} = -8 \text{ kN-m as before.}
\]

\[
\begin{align*}
M_{FB} &= -8 \text{ kN-m} \\
\end{align*}
\]

Fig. 28.14.

For further cycles are necessary since \( m_{EB} \) and \( m_{AB} \) are zero throughout.

\[
\begin{align*}
M_{BE} &= M_{FBE} + 2 m_{BE} + m_{EB} = -8 + 1.6 = -6.40 \text{ kN-m} \\
M_{BA} &= M_{FBA} + 2 m_{BA} + m_{AB} = 0 + (2 \times 3.2) + 0 = +6.40 \text{ kN-m}
\end{align*}
\]
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and \[ M_{AB} = M_{FAB} + 2m_{AB} + m_{BA} = 0 + 0 + 3.2 = +3.2 \text{ kN-m}. \]

We observe that though all the three values are the same as found earlier in Example 28.4, the labour has very much been reduced. By symmetry,

\[ M_{CB} = M_{BC} = -6.4 \text{ kN-m}; M_{CD} = M_{DC} = +6.4; M_{DC} = M_{AB} = +3.2 \text{ kN-m}. \]

28.4. FRAMES WITH SIDE SWAY

Frames may sway due to one of the following reasons:

(i) Eccentric or unsymmetrical loading on the portal frame.

(ii) Unsymmetrical out-line of portal frame.

(iii) Different end conditions of the columns of the portal frame.

(iv) Non-uniform section of the members of the frame.

(v) Horizontal loading on the columns of the frame.

(vi) Settlement of the supports of the frame.

(vii) A combination of the above.

Such frames undergo both joint rotation as well as joint displacement.

Fig. 28.15.

The slope deflection equations for member AB is

\[ M_{AB} = M_{FAB} + 2 \frac{E I}{L_{AB}} (2 \theta_A + \theta_B - \frac{3 \theta_{AB}}{L_{AB}}) \]

or

\[ M_{AB} = M_{FAB} + 2 E K_{AB} (2 \theta_A + \theta_B - \frac{3 \theta_{AB}}{L_{AB}}) \]

or

\[ M_{AB} = M_{FAB} + 2 m_{AB} + m_{BA} + m'_{AB} \] ... (28.10)

where \( m_{AB} = 2 E K_{AB} \cdot \theta_A \)

\( m_{BA} = 2 E K_{BA} \cdot \theta_B \)

and \( m'_{AB} = - \frac{6 E K_{AB} d_{AB}}{L_{AB}} \) ... (28.11)

KANIS'S METHOD

Here \( m_{AB} \) and \( m_{BA} \), are rotation contribution of ends A and B to the total moments \( M_{AB} \) while \( m'_{AB} \), known as displacement contribution, constitute the contribution to \( M_{AB} \) by the lateral displacement \( d_{AB} \).

For generalisation, replacing \( A \) by \( i \) and \( B \) by \( j \), we have

\[ M_i = M_{Fi} + 2 m_j + m_j + m'_j \] ... (III) ... (28.10)

\( m_j = 2 E K_j \theta_j \); \( m'_j = 2 E K_j \theta_j \)

and \( m'_{ij} = - \frac{6 E K_{ij} d_{ij}}{L_{ij}} \) ... (28.11)

Now, for the equilibrium of joint \( i \),

\[ \sum M_i = 0 \]

Hence from Eq. 28.10,

\[ \sum M_i + \sum (2 m_j + m_j + m'_{ij}) = 0 \] ... (28.12)

Introducing \( M_F = \sum M_i = \) resultant restraint moment at \( i \),

we get

\[ M_F + \sum (2 m_j + m_j + m'_{ij}) = 0 \]

or \( \sum m_j = - \frac{1}{2} \left[ M_F + \sum (m_j + m'_{ij}) \right] \) ... (28.13)

Eq. 28.13 is similar to Eq. 28.6, except that the term \( m'_{ij} \) (displacement contribution) has been included.

Eq. 28.13 gives the algebraic sum of rotation contributions of all members meeting at joint \( i \). Now we know that \( m_j \) for any member is proportional to its \( K \)-value. The individual share of the members in the total rotation contribution can be found by distributing the total rotation contribution in proportion to their respective \( K \)-values. Thus, for member \( ij \),

\[ m_j = \frac{K_{ij}}{\sum K_i} \sum m_j \]

\( \therefore \)

\[ m'_j = - \frac{1}{2} \left[ \frac{K_{ij}}{\sum K_i} \left( M_F + \sum (m_j + m'_{ij}) \right) \right] \]

or \( m'_j = R_{ij} \left[ M_F + \sum (m_j + m'_{ij}) \right] \) ... (IV) ... (28.14)
where \( R_{ij} \) = rotation factor for \( ij = -\frac{1}{2} \left( \frac{K_{ij}}{\sum K_{ij}} \right) \)

Thus, from Eq. 28.14, rotation contribution \( m_{ij} \) can be computed, provided the displacement contribution \( m_{ij} \) is known. Let us, therefore, discuss the method for computing displacement contribution \( m_{ij} \) by taking several cases.

CASE 1: FRAME WITH COLUMNS OF EQUAL HEIGHT AND SUBJECTED TO VERTICAL LOADS ONLY

Fig. 28.16 shows the actual case of frame with vertical loading, which causes side sway. As done in the moment distribution method, the solution is accomplished in two steps.

Step 1. No Sway: Artificial restraints are applied so that the frame does not sway, and the frame is analysed as discussed previously.

Step 2. Sway forces: Sway forces are now applied, and the frame is analysed.

The total solution may then be obtained as the algebraic sum of the solutions obtained in the above two steps.

Development of expression for displacement contribution \( m_{ij} \)

Let us pass a horizontal section \( x-x \). For the horizontal equilibrium of the part of the frame above the section \( x-x \), the algebraic sum of column shears must vanish.

\[
\sum Q_{q} = 0
\]

where \( \sum \) denotes the sum of shears for all columns in \( i^{th} \) storey and \( Q_{ij} \) = shear in column \( ij \) of \( i^{th} \) storey.

If \( h_{r} \) is the height of \( i^{th} \) storey,

\[
Q_{q} = \frac{M_{q} + M_{h}}{h_{r}}
\]

But

\[
M_{q} = M_{eq} + 2m_{q} + m_{q} + m'_{q},
\]

\[
M_{h} = M_{eh} + 2m_{h} + m_{h} + m'_{h}
\]

Substituting these values in Eq. 28.15, and noting the fact that fixed end moment in the column ends are zero because of absence of intermediate horizontal loads, and also noting that

\[
m'_{q} = m'_{h} = -\frac{6E}{I_{q}} \frac{d_{ij}}{h_{r}},
\]

we get

\[
Q_{q} = \frac{1}{h_{r}} \left( 3m_{q} + 3m_{h} + 2m'_{q} \right)
\]

\[
\sum Q_{q} = \sum \frac{1}{h_{r}} \left[ 3(m_{q} + m_{h}) + 2m'_{q} \right] = 0
\]

\[
\sum m'_{ij} = -\frac{3}{2} \sum (m_{q} + m_{h})
\]

This shows that the algebraic sum of displacement contribution of all the columns of \( i^{th} \) storey is equal to -1.5 times that of the rotation contributions of the two ends of the same columns.

Now since the \( m'_{ij} \) for any column is proportional to its \( K \)-value, the individual share of the members in the displacement contribution can be found by distributing the total displacement contribution in proportion to their respective \( K \)-values. Thus, for any column \( ij \),

\[
m'_{ij} = -1.5 \left( \frac{K_{ij}}{\sum K_{ij}} \right) \sum (m_{q} + m_{h})
\]

Introducing a displacement factor \( D_{ij} \) for any column \( ij \) by the expression

\[
D_{ij} = -1.5 \left( \frac{K_{ij}}{\sum K_{ij}} \right)
\]

We get from Eq. 28.17,

\[
m'_{ij} = D_{ij} \sum (m_{q} + m_{h})
\]
From Eqs. 28.14 and 28.19, we note that the rotation contribution ($m_{ij}$ and $m_{ji}$) and the displacement contribution $m_{ij}$ are interdependent. Equation 28.19 can also be solved by Gauss-Seidel iteration in conjunction with Eq. 28.14.

When once the rotation contributions and the displacement contribution are known, the final moments at various joints can be found by Eq. 28.10. Example 28.6 illustrates the complete procedure wherein solution has been obtained for a frame with sway in a single table.

**CASE 2. FRAMES WITH COLUMNS OF EQUAL HEIGHT AND SUBJECTED TO HORIZONTAL LOADS**

The procedure is the same as in the previous case, except that the shear equation is changed. For the horizontal equilibrium of the part of the frame above the section $x-x$, the storey shear must be equal to applied horizontal forces above the section.

![Diagram](image)

**Fig. 28.17.**

$Q_r = H_1 + H_2 + \ldots + H_r$

Hence the horizontal shear equation for $i^{th}$ storey may be written as

$$Q + \sum_r \frac{1}{K_r} (M_i + M_j) = 0 \quad \ldots(28.20)$$

Here $ij$ denotes the columns of $i^{th}$ storey.

But $M_i = M_{i0} + 2 m_{ij} + m_{ji} + m_{ij}'$

and $M_j = M_{j0} + 2 m_{ji} + m_{ij} + m_{ji}'$

**KANIS'S METHOD**

Noting that the fixed end moments are zero in absence of intermediate horizontal loads, and also noting that $m_{ij}' = m_{ji}'$ we get

$$Q_r h_r + \sum_r \left[ 3(m_{ij} + m_{ji}) + 2m_{ij}' \right] = 0$$

or

$$\sum_r m_{ij}' = -\frac{3}{2} \left[ \frac{Q_r h_r}{3} + \sum_r (m_{ij} + m_{ji}) \right] \quad \ldots(28.21)$$

It should be noted that Eq. 28.21 differs from Eq. 28.16 of case 1, only by the extra term $\frac{Q_r h_r}{3} = M_r = \text{storey moment}$.

Now since the $m_{ij}'$ for any column is proportional to its $K$-value, the individual share of the members in the displacement contribution can be found by distributing the total displacement contribution in proportion to their respective $K$-values. Thus, for any column $ij$, $m_{ij} = \frac{K_r}{\sum K_r} \cdot m_{ij}'$

$$m_{ij}' = -\frac{3}{2} \frac{K_r}{\sum K_r} \left[ \frac{Q_r h_r}{3} + \sum_r (m_{ij} + m_{ji}) \right]$$

or

$$m_{ij}' = D_{ij} \left[ M_r + \sum_r (m_{ij} + m_{ji}) \right] \quad \ldots(VI) \quad \ldots(28.22)$$

where $M_r = \text{storey moment} = \frac{Q_r h_r}{3}$. Comparing this with Eq. 28.19, we observe that an extra term $M_r$ has been introduced here, which needs to be calculated and added to the rotation contribution before $m_{ij}'$ is calculated.

**Column of equal height with hinged base.**

If the columns of bottom storey, all of the same height, are hinged at the base instead of being fixed, these columns can be replaced by columns with fixed bases but with $K$-values $3/4$ of those of corresponding actual members. Naturally, the factor $-\frac{3}{2}$ will be replaced by $\left( -\frac{3}{2} \times \frac{4}{3} \right) = -2$ in Eq. 28.18.

$$Hence \quad D_{ij} = -\frac{2}{\sum K_r} \quad \ldots(28.18)(a)$$

If, however, some columns are hinged while others are fixed, in a given storey, it can be shown that
where \( m = \frac{3}{4} \) for hinged columns and \( m = 1 \) for fixed columns.

**CASE 3: FRAMES WITH COLUMNS OF UNEQUAL HEIGHT**

It may happen that the columns of any storey mostly the bottom storey are of unequal heights, subjected to horizontal loads.

Let \( h_0 \) be the height of any column \( ij \) of \( ith \) storey which is different from height \( h'_0 \) of any arbitrarily chosen column of the same storey, where \( h'_0 \) is a constant and is called the storey height.

Let \( C_{ij} = \text{reduction factor} \) for the \( i-j \) column = \( \frac{h'_0}{h_0} \) \( \ldots \) \( (28.23) \)

![Diagram](image)

**Fig. 28.18.**

For the \( r \)th storey, the horizontal shear equation is

\[
Q_r + \sum \frac{1}{h_0} (M_y + M_f) = 0
\]

This is same as Eq. 28.20 except that the term \( h_r \), representing constant storey height has been replaced by \( h_{ij} \) which is variable from column to column.

Eq. 28.24 can also be written as

\[
Q_r h'_r + \sum \frac{h'_r}{h_0} (M_y + M_f) = 0
\]

or

\[
Q_r h'_r + \sum C_{ij} (M_y + M_f) = 0
\]

\( \ldots \) \( (28.25) \)

**KANIS METHOD**

\begin{align*}
\text{But} & \quad M_y = M_{eq} + 2m_y + m_y + m'_y \\
& \quad M_f = m_{eq} + 2m_f + m'_y + m'_f
\end{align*}

Noting that the fixed end moments are zero in absence of intermediate horizontal loads, and noting that \( m'_y = m'_f \) we get, as before

\[
\begin{align*}
Q_r h'_r + \sum \frac{C_{ij}}{3} (3m_y + 3m_f + 2m'_y) &= 0 \\
\sum \frac{C_{ij}}{3} m'_y &= -\frac{3}{2} \left[ \frac{Q_r h'_r}{3} + \sum (m_y + m_f) \right] \ldots \text{28.26} \quad (a) \\
\sum C_{ij} m'_y &= -\frac{3}{2} \left[ M_r + \sum C_{ij} (m_y + m_f) \right] \ldots \text{28.26}
\end{align*}
\]

where \( M_r = \text{storey moment} = \frac{Q_r h'_r}{3} \)

Now, by definition, \( m'_y = -\frac{6 E K_q d_q}{h_0} \). Hence, \( m'_y \) is proportional to \( \frac{K_q}{h_0} \) and hence to \( C_{ij} K_q \).

Hence

\[
\sum C_{ij} m'_y = \frac{C_{ij} K_q}{\sum C_{ij} K_q}
\]

\( \ldots \) \( (28.27) \)

Hence from Eqs 28.26 and 28.27

\[
\begin{align*}
m'_y &= -1.5 \frac{C_{ij} K_q}{\sum C_{ij} K_q} \left[ M_r + \sum C_{ij} (m_y + m_f) \right] \ldots \text{28.28} \quad (a) \\
\text{or} \quad m'_y &= D_y \left[ M_r + \sum C_{ij} (m_y + m_f) \right] \ldots \text{28.28}
\end{align*}
\]

where

\[
D_y = -1.5 \frac{C_{ij} K_q}{\sum C_{ij} K_q} = \text{Displacement factor} \ldots \text{28.29}
\]

If some columns are hinged and some are fixed, it can be shown that

\[
D_y = -1.5 \frac{C_{ij} K_q}{\sum m C_{ij} K_q} \ldots \text{28.30}
\]
where \( m = \frac{3}{4} \) for hinged columns and \( m = 1 \) for columns with fixed ends. If, however, all the columns of the storey are hinged, \( m = \frac{3}{4} \) for all the columns, and hence

\[
D_j = -2.0 \frac{C_y K_y}{\sum_r C_y^2 K_y} \quad \text{(28.31)}
\]

**Example 28.6.**

Using Kani's method, analyse the portal frame shown in Fig. 28.19.

**Solution**

![Fig. 28.19.](image)

The frame is geometrically asymmetrical, and is asymmetrically loaded. Hence it will sway. However, since the legs are of equal length, and since there is no imposed horizontal load, it falls under case 1.

**Step 1. Fixed End moments**

\[
M_{FBC} = -20 \times 5 \left( \frac{15}{20} \right)^2 = -56.25 \text{ kN-m}
\]

\[
M_{FCD} = +20 \times 5 \left( \frac{15}{20} \right)^2 = +18.75 \text{ kN-m}
\]

**Step 2 Rotation factors**

Now \( K_j = \frac{I_j}{L_j} \) and \( R_j = -0.5 \frac{K_j}{\sum_j K_j} \)

**Step 3. Displacement factors**

\[
D_j = -1.5 \frac{K_j}{\sum_r K_y}
\]

\[
R_{BA} = -0.5 \frac{2I/10}{2I/10 + 3I/10} = -0.5 \times \frac{2}{5} = -0.2
\]

\[
R_{BC} = -0.5 \frac{3I/10}{2I/10 + 3I/10} = -0.5 \times \frac{3}{5} = -0.3
\]

\[
R_{CD} = -0.5 \frac{I/10}{3I/10 + I/10} = -0.5 \times \frac{1}{4} = -0.125
\]

**Step 4. Resultant restraint moments**

\[
M_{FB} = -56.25 \text{ and } M_{FC} = +18.75
\]

**Steps. Kani's Iteration cycles**

The fixed end moments, restraint moments and rotation factors are entered as usual. The displacement factors and displacement contributions are entered transverse to each column, as shown in Fig. 28.20.

Cycles. The rotation contribution \( m_{ij} \) at joint \( i \) is given by Eq. 28.14.
To start with neither \( m_{ij} \) nor \( m_i \) are known, and hence these can be assumed to be zero.

At B, \( m_{BC} = -0.3 [-56.25 + 0 + 0] = +16.88 \)
\( m_{BA} = -0.2 [-56.25 + 0 + 1] = +11.25 \)

At C, \( m_{CB} = R_{CB} \left[ M_{FC} + m_{BC} + m_{DC} + m_i \right] \)
\( = -0.375 [ +18.25 + 16.88 + 0 + 0 ] = -13.17 \)
\( m_{CD} = R_{CD} \left[ M_{FC} + m_{EC} + m_{DC} + m_i \right] \)
\( = -0.125 [ +18.25 + 16.88 + 0 + 0 ] = -4.39 \)

Displacement contribution The displacement contributions are found by Eq. 28.19, according to which the displacement contribution of any column is equal to displacement factor of that column multiplied by algebraic sum of rotation contributions of all the columns of that storey. Hence

\[
m'_{BA} = D_{BA} \left[ m_{BA} + m_{AB} + m_{CD} + m_{DC} \right] \\
= -1.0 \left[ 11.25 + 0 - 4.39 + 0 \right] = -6.86
\]
\[
m'_{CD} = D_{CD} \left[ m_{BA} + m_{AB} + m_{CD} + m_{DC} \right] \\
= -0.5 \left[ 11.25 + 0 - 4.39 + 0 \right] = -3.43
\]

This constitutes the first cycle.

Cycle 2

(a) Rotation factors

Joint B : \( m_{BC} = R_{BC} \left[ M_{FB} + m_{CB} + m_{AB} + m'_{BA} \right] \)
\( = -0.3 \left[ -56.25 - 13.17 + 0 - 6.86 \right] = +22.84 \)
\( m_{BC} = R_{BA} \left[ M_{FB} + m_{CB} + m_{AB} + m'_{BA} \right] \)
\( = -0.2 \left[ -56.25 - 33.17 + 0 - 6.86 \right] = +15.26 \)

Joint C : \( m_{CB} = R_{CB} \left[ M_{FC} + m_{BC} + m_{DC} + m_{CD} \right] \)
\( = -0.375 [ +18.25 + 22.84 + 0 - 3.43 ] = -14.12 \)
\( m_{CD} = R_{CD} \left[ M_{FC} + m_{BC} + m_{DC} + m_{CD} \right] \)
\( = -0.125 [ +18.25 + 22.84 + 0 - 3.43 ] = -4.71 \)

(b) Displacement factors

\( m_{BA} = -1.0 \left[ 15.26 + 0 - 4.71 + 0 \right] = -10.55 \)
\( m_{CD} = -0.5 \left[ 15.26 + 0 - 4.71 + 0 \right] = -5.28 \)

Cycle 3

(a) Rotation factors

Joint B : \( m_{BC} = -0.3 \left[ -56.25 - 14.12 + 0 - 10.55 \right] = +24.28 \)
\( m_{BA} = -0.2 \left[ -56.25 - 14.12 + 0 - 10.55 \right] = +16.18 \)

Joint C : \( m_{CB} = -0.375 \left[ +18.25 + 24.28 + 0 - 5.28 \right] = -13.97 \)
\( m_{CD} = -0.125 \left[ +18.25 + 22.84 + 0 - 5.28 \right] = -4.66 \)
(b) Displacement factors

\[ m'_{BA} = -1.0[16.18 + 0 - 4.66] = -11.52 \]
\[ m'_{CD} = -1.5[16.18 + 0 - 4.66] = -5.76 \]

**Cycle 4**

(a) Rotation Factors

Joint B : \[ m_{BC} = -0.3[-56.25 - 13.97 + 0 - 11.52] = +24.52 \]
\[ m_{BA} = -0.2[-56.25 - 13.93 + 0 - 11.52] = +16.35 \]

Joint C : \[ m_{CB} = -0.375[+18.25 + 24.52 + 0 - 5.76] = -13.88 \]
\[ m_{CD} = -0.125[18.25 + 24.52 + 0 - 5.76] = -4.63 \]

(b) Displacement Factors

\[ m'_{BA} = -1.0[16.35 + 0 - 4.63] = -11.72 \]
\[ m'_{CD} = -0.5[16.35 + 0 - 4.63] = -5.86 \]

**Cycle 5**

(a) Rotation factors

Joint B : \[ m_{BC} = -0.3[-56.25 - 13.88 + 0 - 11.72] = +24.56 \]
\[ m_{BA} = -0.2[-56.25 - 13.88 + 0 - 11.72] = +16.37 \]

Joint C : \[ m_{CB} = -0.375[+18.25 + 24.56 + 0 - 5.86] = -13.86 \]
\[ m_{CD} = -0.125[18.25 + 24.56 + 0 - 5.86] = -4.62 \]

(b) Displacement Factors

\[ m'_{BA} = -1.0[16.35 + 0 - 4.62] = -11.73 \]
\[ m'_{CD} = -0.5[16.35 + 0 - 4.62] = -5.87 \]

The iteration may be terminated now because the difference between the successive values is very small.

**Step 6. Final moments**

The final moments can be found by Eq. 28.10:

\[ M_y = M_{Fy} + 2m_y + m_y + m_y' \]

The computations are arranged in Tabular form below.

<table>
<thead>
<tr>
<th>( M_y )</th>
<th>( M_{Fy} )</th>
<th>( 2m_y )</th>
<th>( m_y )</th>
<th>( m_y' )</th>
<th>Total (kN-m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_{BA} )</td>
<td>0</td>
<td>+32.74</td>
<td>0</td>
<td>-11.73</td>
<td>+21.01</td>
</tr>
<tr>
<td>( M_{BC} )</td>
<td>-56.25</td>
<td>+49.12</td>
<td>-13.86</td>
<td>-20.99</td>
<td></td>
</tr>
<tr>
<td>( M_{CB} )</td>
<td>+18.25</td>
<td>-27.72</td>
<td>+24.56</td>
<td>+15.09</td>
<td></td>
</tr>
<tr>
<td>( M_{CD} )</td>
<td>0</td>
<td>-9.24</td>
<td>0</td>
<td>-5.87</td>
<td>-15.11</td>
</tr>
<tr>
<td>( M_{AB} )</td>
<td>0</td>
<td>0</td>
<td>+16.37</td>
<td>-11.73</td>
<td>+4.64</td>
</tr>
<tr>
<td>( M_{DC} )</td>
<td>0</td>
<td>0</td>
<td>-4.62</td>
<td>-5.87</td>
<td>-10.49</td>
</tr>
</tbody>
</table>

The B.M.D. for the frame is shown in Fig. 28.21.

Example 28.7.

Draw the bending moment diagram and sketch the deflected shape of the frame shown in Fig. 28.22.

Solution: The frame will evidently sway.

**Step 1. Fixed end moments**

\[ M_{FBC} = -\frac{6(2)^2}{12} = -2.0 \text{ kN-m} \]
\[ M_{FCB} = +2.0 \text{ kN-m} \]
Step 2. Rotation Factors

\[ K_y = \frac{I_y}{L_y} \quad \text{and} \quad R_y = -0.5 \sum K_y \]

\[ K_{BA} = \frac{2I}{3} \quad \text{;} \quad K_{BC} = \frac{I}{2} \quad \text{;} \quad K_{CD} = \frac{I}{2} \]

\[ R_{BA} = -0.5 \times \frac{2I/3}{2I/3 + 1/2} = -0.5 \times \frac{2}{3} \times \frac{6}{7} = -0.2857 \]

\[ R_{BC} = -0.5 \times \frac{1/2}{3/2 + 1/2} = -0.5 \times \frac{1}{2} \times \frac{6}{7} = -0.2143 \]

\[ R_{CB} = -0.5 \times \frac{1/2}{1/2 + 1/2} = -0.5 \times \frac{1}{2} \times 1 = -0.25 \]

\[ R_{CD} = -0.5 \times \frac{1/2}{1/2 + 1/2} = -0.5 \times \frac{1}{2} \times 1 = -0.25 \]

Step 3. Displacement Factors

\[ D_y = -1.5 \times \frac{C_y K_y}{\sum C_y K_y}, \quad \text{because of unequal legs.} \]

Let the reference height be taken as 3 m, so that

\[ C_{BA} = \frac{3}{3} = 1 \quad \text{and} \quad C_{CD} = \frac{3}{2} = 1.5 \]

KANIS METHOD

\[ D_{BA} = -1.5 \times \frac{C_{BA} \cdot K_{BA}}{(C_{ab})^2 \cdot K_{ab} + (C_{cd})^2 \cdot K_{cd}} = -1.5 \times \frac{1.5 \times \frac{2I}{3}}{(1/2)^2 \times \frac{2I}{3} + (1.5/2)^2 \times \frac{2I}{2}} \]

\[ = -0.5581 \]

\[ D_{CD} = -1.5 \times \frac{C_{CD} \cdot K_{CD}}{(C_{ab})^2 \cdot K_{ab} + (C_{cd})^2 \cdot K_{cd}} = -1.5 \times \frac{1.5}{(1/2)^2 \times \frac{2I}{3} + (1.5/2)^2 \times \frac{2I}{2}} \]

\[ = -0.6279 \]

Step 4. Resultant Restraint Moments

\[ M_{RB} = -2.0 \quad \text{;} \quad M_{RC} = +2.0 \quad \text{kN-m.} \]

Step 5. Kani's Iteration cycles

Cycle 1.

The rotation contribution \( m_y \) at a joint i is given by Eq 28.14

\[ m_y = R_y \left[ M_i + \sum_j (m_j + m'_j) \right] \]

To start with, neither \( m_j \) nor \( m'_j \) are known, and hence these can be assumed to be zero.

At B:\n
\[ m_{RB} = R_{BC} \left[ M_{RB} + m_{CB} + m_{DB} + m_{EA} \right] \]

\[ = -0.2143 \left[ -2.0 + 0 + 0 + 0 \right] = +0.429 \]

\[ m_{EA} = -0.2857 \left[ -2.0 + 0 + 0 + 0 \right] = +0.571 \]

At C:\n
\[ m_{CB} = R_{CB} \left[ M_{RC} + m_{BC} + m_{CD} + m_{ED} \right] \]

\[ = -0.25 \left[ +2.0 + 0.429 + 0 + 0 \right] = -0.607 \]

\[ m_{ED} = R_{CD} \left[ M_{RC} + m_{BC} + m_{CD} + m_{ED} \right] \]

\[ = -0.25 \left[ +2.0 + 0.429 + 0 + 0 \right] = -0.607 \]

Displacement Contribution

The displacement contributions given by Eq. 28.28, taking \( M_r = 0 \):

\[ m_y = D_y \left[ \sum C_y \left( m_y + m'_y \right) \right] \]

\[ m_{EA} = D_{EA} \left[ C_{EA} \left( m_{EA} + m'_E \right) + C_{EC} \left( m_{EC} + m_{DE} \right) \right] \]

\[ = -0.5581 \left[ 1 \left( 0.571 + 0 \right) + 1.5 \left( -0.607 + 0 \right) \right] = +0.189 \]

\[ m_{CD} = D_{CD} \left[ 1 \left( m_{BA} + m_{AB} \right) + 1.5 \left( m_{CD} + m_{DE} \right) \right] \]

\[ = -0.6279 \left[ 1 \left( 0.571 + 0 \right) + 1.5 \left( -0.607 + 0 \right) \right] = +0.213 \]

Fig. 28.22
Joint C : $m_{BC} = -0.25 \ [2.0 + 0.535 + 0 + 0.209] = -0.686$
$m_{CD} = -0.25 \ [2.0 + 0.535 + 0 + 0.209] = -0.686$

(b) Displacement Factors
$m_{BA} = -0.5581 \ [1 \times 0.713 + 0 - 1.5 \times 0.686 + 0] = +0.176$
$m_{CD} = -0.6279 \ [1 \times 0.713 + 0 - 1.5 \times 0.686 + 0] = +0.198$

Cycle 4 (a) Rotation Factors
Joint B : $m_{BC} = -0.2143 \ [-2.0 - 0.686 + 0 + 0.176] = +0.538$
$m_{BA} = -0.2857 \ [-2.0 - 0.686 + 0 + 0.176] = +0.717$
Joint C : $m_{CB} = -0.25 \ [2.0 + 0.538 + 0 + 0.198] = -0.684$
$m_{CD} = -0.25 \ [2.0 + 0.538 + 0 + 0.198] = -0.684$

(b) Displacement Factors
$m_{BA} = -0.5581 \ [1 \times 0.717 + 0 - 1.5 \times 0.684 + 0] = +0.172$
$m_{CD} = -0.6279 \ [1 \times 0.717 + 0 - 1.5 \times 0.684 + 0] = +0.194$

Cycle 5 (a) Rotation Factors
Joint B : $m_{BC} = -0.2143 \ [-2.0 - 0.684 + 0 + 0.172] = +0.538$
$m_{BA} = -0.2857 \ [-2.0 - 0.684 + 0 + 0.172] = +0.718$
Joint C : $m_{CB} = -0.25 \ [2.0 + 0.538 + 0 + 0.194] = -0.683$
$m_{CD} = -0.25 \ [2.0 + 0.538 + 0 + 0.194] = -0.683$

(b) Displacement Factors
$m_{BA} = -0.5581 \ [1 \times 0.717 + 0 - 1.5 \times 0.683 + 0] = +0.171$
$m_{CD} = -0.6279 \ [1 \times 0.717 + 0 - 1.5 \times 0.684 + 0] = +0.192$

Step 6. Final moments
The final moment at any joint is found by Eq. 28.10:
$M_y = M_{xy} + 2m_y + m_x + m_y$
The computations are arranged in Table 28.12.

<table>
<thead>
<tr>
<th>$M_x$</th>
<th>$M_y$</th>
<th>$2m_y$</th>
<th>$m_{ij}$</th>
<th>$m_i$</th>
<th>$Total$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{BA}$</td>
<td>0</td>
<td>0</td>
<td>1.436</td>
<td>0</td>
<td>+ 1.608</td>
</tr>
<tr>
<td>$M_{BC}$</td>
<td>- 2.0</td>
<td>+ 1.076</td>
<td>- 0.683</td>
<td>-</td>
<td>- 1.607</td>
</tr>
<tr>
<td>$M_{CB}$</td>
<td>+ 2.0</td>
<td>- 1.366</td>
<td>+ 0.538</td>
<td>-</td>
<td>+ 1.172</td>
</tr>
<tr>
<td>$M_{CD}$</td>
<td>0</td>
<td>- 1.366</td>
<td>0</td>
<td>+ 0.192</td>
<td>- 1.174</td>
</tr>
<tr>
<td>$M_{AB}$</td>
<td>0</td>
<td>0</td>
<td>+ 0.718</td>
<td>+ 0.172</td>
<td>+ 0.89</td>
</tr>
<tr>
<td>$M_{AC}$</td>
<td>0</td>
<td>0</td>
<td>- 0.683</td>
<td>+ 0.192</td>
<td>- 0.491</td>
</tr>
</tbody>
</table>

The B.M.D. and the deflected shape are shown in Fig. 28.24(a) and (b) respectively.

![Fig. 28.24.](image)

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